Supplementary Material for "Gradual Typing for Extensibility by Rows"

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December 30, 2024

1 Definition

1.1 Statically typed language $\mathbf{F}^\rho$

1.1.1 Syntax

$$
\begin{array}{l}
\textbf{Variables for types and rows} \quad X \qquad \textbf{Kinds} \quad K ::= \mathsf{T} \mid \mathsf{R} \\
\textbf{Base types} \quad \iota ::= \mathsf{bool} \mid \mathsf{int} \mid \ldots \qquad \textbf{Constants} \quad \kappa ::= \mathsf{true} \mid \mathsf{false} \mid 0 \mid + \mid \ldots \\
\begin{array}{llcl}
\textbf{Types and rows} & A, B, C, D, \rho & ::= & X \mid \iota \mid A \rightarrow B \mid \forall X{:}K.\, A \mid [\rho] \mid \langle \rho \rangle \mid \cdot \mid \ell : A; \rho \\
\textbf{Terms} & M & ::= & x \mid \kappa \mid \lambda x{:}A.M \mid M_1\, M_2 \mid \Lambda X{:}K.\, M \mid M\ A \mid \\
 & & & \{\} \mid \{\ell = M_1; M_2\} \mid \mathsf{let}\, \{\ell = x; y\} = M_1 \,\mathsf{in}\, M_2 \mid \\
 & & & \ell\, M \mid \mathsf{case}\, M \,\mathsf{with}\, \langle \ell\, x \rightarrow M_1; y \rightarrow M_2 \rangle \mid \uparrow\! \langle \ell : A \rangle\, M
\end{array} \\
\textbf{Values} \quad w \ ::= \ \kappa \mid \lambda x{:}A.M \mid \Lambda X{:}K.\, M \mid \{\} \mid \{\ell = w_1; w_2\} \mid w^\ell \qquad w^\ell \ ::= \ \ell\, w \mid \uparrow\! \langle \ell : A \rangle\, w^\ell \\
\begin{array}{llcl}
\textbf{Evaluation contexts} & F & ::= & [\,] \mid F\, M_2 \mid w_1\, F \mid F\ A \mid \\
 & & & \{\ell = F; M_2\} \mid \{\ell = w_1; F\} \mid \mathsf{let}\, \{\ell = x; y\} = F \,\mathsf{in}\, M_2 \mid \\
 & & & \ell\, F \mid \mathsf{case}\, F \,\mathsf{with}\, \langle \ell\, x \rightarrow M_1; y \rightarrow M_2 \rangle \mid \uparrow\! \langle \ell : A \rangle\, F \\
\textbf{Typing contexts} & \Gamma & ::= & \emptyset \mid \Gamma, x{:}A \mid \Gamma, X{:}K
\end{array}
\end{array}
$$

Definition 1 (Free type variables and type substitution)**.** *The set $ftv(A)$ of free variables for types and rows in A is defined as usual. Substitution $A[B/X]$ of B for X in A is defined in a capture-avoiding manner.*

Definition 2 (Domain of typing contexts)**.** *We define $dom\,(\Gamma)$ as follows.*

$$
\begin{array}{lcl}
dom\,(\emptyset) & \stackrel{\text{def}}{=} & \emptyset \\
dom\,(\Gamma, x{:}A) & \stackrel{\text{def}}{=} & dom\,(\Gamma) \cup \{x\} \\
dom\,(\Gamma, X{:}K) & \stackrel{\text{def}}{=} & dom\,(\Gamma) \cup \{X\}
\end{array}
$$

Assumption 1. *We suppose that each constant κ is assigned a first-order type $ty(\kappa)$ of the form $\iota_1 \rightarrow \cdots \rightarrow \iota_n$.*

Suppose that, for any ι, there is a set $\mathbb{K}_\iota$ of constants of ι. For any constant κ, $ty(\kappa) = \iota$ if and only if $\kappa \in \mathbb{K}_\iota$. The function ζ gives a denotation to pairs of constants. In particular, for any constants κ_1 and κ_2: (1) $\zeta(\kappa_1, \kappa_2)$ is defined if and only if $ty(\kappa_1) = \iota \rightarrow A$ and $ty(\kappa_2) = \iota$ for some A; and (2) if $\zeta(\kappa_1, \kappa_2)$ is defined, $\zeta(\kappa_1, \kappa_2)$ is a constant and $ty(\zeta(\kappa_1, \kappa_2)) = A$ where $ty(\kappa_1) = \iota \rightarrow A$.

We use the notation and the assumption above also in $\mathrm{F}^\rho_{\mathrm{G}}$ and $\mathrm{F}^\rho_{\mathrm{C}}$.

1.1.2 Semantics

Definition 3 (Record splitting)**.** *$w \rhd_\ell w_1, w_2$ is defined as follows:*

$$\begin{array}{l} \{\ell = w_1; w_2\} \rhd_\ell w_1, w_2 \\ \{\ell' = w_1; w_2\} \rhd_\ell w_{21}, \{\ell' = w_1; w_{22}\} \quad \textit{(if } \ell \neq \ell' \textit{ and } w_2 \rhd_\ell w_{21}, w_{22}\textit{)} \end{array}$$

Reduction rules $\boxed{M_1 \rightsquigarrow^s M_2}$

$$\begin{array}{rcllr} \kappa_1\,\kappa_2 & \rightsquigarrow^s & \zeta(\kappa_1, \kappa_2) & & \text{Rs\_Const} \\ (\lambda x{:}A.M)\,w & \rightsquigarrow^s & M\,[w/x] & & \text{Rs\_Beta} \\ (\Lambda X{:}K.\,M)\,A & \rightsquigarrow^s & M\,[A/X] & & \text{Rs\_TyBeta} \\ \mathsf{let}\,\{\ell = x; y\} = w\,\mathsf{in}\,M_2 & \rightsquigarrow^s & M_2\,[w_1/x, w_2/y] & (\text{if } w \rhd_\ell w_1, w_2) & \text{Rs\_Record} \\ \uparrow\langle \ell : A\rangle\,(w^{\ell'}) & \rightsquigarrow^s & w^{\ell'} & (\text{if } \ell \neq \ell') & \text{Rs\_Embed} \\ \mathsf{case}\,(\ell\,w)\,\mathsf{with}\,\langle \ell\,x \to M_1; y \to M_2\rangle & \rightsquigarrow^s & M_1\,[w/x] & & \text{Rs\_CaseL} \\ \mathsf{case}\,\uparrow\langle \ell : A\rangle\,(w^{\ell})\,\mathsf{with}\,\langle \ell\,x \to M_1; y \to M_2\rangle & \rightsquigarrow^s & M_2\,[w^{\ell}/y] & & \text{Rs\_CaseR1} \\ \mathsf{case}\,w^{\ell'}\,\mathsf{with}\,\langle \ell\,x \to M_1; y \to M_2\rangle & \rightsquigarrow^s & M_2\,[w^{\ell'}/y] & (\text{if } \ell \neq \ell') & \text{Rs\_CaseR2} \end{array}$$

Evaluation rule $\boxed{M_1 \longrightarrow^s M_2}$

$$\frac{M_1 \rightsquigarrow^s M_2}{F[M_1] \longrightarrow^s F[M_2]}\ \text{Es\_Red}$$

Figure 1: Semantics of F^ρ.

Definition 4 (Semantics)**.** *The reduction relation $\rightsquigarrow^s$ and the evaluation relation $\longrightarrow^s$ of F^ρ are defined by the rules given in Figure 1.*

1.1.3 Type system

Type-and-row equivalence rules $\boxed{A \equiv B}$

$$\frac{}{A \equiv A}\ \text{Eq\_Refl} \qquad \frac{A \equiv C \quad C \equiv B}{A \equiv B}\ \text{Eq\_Trans} \qquad \frac{B \equiv A}{A \equiv B}\ \text{Eq\_Sym}$$

$$\frac{A_1 \equiv A_2 \quad B_1 \equiv B_2}{A_1 \to B_1 \equiv A_2 \to B_2}\ \text{Eq\_Fun} \qquad \frac{A_1 \equiv A_2}{\forall X{:}K.\,A_1 \equiv \forall X{:}K.\,A_2}\ \text{Eq\_Poly} \qquad \frac{\rho_1 \equiv \rho_2}{[\rho_1] \equiv [\rho_2]}\ \text{Eq\_Record}$$

$$\frac{\rho_1 \equiv \rho_2}{\langle\rho_1\rangle \equiv \langle\rho_2\rangle}\ \text{Eq\_Variant} \qquad \frac{\rho_1 \equiv \rho_2}{\ell : A; \rho_1 \equiv \ell : A; \rho_2}\ \text{Eq\_Cons} \qquad \frac{\ell \neq \ell'}{\ell : A; \ell' : B; \rho \equiv \ell' : B; \ell : A; \rho}\ \text{Eq\_Swap}$$

Figure 2: Type-and-row equivalence of F^ρ.

Definition 5 (Type-and-row equivalence)**.** *Type-and-row equivalence $\equiv$ is the smallest relation satisfying the rules given by Figure 2.*

Definition 6 (Typing)**.** *The well-formedness judgments $\vdash^s \Gamma$ and $\Gamma \vdash^s A : K$, and the typing judgment $\Gamma \vdash^s M : A$ of F^ρ are the smallest relations satisfying the rules given by Figure 3.*

Well-formedness rules for typing contexts $\boxed{\vdash^s \Gamma}$

$$\frac{}{\vdash^s \emptyset}\ \textsc{WFs\_Empty} \qquad \frac{x \notin dom(\Gamma) \quad \vdash^s \Gamma \quad \Gamma \vdash^s A : \mathsf{T}}{\vdash^s \Gamma, x{:}A}\ \textsc{WFs\_ExtVar} \qquad \frac{\vdash^s \Gamma \quad X \notin dom(\Gamma)}{\vdash^s \Gamma, X{:}K}\ \textsc{WFs\_ExtTyVar}$$

Well-formedness rules for types and rows $\boxed{\Gamma \vdash^s A : K}$

$$\frac{\vdash^s \Gamma \quad X{:}K \in \Gamma}{\Gamma \vdash^s X : K}\ \textsc{WFs\_TyVar} \qquad \frac{\vdash^s \Gamma}{\Gamma \vdash^s \iota : \mathsf{T}}\ \textsc{WFs\_Base} \qquad \frac{\Gamma \vdash^s A : \mathsf{T} \quad \Gamma \vdash^s B : \mathsf{T}}{\Gamma \vdash^s A \to B : \mathsf{T}}\ \textsc{WFs\_Fun}$$

$$\frac{\Gamma, X{:}K \vdash^s A : \mathsf{T}}{\Gamma \vdash^s \forall X{:}K.\, A : \mathsf{T}}\ \textsc{WFs\_Poly} \qquad \frac{\Gamma \vdash^s \rho : \mathsf{R}}{\Gamma \vdash^s [\rho] : \mathsf{T}}\ \textsc{WFs\_Record} \qquad \frac{\Gamma \vdash^s \rho : \mathsf{R}}{\Gamma \vdash^s \langle\rho\rangle : \mathsf{T}}\ \textsc{WFs\_Variant}$$

$$\frac{\vdash^s \Gamma}{\Gamma \vdash^s \cdot : \mathsf{R}}\ \textsc{WFs\_REmp} \qquad \frac{\Gamma \vdash^s A : \mathsf{T} \quad \Gamma \vdash^s \rho : \mathsf{R}}{\Gamma \vdash^s \ell : A; \rho : \mathsf{R}}\ \textsc{WFs\_Cons}$$

Typing rules $\boxed{\Gamma \vdash^s M : A}$

$$\frac{\vdash^s \Gamma \quad x{:}A \in \Gamma}{\Gamma \vdash^s x : A}\ \textsc{Ts\_Var} \qquad \frac{\vdash^s \Gamma}{\Gamma \vdash^s \kappa : ty(\kappa)}\ \textsc{Ts\_Const} \qquad \frac{\Gamma, x{:}A \vdash^s M : B}{\Gamma \vdash^s \lambda x{:}A.M : A \to B}\ \textsc{Ts\_Lam}$$

$$\frac{\Gamma \vdash^s M_1 : A \to B \quad \Gamma \vdash^s M_2 : A}{\Gamma \vdash^s M_1\, M_2 : B}\ \textsc{Ts\_App} \qquad \frac{\Gamma, X{:}K \vdash^s M : A}{\Gamma \vdash^s \Lambda X{:}K.\, M : \forall X{:}K.\, A}\ \textsc{Ts\_TLam}$$

$$\frac{\Gamma \vdash^s M : \forall X{:}K.\, A \quad \Gamma \vdash^s B : K}{\Gamma \vdash^s M\, B : A[B/X]}\ \textsc{Ts\_TApp} \qquad \frac{\vdash^s \Gamma}{\Gamma \vdash^s \{\} : [\cdot]}\ \textsc{Ts\_REmp}$$

$$\frac{\Gamma \vdash^s M_1 : A \quad \Gamma \vdash^s M_2 : [\rho]}{\Gamma \vdash^s \{\ell = M_1; M_2\} : [\ell : A; \rho]}\ \textsc{Ts\_RExt} \qquad \frac{\Gamma \vdash^s M_1 : [\ell : A; \rho] \quad \Gamma, x{:}A, y{:}[\rho] \vdash^s M_2 : B}{\Gamma \vdash^s \mathsf{let}\, \{\ell = x; y\} = M_1\, \mathsf{in}\, M_2 : B}\ \textsc{Ts\_RLet}$$

$$\frac{\Gamma \vdash^s M : A \quad \Gamma \vdash^s \rho : \mathsf{R}}{\Gamma \vdash^s \ell\, M : \langle \ell : A; \rho\rangle}\ \textsc{Ts\_VInj} \qquad \frac{\Gamma \vdash^s M : \langle\rho\rangle \quad \Gamma \vdash^s A : \mathsf{T}}{\Gamma \vdash^s {\uparrow}\langle \ell : A\rangle\, M : \langle \ell : A; \rho\rangle}\ \textsc{Ts\_VLift}$$

$$\frac{\Gamma \vdash^s M : \langle \ell : A; \rho\rangle \quad \Gamma, x{:}A \vdash^s M_1 : B \quad \Gamma, y{:}\langle\rho\rangle \vdash^s M_2 : B}{\Gamma \vdash^s \mathsf{case}\, M\, \mathsf{with}\, \langle \ell\, x \to M_1; y \to M_2\rangle : B}\ \textsc{Ts\_VCase}$$

$$\frac{\Gamma \vdash^s M : A \quad A \equiv B \quad \Gamma \vdash^s B : \mathsf{T}}{\Gamma \vdash^s M : B}\ \textsc{Ts\_Equiv}$$

Figure 3: Typing of F^ρ.

1.2 Gradually typed language $\mathbf{F}^{\rho}_{\mathbf{G}}$

1.2.1 Syntax

$$
\begin{array}{llll}
\textbf{Variables for types and rows} & X & \textbf{Kinds} & K ::= \mathsf{T} \mid \mathsf{R} \\
\textbf{Type-and-row names} & \alpha & & \\
\textbf{Base types} & \iota ::= \mathsf{bool} \mid \mathsf{int} \mid \ldots & \textbf{Constants} & \kappa ::= \mathsf{true} \mid \mathsf{false} \mid 0 \mid + \mid \ldots
\end{array}
$$

$$
\begin{array}{llcl}
\textbf{Types and rows} & A, B, C, D, \rho & ::= & X \mid \alpha \mid \star \mid \iota \mid A \rightarrow B \mid \forall X{:}K.\, A \mid [\rho] \mid \langle \rho \rangle \mid \cdot \mid \ell : A; \rho \\
\textbf{Terms} & M & ::= & x \mid \kappa \mid \lambda x{:}A.M \mid M_1\, M_2 \mid \Lambda X{:}K.\, M \mid M\ A \mid \\
& & & \{\} \mid \{\ell = M_1; M_2\} \mid \mathsf{let}\, \{\ell = x; y\} = M_1 \;\mathsf{in}\; M_2 \mid \\
& & & \ell\, M \mid \mathsf{case}\; M \;\mathsf{with}\; \langle \ell\, x \rightarrow M_1; y \rightarrow M_2 \rangle \mid \uparrow\!\langle \ell : A \rangle\, M \\
\textbf{Typing contexts} & \Gamma & ::= & \emptyset \mid \Gamma, x{:}A \mid \Gamma, X{:}K
\end{array}
$$

Assumption 2. *We assume that operation $A \oplus B$ that produces a type is available. Assumptions for $\oplus$ are stated in the beginnings of subsections of proving properties (Section 2.3 and Section 2.4).*

1.2.2 Typing

Definition 7 (Type-and-row equivalence)**.** *Type-and-row equivalence $\equiv$ is the smallest relation satisfying the rules given by Figure 2.*

Consistency rules $\boxed{A \sim B}$

$$\frac{}{A \sim A}\ \textsc{C\_Refl} \qquad \frac{}{\star \sim A}\ \textsc{C\_DynL} \qquad \frac{}{A \sim \star}\ \textsc{C\_DynR}$$

$$\frac{A_1 \sim A_2 \quad B_1 \sim B_2}{A_1 \rightarrow B_1 \sim A_2 \rightarrow B_2}\ \textsc{C\_Fun} \qquad \frac{A_1 \sim A_2}{\forall X{:}K.\, A_1 \sim \forall X{:}K.\, A_2}\ \textsc{C\_Poly}$$

$$\frac{\mathbf{QPoly}\,(A_2) \quad X \notin \mathit{ftv}(A_2) \quad A_1 \sim A_2}{\forall X{:}K.\, A_1 \sim A_2}\ \textsc{C\_PolyL} \qquad \frac{\mathbf{QPoly}\,(A_1) \quad X \notin \mathit{ftv}(A_1) \quad A_1 \sim A_2}{A_1 \sim \forall X{:}K.\, A_2}\ \textsc{C\_PolyR}$$

$$\frac{\rho_1 \sim \rho_2}{[\rho_1] \sim [\rho_2]}\ \textsc{C\_Record} \qquad \frac{\rho_1 \sim \rho_2}{\langle \rho_1 \rangle \sim \langle \rho_2 \rangle}\ \textsc{C\_Variant} \qquad \frac{A_1 \sim A_2 \quad \rho_1 \sim \rho_2}{\ell : A_1; \rho_1 \sim \ell : A_2; \rho_2}\ \textsc{C\_Cons}$$

$$\frac{\ell \notin \mathit{dom}\,(\rho_2) \quad \rho_2 \text{ ends with } \star \quad \rho_1 \sim \rho_2}{\ell : A; \rho_1 \sim \rho_2}\ \textsc{C\_ConsL} \qquad \frac{\ell \notin \mathit{dom}\,(\rho_1) \quad \rho_1 \text{ ends with } \star \quad \rho_1 \sim \rho_2}{\rho_1 \sim \ell : A; \rho_2}\ \textsc{C\_ConsR}$$

Figure 4: Consistency.

Definition 8 (Quasi-universal types)**.** *The predicate* **QPoly** (A) *is defined by:* **QPoly** (A) *if and only if*

- $A \neq \forall X{:}K.\, B$ *for any* X*,* K*, and* B*,*
- $A \neq \cdot$*,*
- $A \neq \ell : B; \rho$ *for any* ℓ*,* B*, and* ρ*, and*
- $\star$ *occurs somewhere in* A*.*

Type A *is a quasi-universal type if and only if* **QPoly** (A)*.*

Definition 9 (Labels in row)**.** *We define $dom(\rho)$, the set of the field labels in ρ, as follows.*

$$\begin{array}{lcl} dom(\cdot) & \stackrel{\text{def}}{=} & \emptyset \\ dom(\star) & \stackrel{\text{def}}{=} & \emptyset \\ dom(X) & \stackrel{\text{def}}{=} & \emptyset \\ dom(\alpha) & \stackrel{\text{def}}{=} & \emptyset \\ dom(\ell : A; \rho) & \stackrel{\text{def}}{=} & dom(\rho) \cup \{\ell\} \end{array}$$

Definition 10 (Row concatenation)**.** *Row concatenation $\rho_1 \odot \rho_2$ is defined as follows:*

$$\begin{array}{lcl} \cdot \odot \rho_2 & \stackrel{\text{def}}{=} & \rho_2 \\ (\ell : A; \rho_1) \odot \rho_2 & \stackrel{\text{def}}{=} & \ell : A; (\rho_1 \odot \rho_2) \end{array}$$

Definition 11 (Rows ending with $\star$)**.** *Row type ρ ends with $\star$ if and only if $\rho = \rho' \odot \star$ for some ρ'.*

Definition 12 (Consistency)**.** *Consistency $A \sim B$ is the smallest relation satisfying the rules given by Figure 4.*

Consistent equivalence rules $\boxed{A \simeq B}$

$$\frac{}{A \simeq A}\ \text{CE\_Refl} \qquad \frac{}{\star \simeq A}\ \text{CE\_DynL} \qquad \frac{}{A \simeq \star}\ \text{CE\_DynR}$$

$$\frac{A_1 \simeq A_2 \quad B_1 \simeq B_2}{A_1 \to B_1 \simeq A_2 \to B_2}\ \text{CE\_Fun} \qquad \frac{A_1 \simeq A_2}{\forall X{:}K.\, A_1 \simeq \forall X{:}K.\, A_2}\ \text{CE\_Poly}$$

$$\frac{\mathbf{QPoly}\,(A_2) \quad X \notin \mathit{ftv}(A_2) \quad A_1 \simeq A_2}{\forall X{:}K.\, A_1 \simeq A_2}\ \text{CE\_PolyL} \qquad \frac{\mathbf{QPoly}\,(A_1) \quad X \notin \mathit{ftv}(A_1) \quad A_1 \simeq A_2}{A_1 \simeq \forall X{:}K.\, A_2}\ \text{CE\_PolyR}$$

$$\frac{\rho_1 \simeq \rho_2}{[\rho_1] \simeq [\rho_2]}\ \text{CE\_Record} \qquad \frac{\rho_1 \simeq \rho_2}{\langle\rho_1\rangle \simeq \langle\rho_2\rangle}\ \text{CE\_Variant}$$

$$\frac{\rho_2 \triangleright_\ell B, \rho_2' \quad A \simeq B \quad \rho_1 \simeq \rho_2'}{\ell : A; \rho_1 \simeq \rho_2}\ \text{CE\_ConsL} \qquad \frac{\rho_1 \triangleright_\ell A, \rho_1' \quad A \simeq B \quad \rho_1' \simeq \rho_2}{\rho_1 \simeq \ell : B; \rho_2}\ \text{CE\_ConsR}$$

Figure 5: Consistent equivalence.

Definition 13 (Row splitting)**.** *Row splitting $\rho_1 \triangleright_\ell A, \rho_2$ is defined as follows.*

$$\begin{array}{lcll} \ell : A; \rho & \triangleright_\ell & A, \rho & \\ \ell' : B; \rho_1 & \triangleright_\ell & A, (\ell' : B; \rho_2) & (\text{if } \ell \neq \ell' \text{ and } \rho_1 \triangleright_\ell A, \rho_2) \\ \star & \triangleright_\ell & \star, \star & \end{array}$$

Definition 14 (Consistent equivalence)**.** *Consistency equivalence $A \simeq B$ is the smallest relation satisfying the rules given by Figure 5.*

Definition 15 (Type matching)**.** *Type matching $A \triangleright B$ is the smallest relation satisfying the rules given by Figure 6.*

Definition 16 (Typing)**.** *The well-formedness judgments $\vdash \Gamma$ and $\Gamma \vdash A : K$, and the typing judgment $\Gamma \vdash M : A$ of F_G^ρ are the smallest relations satisfying the rules given by Figure 7.*

Type matching rules $\boxed{A \triangleright B}$

$$A \to B \triangleright A \to B \qquad \star \triangleright \star \to \star$$

$$\forall X{:}K.\, A \triangleright \forall X{:}K.\, A \qquad \star \triangleright \forall X{:}K.\, \star$$

$$[\rho] \triangleright [\rho] \qquad \star \triangleright [\star]$$

$$\langle\rho\rangle \triangleright \langle\rho\rangle \qquad \star \triangleright \langle\star\rangle$$

Figure 6: Type matching.

Well-formedness rules for typing contexts $\boxed{\vdash \Gamma}$

$$\frac{}{\vdash \emptyset}\ \textsc{WFg\_Empty} \qquad \frac{\vdash \Gamma \quad x \notin \mathit{dom}(\Gamma) \quad \Gamma \vdash A : \mathsf{T}}{\vdash \Gamma, x{:}A}\ \textsc{WFg\_ExtVar}$$

$$\frac{\vdash \Gamma \quad X \notin \mathit{dom}(\Gamma)}{\vdash \Gamma, X{:}K}\ \textsc{WFg\_ExtTyVar}$$

Well-formedness rules for types and rows $\boxed{\Gamma \vdash A : K}$

$$\frac{\vdash \Gamma \quad X{:}K \in \Gamma}{\Gamma \vdash X : K}\ \textsc{WFg\_TyVar} \qquad \frac{\vdash \Gamma}{\Gamma \vdash \star : K}\ \textsc{WFg\_Dyn} \qquad \frac{\vdash \Gamma}{\Gamma \vdash \iota : \mathsf{T}}\ \textsc{WFg\_Base}$$

$$\frac{\Gamma \vdash A : \mathsf{T} \quad \Gamma \vdash B : \mathsf{T}}{\Gamma \vdash A \to B : \mathsf{T}}\ \textsc{WFg\_Fun} \qquad \frac{\Gamma, X{:}K \vdash A : \mathsf{T}}{\Gamma \vdash \forall X{:}K.\, A : \mathsf{T}}\ \textsc{WFg\_Poly} \qquad \frac{\Gamma \vdash \rho : \mathsf{R}}{\Gamma \vdash [\rho] : \mathsf{T}}\ \textsc{WFg\_Record}$$

$$\frac{\Gamma \vdash \rho : \mathsf{R}}{\Gamma \vdash \langle \rho \rangle : \mathsf{T}}\ \textsc{WFg\_Variant} \qquad \frac{\vdash \Gamma}{\Gamma \vdash \cdot : \mathsf{R}}\ \textsc{WFg\_REmp} \qquad \frac{\Gamma \vdash A : \mathsf{T} \quad \Gamma \vdash \rho : \mathsf{R}}{\Gamma \vdash \ell : A; \rho : \mathsf{R}}\ \textsc{WFg\_Cons}$$

Typing rules $\boxed{\Gamma \vdash M : A}$

$$\frac{\vdash \Gamma \quad x{:}A \in \Gamma}{\Gamma \vdash x : A}\ \textsc{Tg\_Var} \qquad \frac{\vdash \Gamma}{\Gamma \vdash \kappa : \mathit{ty}(\kappa)}\ \textsc{Tg\_Const} \qquad \frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : A \to B}\ \textsc{Tg\_Lam}$$

$$\frac{\Gamma \vdash M_1 : A_1 \quad \Gamma \vdash M_2 : A_2 \quad A_1 \rhd A_{11} \to A_{12} \quad A_2 \simeq A_{11}}{\Gamma \vdash M_1\, M_2 : A_{12}}\ \textsc{Tg\_App}$$

$$\frac{\Gamma, X{:}K \vdash M : A}{\Gamma \vdash \Lambda X{:}K.\, M : \forall X{:}K.\, A}\ \textsc{Tg\_TLam} \qquad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : K \quad A \rhd \forall X{:}K.\, C}{\Gamma \vdash M\, B : C[B/X]}\ \textsc{Tg\_TApp}$$

$$\frac{\vdash \Gamma}{\Gamma \vdash \{\} : [\cdot]}\ \textsc{Tg\_REmp} \qquad \frac{\Gamma \vdash M_1 : A \quad \Gamma \vdash M_2 : B \quad B \rhd [\rho]}{\Gamma \vdash \{\ell = M_1; M_2\} : [\ell : A; \rho]}\ \textsc{Tg\_RExt}$$

$$\frac{\Gamma \vdash M_1 : A \quad A \rhd [\rho] \quad \rho \rhd_\ell B, \rho' \quad \Gamma, x{:}B, y{:}[\rho'] \vdash M_2 : C}{\Gamma \vdash \mathsf{let}\, \{\ell = x; y\} = M_1\, \mathsf{in}\, M_2 : C}\ \textsc{Tg\_RLet}$$

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash \rho : \mathsf{R}}{\Gamma \vdash \ell\, M : \langle \ell : A; \rho \rangle}\ \textsc{Tg\_VInj} \qquad \frac{\Gamma \vdash M : B \quad B \rhd \langle \rho \rangle \quad \Gamma \vdash A : \mathsf{T}}{\Gamma \vdash \uparrow\langle \ell : A \rangle\, M : \langle \ell : A; \rho \rangle}\ \textsc{Tg\_VLift}$$

$$\frac{\Gamma \vdash M : A \quad A \rhd \langle \rho \rangle \quad \rho \rhd_\ell B, \rho' \quad \Gamma, x{:}B \vdash M_1 : C \quad \Gamma, y{:}\langle \rho' \rangle \vdash M_2 : D}{\Gamma \vdash \mathsf{case}\, M\, \mathsf{with}\, \langle \ell\, x \to M_1; y \to M_2 \rangle : C \oplus D}\ \textsc{Tg\_VCase}$$

Figure 7: Typing of F_G^ρ.

1.3 Blame calculus $\mathbf{F}_{\mathbf{C}}^{\rho}$

1.3.1 Syntax

$$
\begin{array}{lll}
\textbf{Blame labels} \quad p, q \quad \textbf{Type-and-row names} \quad \alpha \quad \textbf{Conversion labels} \quad \Phi ::= +\alpha \mid -\alpha \\
\textbf{Types and rows} \quad A, B, C, D, \rho & ::= & X \mid \alpha \mid \star \mid \iota \mid A \to B \mid \forall X{:}K.\, A \mid [\rho] \mid \langle \rho \rangle \mid \cdot \mid \ell : A; \rho \\
\textbf{Ground types} \quad G, H & ::= & \alpha \mid \iota \mid \star \to \star \mid [\star] \mid \langle \star \rangle \\
\textbf{Ground row types} \quad \gamma & ::= & \alpha \mid \cdot \mid \ell : \star; \star \\
\textbf{Terms} \quad e & ::= & x \mid \kappa \mid \lambda x{:}A.e \mid e_1\, e_2 \mid \Lambda X{:}K.e :: A \mid e\, A \mid \\
 & & \{\} \mid \{\ell = e_1; e_2\} \mid \mathsf{let}\, \{\ell = x; y\} = e_1\, \mathsf{in}\, e_2 \mid \\
 & & \ell\, e \mid \mathsf{case}\, e\, \mathsf{with}\, \langle \ell\, x \to e_1; y \to e_2 \rangle \mid \uparrow\langle \ell : A \rangle\, e \mid \\
 & & e : A \stackrel{p}{\Rightarrow} B \mid e : A \stackrel{\Phi}{\Rightarrow} B \mid \mathsf{blame}\, p \\
\textbf{Values} \quad v & ::= & \kappa \mid \lambda x{:}A.e \mid \Lambda X{:}K.e :: A \mid \{\} \mid \{\ell = v_1; v_2\} \mid \ell\, v \mid \uparrow\langle \ell : A \rangle\, v \mid \\
 & & v : G \stackrel{p}{\Rightarrow} \star \mid v : [\gamma] \stackrel{p}{\Rightarrow} [\star] \mid v : \langle \gamma \rangle \stackrel{p}{\Rightarrow} \langle \star \rangle \mid \\
 & & v : A \stackrel{-\alpha}{\Rightarrow} \alpha \mid v : [\rho] \stackrel{-\alpha}{\Rightarrow} [\alpha] \mid v : \langle \rho \rangle \stackrel{-\alpha}{\Rightarrow} \langle \alpha \rangle \\
\textbf{Evaluation contexts} \quad E & ::= & [\,] \mid E\, e_2 \mid v_1\, E \mid E\, A \mid \{\ell = E; e_2\} \mid \{\ell = v_1; E\} \mid \\
 & & \mathsf{let}\, \{\ell = x; y\} = E\, \mathsf{in}\, e_2 \mid \\
 & & \ell\, E \mid \mathsf{case}\, E\, \mathsf{with}\, \langle \ell\, x \to e_1; y \to e_2 \rangle \mid \uparrow\langle \ell : A \rangle\, E \mid \\
 & & E : A \stackrel{p}{\Rightarrow} B \mid E : A \stackrel{\Phi}{\Rightarrow} B \\
\textbf{Name stores} \quad \Sigma & ::= & \emptyset \mid \Sigma, \alpha{:}K := A
\end{array}
$$

Figure 8: Syntax of $\mathrm{F}_{\mathrm{C}}^{\rho}$.

Definition 17 (Comparison between name stores)**.** *We write* $\Sigma \subseteq \Sigma'$ *if and only if, for any* α, K, *and* A, *if* $\alpha{:}K := A \in \Sigma$, *then* $\alpha{:}K := A \in \Sigma'$.

Definition 18 (Substitution)**.** *Type substitution* $e[A/X]$ *of* A *for* X *in* e *is defined in a capture-avoiding manner as usual. Value substitution* $e[v/x]$ *is also defined similarly.*

1.3.2 Semantics

Definition 19 (Record splitting)**.** $v \triangleright_\ell v_1, v_2$ *is defined as follows:*

$$
\begin{array}{l}
\{\ell = v_1; v_2\} \triangleright_\ell v_1, v_2 \\
\{\ell' = v_1; v_2\} \triangleright_\ell v_{21}, \{\ell' = v_1; v_{22}\} \quad \textit{(where } \ell \neq \ell' \textit{ and } v_2 \triangleright_\ell v_{21}, v_{22}\textit{)}
\end{array}
$$

Definition 20 (Field postpending)**.** *Field postpending* $\rho \mathbin{@} \ell : A$ *is defined as follows:*

$$
\begin{array}{lcl}
(\ell' : B; \rho) \mathbin{@} \ell : A & \stackrel{\mathrm{def}}{=} & \ell' : B; (\rho \mathbin{@} \ell : A) \\
\star \mathbin{@} \ell : A & \stackrel{\mathrm{def}}{=} & \ell : A; \star
\end{array}
$$

Definition 21 (Ground row types of rows)**.**

$$
\begin{array}{lcl}
\mathit{grow}(\cdot) & \stackrel{\mathrm{def}}{=} & \cdot \\
\mathit{grow}(\alpha) & \stackrel{\mathrm{def}}{=} & \alpha \\
\mathit{grow}(\ell : A; \rho) & \stackrel{\mathrm{def}}{=} & \ell : \star; \star
\end{array}
$$

Definition 22 (Row embedding)**.** *Row embedding* $\uparrow\rho\, e$ *is defined as follows:*

$$
\begin{array}{lcl}
\uparrow(\ell : A; \rho)\, e & \stackrel{\mathrm{def}}{=} & \uparrow\langle \ell : A \rangle\, (\uparrow\rho\, e) \\
\uparrow\rho\, e & \stackrel{\mathrm{def}}{=} & e \quad \textit{(if } \rho \neq (\ell : A; \rho')\textit{)}
\end{array}
$$

Definition 23 (Field insertion)**.** *Function* $\downarrow^{\rho}_{\langle \ell : A \rangle} e$ *embeds a term* e *of type* $\langle \rho \odot \rho' \rangle$ *into* $\langle \rho \odot (\ell : A; \cdot) \odot \rho' \rangle$*. Formally, it is defined as follows:*

$$\begin{array}{lcl} \downarrow^{(\ell' : B' ; \rho)}_{\langle \ell : A \rangle} e & \stackrel{\text{def}}{=} & \mathsf{case}\; e \;\mathsf{with}\; \langle \ell' \, x \rightarrow \ell' \, x ; y \rightarrow \uparrow \langle \ell' : B' \rangle \, (\downarrow^{\rho}_{\langle \ell : A \rangle} y) \rangle \\ \downarrow^{\rho}_{\langle \ell : A \rangle} e & \stackrel{\text{def}}{=} & \uparrow \langle \ell : A \rangle \, e \quad (\textit{if } \rho \neq (\ell' : B' ; \rho') \textit{ for any } \ell', B', \textit{and } \rho') \end{array}$$

Definition 24 (Name in conversion label)**.** *We define* $\mathit{name}(+\alpha)$ *and* $\mathit{name}(-\alpha)$ *to be* α*.*

Reduction rules $\boxed{e_1 \rightsquigarrow e_2}$

$$\begin{array}{rcll}
\kappa_1 \, \kappa_2 & \rightsquigarrow & \zeta(\kappa_1, \kappa_2) & \text{R_Const} \\
(\lambda x{:}A.e) \, v & \rightsquigarrow & e[v/x] & \text{R_Beta} \\
\mathsf{let}\, \{\ell = x ; y\} = \{\ell = v_1 ; v_2\} \,\mathsf{in}\, e_2 & \rightsquigarrow & e[v_1/x, v_2/y] & \text{R_Record} \\
\mathsf{case}\, (\ell \, v) \,\mathsf{with}\, \langle \ell \, x \rightarrow e_1 ; y \rightarrow e_2 \rangle & \rightsquigarrow & e_1[v/x] & \text{R_CaseL} \\
\mathsf{case}\, \uparrow \langle \ell : A \rangle \, v \,\mathsf{with}\, \langle \ell \, x \rightarrow e_1 ; y \rightarrow e_2 \rangle & \rightsquigarrow & e_2[v/y] & \text{R_CaseR} \\
v : \star \stackrel{p}{\Rightarrow} \star & \rightsquigarrow & v & \text{R_IdDyn} \\
v : A \stackrel{p}{\Rightarrow} \star & \rightsquigarrow & v : A \stackrel{p}{\Rightarrow} G \stackrel{p}{\Rightarrow} \star & \text{R_ToDyn} \\
 & & \text{(if } A \simeq G \text{ and } A \neq G \text{ and } A \neq \star \text{ and } A \neq \forall X{:}K.\, B) & \\
v : \star \stackrel{p}{\Rightarrow} A & \rightsquigarrow & v : \star \stackrel{p}{\Rightarrow} G \stackrel{p}{\Rightarrow} A & \text{R_FromDyn} \\
 & & \text{(if } A \simeq G \text{ and } A \neq G \text{ and } A \neq \star \text{ and } A \neq \forall X{:}K.\, B) & \\
v : G \stackrel{p}{\Rightarrow} \star \stackrel{q}{\Rightarrow} G & \rightsquigarrow & v & \text{R_Ground} \\
v : G \stackrel{p}{\Rightarrow} \star \stackrel{q}{\Rightarrow} H & \rightsquigarrow & \mathsf{blame}\, q \quad (\text{if } G \neq H) & \text{R_Blame} \\
v : \iota \stackrel{p}{\Rightarrow} \iota & \rightsquigarrow & v & \text{R_IdBase} \\
v : \alpha \stackrel{p}{\Rightarrow} \alpha & \rightsquigarrow & v & \text{R_IdName} \\
v : A_1 \rightarrow B_1 \stackrel{p}{\Rightarrow} A_2 \rightarrow B_2 & \rightsquigarrow & \lambda x{:}A_2.v\,(x : A_2 \stackrel{\bar{p}}{\Rightarrow} A_1) : B_1 \stackrel{p}{\Rightarrow} B_2 & \text{R_Wrap} \\
v : \forall X{:}K.\, A_1 \stackrel{p}{\Rightarrow} \forall X{:}K.\, A_2 & \rightsquigarrow & \Lambda X{:}K.(v\, X : A_1 \stackrel{p}{\Rightarrow} A_2) :: A_2 & \text{R_Content} \\
v : \forall X{:}K.\, A \stackrel{p}{\Rightarrow} B & \rightsquigarrow & (v\, \star) : A[\star/X] \stackrel{p}{\Rightarrow} B \quad (\text{if } \mathbf{QPoly}\,(B)) & \text{R_Inst} \\
v : A \stackrel{p}{\Rightarrow} \forall X{:}K.\, B & \rightsquigarrow & \Lambda X{:}K.(v : A \stackrel{p}{\Rightarrow} B) :: B \quad (\text{if } \mathbf{QPoly}\,(A)) & \text{R_Gen} \\
v : A \stackrel{-\alpha}{\Rightarrow} \alpha \stackrel{+\alpha}{\Rightarrow} A & \rightsquigarrow & v & \text{R_CName} \\
v : \star \stackrel{\Phi}{\Rightarrow} \star & \rightsquigarrow & v & \text{R_CIdDyn} \\
v : \alpha \stackrel{\Phi}{\Rightarrow} \alpha & \rightsquigarrow & v \quad (\text{if } \mathit{name}(\Phi) \neq \alpha) & \text{R_CIdName} \\
v : \iota \stackrel{\Phi}{\Rightarrow} \iota & \rightsquigarrow & v & \text{R_CIdBase} \\
v : A_1 \rightarrow B_1 \stackrel{\Phi}{\Rightarrow} A_2 \rightarrow B_2 & \rightsquigarrow & \lambda x{:}A_2.v\,(x : A_2 \stackrel{\bar{\Phi}}{\Rightarrow} A_1) : B_1 \stackrel{\Phi}{\Rightarrow} B_2 & \text{R_CFun} \\
v : \forall X{:}K.\, A_1 \stackrel{\Phi}{\Rightarrow} \forall X{:}K.\, A_2 & \rightsquigarrow & \Lambda X{:}K.(v\, X : A_1 \stackrel{\Phi}{\Rightarrow} A_2) :: A_2 & \text{R_CForall}
\end{array}$$

Figure 9: Reduction rules of $\mathrm{F}^{\rho}_{\mathrm{C}}$.

Definition 25. *Relations* $\longrightarrow$ *and* $\rightsquigarrow$ *are the smallest relations satisfying the rules in Figures 9, 10, 11, and 12.*

Definition 26 (Multi-step evaluation)**.** *Binary relation* $\longrightarrow^*$ *over terms is the reflexive and transitive closure of* $\longrightarrow$*.*

1.4 Typing

Definition 27. *Judgments* $\Sigma \vdash \Gamma$*,* $\Sigma ; \Gamma \vdash A : K$*, and* $\Sigma ; \Gamma \vdash e : A$ *are the smallest relations satisfying the rules in Figures 14 and 15.*

Cast and conversion reduction rules for records $\boxed{e_1 \rightsquigarrow e_2}$

$$
\begin{array}{rcll}
v : [\cdot] \overset{p}{\Rightarrow} [\cdot] & \rightsquigarrow & v & \text{R\_REmp} \\
v : [\alpha] \overset{p}{\Rightarrow} [\alpha] & \rightsquigarrow & v & \text{R\_RIdName} \\
v : [\rho] \overset{p}{\Rightarrow} [\star] & \rightsquigarrow & v : [\rho] \overset{p}{\Rightarrow} [\mathit{grow}(\rho)] \overset{p}{\Rightarrow} [\star] \quad (\text{if } \rho \neq \mathit{grow}(\rho)) & \text{R\_RToDyn} \\
v : [\gamma] \overset{p}{\Rightarrow} [\star] \overset{q}{\Rightarrow} [\rho] & \rightsquigarrow & v : [\gamma] \overset{q}{\Rightarrow} [\rho] \quad (\text{if } \gamma \simeq \rho) & \text{R\_RFromDyn} \\
v : [\gamma] \overset{p}{\Rightarrow} [\star] \overset{q}{\Rightarrow} [\rho] & \rightsquigarrow & \mathsf{blame}\, q \quad (\text{if } \gamma \not\simeq \rho) & \text{R\_RBlame} \\
v : [\rho_1] \overset{p}{\Rightarrow} [\ell : B; \rho_2] & \rightsquigarrow & \{\ell = (v_1 : A \overset{p}{\Rightarrow} B); v_2 : [\rho_1'] \overset{p}{\Rightarrow} [\rho_2]\} & \\
& & \qquad (\text{if } v \triangleright_\ell v_1, v_2 \text{ and } \rho_1 \triangleright_\ell A, \rho_1') & \text{R\_RRev} \\
v : [\rho_1] \overset{p}{\Rightarrow} [\ell : B; \rho_2] & \rightsquigarrow & v : [\rho_1] \overset{p}{\Rightarrow} [\rho_1 \mathbin{@} \ell : B] \overset{p}{\Rightarrow} [\ell : B; \rho_2] & \\
& & \qquad (\text{if } \ell \notin \mathit{dom}(\rho_1) \text{ and } \rho_1 \neq \star) & \text{R\_RCon} \\
\\
v : [\rho] \overset{-\alpha}{\Rightarrow} [\alpha] \overset{+\alpha}{\Rightarrow} [\rho] & \rightsquigarrow & v & \text{R\_CRName} \\
v : [\cdot] \overset{\Phi}{\Rightarrow} [\cdot] & \rightsquigarrow & v & \text{R\_CREmp} \\
v : [\ell : A; \rho_1] \overset{\Phi}{\Rightarrow} [\ell : B; \rho_2] & \rightsquigarrow & & \\
& & \mathsf{let}\, \{\ell = x; y\} = v \,\mathsf{in}\, \{\ell = x : A \overset{\Phi}{\Rightarrow} B; y : [\rho_1] \overset{\Phi}{\Rightarrow} [\rho_2]\} & \text{R\_CRExt} \\
v : [\star] \overset{\Phi}{\Rightarrow} [\star] & \rightsquigarrow & v & \text{R\_CRIdDyn} \\
v : [\alpha] \overset{\Phi}{\Rightarrow} [\alpha] & \rightsquigarrow & v \quad (\text{if } \mathit{name}(\Phi) \neq \alpha) & \text{R\_CRIdName}
\end{array}
$$

Figure 10: Cast and conversion reduction rules for record types.

1.5 Translation

Definition 28. *Relation* $\Gamma \vdash M : A \hookrightarrow e$ *is the smallest relation satisfying the rules in Figure 16.*

Cast and conversion reduction rules for variants $\boxed{e_1 \leadsto e_2}$

$$v : \langle\alpha\rangle \stackrel{p}{\Rightarrow} \langle\alpha\rangle \quad\leadsto\quad v \qquad \text{R\_VIdName}$$

$$v : \langle\rho\rangle \stackrel{p}{\Rightarrow} \langle\star\rangle \quad\leadsto\quad v : \langle\rho\rangle \stackrel{p}{\Rightarrow} \langle grow(\rho)\rangle \stackrel{p}{\Rightarrow} \langle\star\rangle \quad (\text{if } \rho \neq grow(\rho)) \qquad \text{R\_VToDyn}$$

$$v : \langle\gamma\rangle \stackrel{p}{\Rightarrow} \langle\star\rangle \stackrel{q}{\Rightarrow} \langle\rho\rangle \quad\leadsto\quad v : \langle\gamma\rangle \stackrel{q}{\Rightarrow} \langle\rho\rangle \quad (\text{if } \gamma \simeq \rho) \qquad \text{R\_VFromDyn}$$

$$v : \langle\gamma\rangle \stackrel{p}{\Rightarrow} \langle\star\rangle \stackrel{q}{\Rightarrow} \langle\rho\rangle \quad\leadsto\quad \mathsf{blame}\, q \quad (\text{if } \gamma \not\simeq \rho) \qquad \text{R\_VBlame}$$

$$(\ell\, v) : \langle \ell : A; \rho_1\rangle \stackrel{p}{\Rightarrow} \langle\rho_2\rangle \quad\leadsto\quad \uparrow\rho_{21}\, (\ell\, (v : A \stackrel{p}{\Rightarrow} B)) \quad (\text{if } \rho_2 = \rho_{21} \odot (\ell : B; \cdot) \odot \rho_{22} \text{ and } \ell \notin dom\,(\rho_{21})) \qquad \text{R\_VRevInj}$$

$$(\uparrow\langle \ell : A\rangle\, v) : \langle \ell : A; \rho_1\rangle \stackrel{p}{\Rightarrow} \langle\rho_2\rangle \quad\leadsto\quad \downarrow^{\rho_{21}}_{\langle \ell : B\rangle}(v : \langle\rho_1\rangle \stackrel{p}{\Rightarrow} \langle\rho_{21} \odot \rho_{22}\rangle) \quad (\text{if } \rho_2 = \rho_{21} \odot (\ell : B; \cdot) \odot \rho_{22} \text{ and } \ell \notin dom\,(\rho_{21})) \qquad \text{R\_VRevLift}$$

$$(\ell\, v) : \langle \ell : A; \rho_1\rangle \stackrel{p}{\Rightarrow} \langle\rho_2\rangle \quad\leadsto\quad \uparrow\rho_2\, (\ell\, v : \langle \ell : A; \star\rangle \stackrel{p}{\Rightarrow} \langle\star\rangle) \quad (\text{if } \ell \notin dom\,(\rho_2) \text{ and } \rho_2 \neq \star) \qquad \text{R\_VConInj}$$

$$(\uparrow\langle \ell : A\rangle\, v) : \langle \ell : A; \rho_1\rangle \stackrel{p}{\Rightarrow} \langle\rho_2\rangle \quad\leadsto\quad (\downarrow^{\rho_2}_{\langle \ell : A\rangle}(v : \langle\rho_1\rangle \stackrel{p}{\Rightarrow} \langle\rho_2\rangle)) : \langle\rho_2 \mathbin{@} \ell : A\rangle \stackrel{p}{\Rightarrow} \langle\rho_2\rangle \quad (\text{if } \ell \notin dom\,(\rho_2) \text{ and } \rho_2 \neq \star) \qquad \text{R\_VConLift}$$

$$v : \langle\rho\rangle \stackrel{-\alpha}{\Rightarrow} \langle\alpha\rangle \stackrel{\pm\alpha}{\Rightarrow} \langle\rho\rangle \quad\leadsto\quad v \qquad \text{R\_CVName}$$

$$v : \langle \ell : A; \rho_1\rangle \stackrel{\Phi}{\Rightarrow} \langle \ell : B; \rho_2\rangle \quad\leadsto\quad \mathsf{case}\, v\, \mathsf{with}\, \langle \ell\, x \to \ell\, (x : A \stackrel{\Phi}{\Rightarrow} B); y \to \uparrow\langle \ell : B\rangle\, (y : \langle\rho_1\rangle \stackrel{\Phi}{\Rightarrow} \langle\rho_2\rangle)\rangle \qquad \text{R\_CVar}$$

$$v : \langle\star\rangle \stackrel{\Phi}{\Rightarrow} \langle\star\rangle \quad\leadsto\quad v \qquad \text{R\_CVIdDyn}$$

$$v : \langle\alpha\rangle \stackrel{\Phi}{\Rightarrow} \langle\alpha\rangle \quad\leadsto\quad v \quad (\text{if } name(\Phi) \neq \alpha) \qquad \text{R\_CVIdName}$$

Figure 11: Cast and conversion reduction rules for variant types.

Evaluation rules $\boxed{\Sigma_1 \mid e_1 \longrightarrow \Sigma_2 \mid e_2}$

$$\frac{e_1 \leadsto e_2}{\Sigma \mid E[e_1] \longrightarrow \Sigma \mid E[e_2]} \ \text{E\_Red} \qquad\qquad \frac{E \neq [\,]}{\Sigma \mid E[\mathsf{blame}\, p] \longrightarrow \Sigma \mid \mathsf{blame}\, p} \ \text{E\_Blame}$$

$$\Sigma \mid E[(\Lambda X{:}K.\, e :: A)\, B] \longrightarrow \Sigma, \alpha{:}K := B \mid E[e[\alpha/X] : A[\alpha/X] \stackrel{\pm\alpha}{\Rightarrow} A[B/X]] \quad \text{E\_TyBeta}$$

Figure 12: Evaluation rules of $\mathrm{F}^{\rho}_{\mathrm{C}}$.

Convertible rules $\boxed{\Sigma \vdash A \prec^{\Phi} B}$

$$\frac{}{\Sigma \vdash \star \prec^{\Phi} \star}\ \textsc{Cv\_Dyn} \qquad \frac{}{\Sigma \vdash X \prec^{\Phi} X}\ \textsc{Cv\_TyVar}$$

$$\frac{name(\Phi) \neq \alpha}{\Sigma \vdash \alpha \prec^{\Phi} \alpha}\ \textsc{Cv\_TyName} \qquad \frac{\Sigma(\alpha) = A}{\Sigma \vdash \alpha \prec^{+\alpha} A}\ \textsc{Cv\_Reveal} \qquad \frac{\Sigma(\alpha) = A}{\Sigma \vdash A \prec^{-\alpha} \alpha}\ \textsc{Cv\_Conceal}$$

$$\frac{}{\Sigma \vdash \iota \prec^{\Phi} \iota}\ \textsc{Cv\_Base} \qquad \frac{\Sigma \vdash A_2 \prec^{\overline{\Phi}} A_1 \quad \Sigma \vdash B_1 \prec^{\Phi} B_2}{\Sigma \vdash A_1 \rightarrow B_1 \prec^{\Phi} A_2 \rightarrow B_2}\ \textsc{Cv\_Fun} \qquad \frac{\Sigma \vdash A_1 \prec^{\Phi} A_2}{\Sigma \vdash \forall X{:}K.\, A_1 \prec^{\Phi} \forall X{:}K.\, A_2}\ \textsc{Cv\_Poly}$$

$$\frac{\Sigma \vdash \rho_1 \prec^{\Phi} \rho_2}{\Sigma \vdash [\rho_1] \prec^{\Phi} [\rho_2]}\ \textsc{Cv\_Record} \qquad \frac{\Sigma \vdash \rho_1 \prec^{\Phi} \rho_2}{\Sigma \vdash \langle \rho_1 \rangle \prec^{\Phi} \langle \rho_2 \rangle}\ \textsc{Cv\_Variant}$$

$$\frac{}{\Sigma \vdash \cdot \prec^{\Phi} \cdot}\ \textsc{Cv\_REmp} \qquad \frac{\Sigma \vdash A_1 \prec^{\Phi} A_2 \quad \Sigma \vdash \rho_1 \prec^{\Phi} \rho_2}{\Sigma \vdash \ell : A_1; \rho_1 \prec^{\Phi} \ell : A_2; \rho_2}\ \textsc{Cv\_Cons}$$

Figure 13: Type convertibility.

Well-formedness rules for typing contexts $\boxed{\Sigma \vdash \Gamma}$

$$\frac{}{\Sigma \vdash \emptyset}\ \textsc{WF\_Empty} \qquad \frac{\Sigma \vdash \Gamma \quad x \notin dom(\Gamma) \quad \Sigma; \Gamma \vdash A : \mathsf{T}}{\Sigma \vdash \Gamma, x{:}A}\ \textsc{WF\_ExtVar}$$

$$\frac{\Sigma \vdash \Gamma \quad X \notin dom(\Gamma)}{\Sigma \vdash \Gamma, X{:}K}\ \textsc{WF\_ExtTyVar}$$

Well-formedness rules for types and rows $\boxed{\Sigma; \Gamma \vdash A : K}$

$$\frac{\Sigma \vdash \Gamma \quad X{:}K \in \Gamma}{\Sigma; \Gamma \vdash X : K}\ \textsc{WF\_TyVar} \qquad \frac{\Sigma \vdash \Gamma \quad \alpha{:}K := A \in \Sigma}{\Sigma; \Gamma \vdash \alpha : K}\ \textsc{WF\_TyName} \qquad \frac{\Sigma \vdash \Gamma}{\Sigma; \Gamma \vdash \star : K}\ \textsc{WF\_Dyn}$$

$$\frac{\Sigma \vdash \Gamma}{\Sigma; \Gamma \vdash \iota : \mathsf{T}}\ \textsc{WF\_Base} \qquad \frac{\Sigma; \Gamma \vdash A : \mathsf{T} \quad \Sigma; \Gamma \vdash B : \mathsf{T}}{\Sigma; \Gamma \vdash A \rightarrow B : \mathsf{T}}\ \textsc{WF\_Fun} \qquad \frac{\Sigma; \Gamma, X{:}K \vdash A : \mathsf{T}}{\Sigma; \Gamma \vdash \forall X{:}K.\, A : \mathsf{T}}\ \textsc{WF\_Poly}$$

$$\frac{\Sigma; \Gamma \vdash \rho : \mathsf{R}}{\Sigma; \Gamma \vdash [\rho] : \mathsf{T}}\ \textsc{WF\_Record} \qquad \frac{\Sigma; \Gamma \vdash \rho : \mathsf{R}}{\Sigma; \Gamma \vdash \langle \rho \rangle : \mathsf{T}}\ \textsc{WF\_Variant}$$

$$\frac{\Sigma \vdash \Gamma}{\Sigma; \Gamma \vdash \cdot : \mathsf{R}}\ \textsc{WF\_REmp} \qquad \frac{\Sigma; \Gamma \vdash A : \mathsf{T} \quad \Sigma; \Gamma \vdash \rho : \mathsf{R}}{\Sigma; \Gamma \vdash \ell : A; \rho : \mathsf{R}}\ \textsc{WF\_Cons}$$

Figure 14: Well-formedness rules of $\mathrm{F}^{\rho}_{\mathrm{C}}$.

Typing rules $\boxed{\Sigma; \Gamma \vdash e : A}$

$$\frac{\Sigma \vdash \Gamma \quad x{:}A \in \Gamma}{\Sigma; \Gamma \vdash x : A}\ \text{T\_VAR} \qquad \frac{\Sigma \vdash \Gamma}{\Sigma; \Gamma \vdash \kappa : ty(\kappa)}\ \text{T\_CONST}$$

$$\frac{\Sigma; \Gamma, x{:}A \vdash e : B}{\Sigma; \Gamma \vdash \lambda x{:}A.e : A \to B}\ \text{T\_LAM} \qquad \frac{\Sigma; \Gamma \vdash e_1 : A \to B \quad \Sigma; \Gamma \vdash e_2 : A}{\Sigma; \Gamma \vdash e_1\, e_2 : B}\ \text{T\_APP}$$

$$\frac{\Sigma; \Gamma, X{:}K \vdash e : A}{\Sigma; \Gamma \vdash \Lambda X{:}K.e :: A : \forall X{:}K.\, A}\ \text{T\_TLAM} \qquad \frac{\Sigma; \Gamma \vdash e : \forall X{:}K.\, A \quad \Sigma; \Gamma \vdash B : K}{\Sigma; \Gamma \vdash e\, B : A[B/X]}\ \text{T\_TAPP}$$

$$\frac{\Sigma \vdash \Gamma}{\Sigma; \Gamma \vdash \{\} : [\cdot]}\ \text{T\_REMP} \qquad \frac{\Sigma; \Gamma \vdash e_1 : A \quad \Sigma; \Gamma \vdash e_2 : [\rho]}{\Sigma; \Gamma \vdash \{\ell = e_1; e_2\} : [\ell : A; \rho]}\ \text{T\_REXT}$$

$$\frac{\Sigma; \Gamma \vdash e_1 : [\ell : A; \rho] \quad \Sigma; \Gamma, x{:}A, y{:}[\rho] \vdash e_2 : B}{\Sigma; \Gamma \vdash \mathsf{let}\, \{\ell = x; y\} = e_1\, \mathsf{in}\, e_2 : B}\ \text{T\_RLET} \qquad \frac{\Sigma; \Gamma \vdash e : A \quad \Sigma; \Gamma \vdash \rho : \mathsf{R}}{\Sigma; \Gamma \vdash \ell\, e : \langle \ell : A; \rho \rangle}\ \text{T\_VINJ}$$

$$\frac{\Sigma; \Gamma \vdash e : \langle \rho \rangle \quad \Sigma; \Gamma \vdash A : \mathsf{T}}{\Sigma; \Gamma \vdash \uparrow \langle \ell : A \rangle\, e : \langle \ell : A; \rho \rangle}\ \text{T\_VLIFT}$$

$$\frac{\Sigma; \Gamma \vdash e : \langle \ell : A; \rho \rangle \quad \Sigma; \Gamma, x{:}A \vdash e_1 : B \quad \Sigma; \Gamma, y{:}\langle \rho \rangle \vdash e_2 : B}{\Sigma; \Gamma \vdash \mathsf{case}\, e\, \mathsf{with}\, \langle \ell\, x \to e_1; y \to e_2 \rangle : B}\ \text{T\_VCASE}$$

$$\frac{\Sigma; \Gamma \vdash A : \mathsf{T}}{\Sigma; \Gamma \vdash \mathsf{blame}\, p : A}\ \text{T\_BLAME}$$

$$\frac{\Sigma; \Gamma \vdash e : A \quad \Sigma; \Gamma \vdash B : \mathsf{T} \quad A \simeq B}{\Sigma; \Gamma \vdash e : A \stackrel{p}{\Rightarrow} B : B}\ \text{T\_CAST} \qquad \frac{\Sigma \vdash \Gamma \quad \Sigma; \emptyset \vdash e : A \quad \Sigma; \emptyset \vdash B : \mathsf{T} \quad \Sigma \vdash A \prec^{\Phi} B}{\Sigma; \Gamma \vdash e : A \stackrel{\Phi}{\Rightarrow} B : B}\ \text{T\_CONV}$$

Figure 15: Typing rules of $\mathrm{F}^{\rho}_{\mathrm{C}}$.

Translation rules $\boxed{\Gamma \vdash M : A \hookrightarrow e}$

$$\frac{\vdash \Gamma \quad x{:}A \in \Gamma}{\Gamma \vdash x : A \hookrightarrow x}\ \textsc{Trans\_Var} \qquad \frac{\vdash \Gamma}{\Gamma \vdash \kappa : ty(\kappa) \hookrightarrow \kappa}\ \textsc{Trans\_Const}$$

$$\frac{\Gamma, x{:}A \vdash M : B \hookrightarrow e}{\Gamma \vdash \lambda x{:}A.M : A \to B \hookrightarrow \lambda x{:}A.e}\ \textsc{Trans\_Lam}$$

$$\frac{\Gamma \vdash M_1 : A_1 \hookrightarrow e_1 \quad \Gamma \vdash M_2 : A_2 \hookrightarrow e_2 \quad A_1 \rhd A_{11} \to A_{12} \quad A_2 \simeq A_{11}}{\Gamma \vdash M_1\, M_2 : A_{12} \hookrightarrow (e_1 : A_1 \stackrel{p}{\Rightarrow} A_{11} \to A_{12})\, (e_2 : A_2 \stackrel{q}{\Rightarrow} A_{11})}\ \textsc{Trans\_App}$$

$$\frac{\Gamma, X{:}K \vdash M : A \hookrightarrow e}{\Gamma \vdash \Lambda X{:}K.\, M : \forall X{:}K.\, A \hookrightarrow \Lambda X{:}K.e :: A}\ \textsc{Trans\_TLam}$$

$$\frac{\Gamma \vdash M : A \hookrightarrow e \quad \Gamma \vdash B : K \quad A \rhd \forall X{:}K.\, C}{\Gamma \vdash M\, B : C[B/X] \hookrightarrow (e : A \stackrel{p}{\Rightarrow} \forall X{:}K.\, C)\, B}\ \textsc{Trans\_TApp}$$

$$\frac{\vdash \Gamma}{\Gamma \vdash \{\} : [\cdot] \hookrightarrow \{\}}\ \textsc{Trans\_REmp}$$

$$\frac{\Gamma \vdash M_1 : A \hookrightarrow e_1 \quad \Gamma \vdash M_2 : B \hookrightarrow e_2 \quad B \rhd [\rho]}{\Gamma \vdash \{\ell = M_1; M_2\} : [\ell : A; \rho] \hookrightarrow \{\ell = e_1; e_2 : B \stackrel{p}{\Rightarrow} [\rho]\}}\ \textsc{Trans\_RExt}$$

$$\frac{\Gamma \vdash M_1 : A \hookrightarrow e_1 \quad A \rhd [\rho] \quad \rho \rhd_\ell B, \rho' \quad \Gamma, x{:}B, y{:}[\rho'] \vdash M_2 : C \hookrightarrow e_2}{\Gamma \vdash \mathsf{let}\, \{\ell = x; y\} = M_1\, \mathsf{in}\, M_2 : C \hookrightarrow \mathsf{let}\, \{\ell = x; y\} = (e_1 : A \stackrel{p}{\Rightarrow} [\ell : B; \rho'])\, \mathsf{in}\, e_2}\ \textsc{Trans\_RLet}$$

$$\frac{\Gamma \vdash M : A \hookrightarrow e \quad \Gamma \vdash \rho : \mathsf{R}}{\Gamma \vdash \ell\, M : \langle \ell : A; \rho \rangle \hookrightarrow \ell\, e}\ \textsc{Trans\_VInj} \qquad \frac{\Gamma \vdash M : B \hookrightarrow e \quad B \rhd \langle \rho \rangle \quad \Gamma \vdash A : \mathsf{T}}{\Gamma \vdash \uparrow \langle \ell : A \rangle\, M : \langle \ell : A; \rho \rangle \hookrightarrow \uparrow \langle \ell : A \rangle\, (e : B \stackrel{p}{\Rightarrow} \langle \rho \rangle)}\ \textsc{Trans\_VLift}$$

$$\frac{\begin{array}{c}\Gamma \vdash M : A \hookrightarrow e \quad A \rhd \langle \rho \rangle \quad \rho \rhd_\ell B, \rho' \quad \Gamma, x{:}B \vdash M_1 : C \hookrightarrow e_1 \quad \Gamma, y{:}\langle \rho' \rangle \vdash M_2 : D \hookrightarrow e_2 \\ e'_1 \;=\; e_1 : C \stackrel{q_1}{\Rightarrow} C \oplus D \quad e'_2 \;=\; e_2 : D \stackrel{q_2}{\Rightarrow} C \oplus D\end{array}}{\Gamma \vdash \mathsf{case}\, M\, \mathsf{with}\, \langle \ell\, x \to M_1; y \to M_2 \rangle : C \oplus D \hookrightarrow \mathsf{case}\, (e : A \stackrel{p}{\Rightarrow} \langle \ell : B; \rho' \rangle)\, \mathsf{with}\, \langle \ell\, x \to e'_1; y \to e'_2 \rangle}\ \textsc{Trans\_VCase}$$

Figure 16: Translation rules.

2 Proofs

2.1 Consistency

Lemma 1. *Suppose* $A \equiv B$. $\mathbf{QPoly}\,(A)$ *if and only if* $\mathbf{QPoly}\,(B)$.

Proof. Straightforward by induction on the derivation of $A \equiv B$. □

Lemma 2. *If* $A \equiv B$*, then* $ftv(A) = ftv(B)$.

Proof. Straightforward by induction on the derivation of $A \equiv B$. □

Lemma 3. *Suppose that* $\rho_1 \equiv \rho_2$. ρ_1 *ends with* $\star$ *if and only if so does* ρ_2.

Proof. Straightforward by induction on the derivation of $A \equiv B$. □

Lemma 4. *If* $\rho_1 \equiv \rho_2$*, then* $dom\,(\rho_1) = dom\,(\rho_2)$.

Proof. Straightforward by induction on the derivation of $\rho_1 \equiv \rho_2$. □

Lemma 5. *Suppose* $A \equiv B$.

1. $A = \star$ *if and only if* $B = \star$.
2. $A = A_1 \rightarrow A_2$ *if and only if* $B = B_1 \rightarrow B_2$*, and* $A_1 \equiv B_1$ *and* $A_2 \equiv B_2$.
3. $A = \forall X{:}K.\,A'$ *if and only if* $B = \forall X{:}K.\,B'$*, and* $A' \equiv B'$.
4. $A = [\rho_1]$ *if and only if* $B = [\rho_2]$*, and* $\rho_1 \equiv \rho_2$.
5. $A = \langle \rho_1 \rangle$ *if and only if* $B = \langle \rho_2 \rangle$*, and* $\rho_1 \equiv \rho_2$.
6. $A = \rho_{11} \odot (\ell : A'; \cdot) \odot \rho_{12}$ *and* $\ell \notin dom\,(\rho_{11})$ *if and only if* $B = \rho_{21} \odot (\ell : B'; \cdot) \odot \rho_{22}$ *and* $\ell \notin dom\,(\rho_{21})$*, and* $A' \equiv B'$ *and* $\rho_{11} \odot \rho_{12} \equiv \rho_{21} \odot \rho_{22}$.

Proof. Straightforward by induction on the derivation of $A \equiv B$. □

Lemma 6. *If* $A \simeq B$*, then* $B \simeq A$.

Proof. Straightforward by induction on the derivation of $A \simeq B$. □

Lemma 7. *If* $\alpha \simeq \rho$*, then* $\rho = \alpha$ *or* $\rho = \star$.

Proof. Straightforward by case analysis on the derivation of $\alpha \simeq \rho$. □

Lemma 8. *If* $\cdot \simeq \rho$*, then* $\rho = \cdot$ *or* $\rho = \star$.

Proof. Straightforward by case analysis on the derivation of $\cdot \simeq \rho$. □

Lemma 9. *If* $\ell : A; \rho_1 \simeq \rho_2$*, then* $\rho_2 \rhd_\ell B, \rho'_2$ *and* $A \simeq B$ *and* $\rho_1 \simeq \rho'_2$.

Proof. By induction on $\ell : A; \rho_1 \simeq \rho_2$.

Case (CE_Refl): Obvious since $\rho_2 = \ell : A; \rho_1$ and $\ell : A; \rho_1 \rhd_\ell A, \rho_1$.

Case (CE_ConsL): Obvious by inversion.

Case (CE_ConsR): We have $\rho_2 = \ell' : B; \rho'_2$ for some ℓ', B, and ρ'_2.

If $\ell = \ell'$, then, since $\ell : A; \rho_1 \rhd_{\ell'} A, \rho_1$, we have $A \simeq B$ and $\rho_1 \simeq \rho'_2$ by inversion. Since $\ell' : B; \rho'_2 \rhd_\ell B, \rho'_2$, we finish.

Otherwise, suppose $\ell \neq \ell'$. Then, by inversion and definition of type matching,

- $\ell : A; \rho_1 \rhd_{\ell'} A', \ell : A; \rho'_1$,
- $\rho_1 \rhd_{\ell'} A', \rho'_1$,

- $A' \simeq B$, and
- $\ell : A; \rho'_1 \simeq \rho'_2$

for some A' and ρ'_1. By the IH, $\rho'_2 \triangleright_\ell B', \rho''_2$ and $A \simeq B'$ and $\rho'_1 \simeq \rho''_2$ for some B' and ρ''_2. Since $\ell \neq \ell'$, we have $\rho_2 = \ell' : B; \rho'_2 \triangleright_\ell B', \ell' : B; \rho''_2$. Since $A \simeq B'$, it suffices to show that $\rho_1 \simeq \ell' : B; \rho''_2$. Here, $\rho_1 \triangleright_{\ell'} A', \rho'_1$ and $A' \simeq B$ and $\rho'_1 \simeq \rho''_2$ (obtained above). Thus, by (CE_ConsR), $\rho_1 \simeq \ell' : B; \rho''_2$.

Case (CE_DynL), (CE_DynR), (CE_Fun), (CE_Poly), (CE_PolyL), (CE_PolyR), (CE_Record), and (CE_Variant): Contradictory.

□

Lemma 10. *If $A_1 \rightarrow A_2 \simeq B_1 \rightarrow B_2$, then $A_1 \simeq B_1$ and $A_2 \simeq B_2$.*

Proof. Straightforward by case analysis on the derivation of $A_1 \rightarrow A_2 \simeq B_1 \rightarrow B_2$. □

Lemma 11. *If $\forall X{:}K.\, A \simeq \forall X{:}K.\, B$, then $A \simeq B$.*

Proof. Straightforward by case analysis on the derivation of $A_1 \rightarrow A_2 \simeq B_1 \rightarrow B_2$. □

Lemma 12. *If $\forall X{:}K.\, A \simeq B$ and* **QPoly** *(B), then $X \notin ftv(B)$ and $A \simeq B$.*

Proof. Straightforward by case analysis on the derivation of $\forall X{:}K.\, A \simeq B$. □

Lemma 13. *If $[\rho_1] \simeq [\rho_2]$, then $\rho_1 \simeq \rho_2$.*

Proof. Straightforward by case analysis on the derivation of $[\rho_1] \simeq [\rho_2]$. □

Lemma 14. *If $\langle \rho_1 \rangle \simeq \langle \rho_2 \rangle$, then $\rho_1 \simeq \rho_2$.*

Proof. Straightforward by case analysis on the derivation of $\langle \rho_1 \rangle \simeq \langle \rho_2 \rangle$. □

Lemma 15 (consistent-decomp-aux)**.** *uppose that $A \sim B$. If $\rho_1 \sim \rho_{21} \odot \rho_{22}$ and $\ell \notin dom(\rho_{21})$, then there exist some ρ_{11} and ρ_{12} such that*

- *$\rho_1 \equiv \rho_{11} \odot \rho_{12}$,*
- *$\rho_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \sim \rho_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$,*
- *$\rho_{11} \odot (\ell : A; \cdot) \odot \rho_3 \odot \rho_{12} \sim \rho_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$ for any ρ_3 such that $dom(\rho_3) \cap dom(\rho_{21} \odot \rho_{22}) = \emptyset$ if $\rho_{21} \odot \rho_{22}$ ends with $\star$, and*
- *$\ell \notin dom(\rho_{11})$.*

Proof. By induction on the derivation of $\rho_1 \sim \rho_{21} \odot \rho_{22}$. Since $\rho_{21} \odot \rho_{22}$ is defined, there are only two cases on ρ_{21} to be considered.

Case $\rho_{21} = \cdot$: Let $\rho_{11} = \cdot$ and $\rho_{12} = \rho_1$. Then, it suffices to show the followings.

- $\rho_1 \equiv \rho_1$. By (Eq_Refl).
- $\ell : A; \rho_1 \sim \ell : B; \rho_{22}$. Since $\rho_1 \sim \rho_{21} \odot \rho_{22} = \rho_{22}$ and $A \sim B$, we prove this by (C_Cons).
- Supposing ρ_{22} ends with $\star$, we have to show $(\ell : A; \cdot) \odot \rho_3 \odot \rho_1 \sim (\ell : B; \cdot) \odot \rho_{22}$ for ρ_3 such that $dom(\rho_3) \cap dom(\rho_{22}) = \emptyset$. Since $\rho_1 \sim \rho_{22}$ and $dom(\rho_3) \cap dom(\rho_{22}) = \emptyset$ and ρ_{22} ends with $\star$, we have $\rho_3 \odot \rho_1 \sim \rho_{22}$ by (C_ConsL). Since $A \sim B$, we have that by (C_Cons).
- $\ell \notin dom(\cdot)$. Trivial.

Case $\rho_{21} = \ell' : C; \rho'_{21}$: We have:

$$\rho_{21} \odot \rho_{22} = \ell' : C; \rho'_{21} \odot \rho_{22} \quad (1)$$
$$\ell \notin dom(\ell' : C; \rho'_{21}) \quad (2)$$

By case analysis on the rule applied last to derive $\rho_1 \sim \ell' : C; \rho'_{21} \odot \rho_{22}$.

Case (C_Refl): We have $\rho_1 = \ell' : C; \rho'_{21} \odot \rho_{22}$. Let $\rho_{11} = \ell' : C; \rho'_{21}$ and $\rho_{12} = \rho_{22}$. Then, it suffices to show the followings.

- $\rho_1 \equiv (\ell' : C; \rho'_{21}) \odot \rho_{22}$. By (Eq_Refl).
- $\ell' : C; \rho'_{21} \odot (\ell : A; \cdot) \odot \rho_{22} \sim \ell' : C; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$. By (C_Refl) and (C_Cons).
- Supposing $(\ell' : C; \rho'_{21}) \odot \rho_{22}$ ends with $\star$, we have to show

$$\ell' : C; \rho'_{21} \odot (\ell : A; \cdot) \odot \rho_3 \odot \rho_{22} \sim \ell' : C; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$$

  for any ρ_3 such that $dom(\rho_3) \cap dom(\ell' : C; \rho'_{21} \odot \rho_{22}) = \emptyset$. By (C_Refl), (C_ConsL), and (C_Cons).
- $\ell \notin dom(\ell' : C; \rho'_{21})$. By (2).

Case (C_DynL): We have $\rho_1 = \star$. Let $\rho_{11} = \cdot$ and $\rho_{12} = \star$. Then, it suffices to show the followings.

- $\star \equiv \star$. By (Eq_Refl).
- $\ell : A; \star \sim \ell' : C; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$. By (C_ConsR) and (C_Cons) with (2).
- Supposing $(\ell' : C; \rho'_{21}) \odot \rho_{22}$ ends with $\star$, we have to show

$$(\ell : A; \cdot) \odot \rho_3 \odot \star \sim \ell' : C; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$$

  for any ρ_3 such that $dom(\rho_3) \cap dom(\ell' : C; \rho'_{21} \odot \rho_{22}) = \emptyset$. By (C_ConsR), (C_ConsL), and (C_Cons).
- $\ell \notin dom(\cdot)$. Trivial.

Case (C_Cons): We have $\rho_1 = \ell' : D; \rho'_1$ and, by inversion, $D \sim C$ and $\rho'_1 \sim \rho'_{21} \odot \rho_{22}$ for some D and ρ'_1. By the IH, there exist some ρ'_{11} and ρ_{12} such that

(a) $\rho'_1 \equiv \rho'_{11} \odot \rho_{12}$,
(b) $\rho'_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \sim \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$,
(c) $\rho'_{11} \odot (\ell : A; \cdot) \odot \rho_3 \odot \rho_{12} \sim \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$ for any ρ_3 such that $dom(\rho_3) \cap dom(\rho'_{21} \odot \rho_{22}) = \emptyset$ if $\rho'_{21} \odot \rho_{22}$ ends with $\star$ for some ρ_2, and
(d) $\ell \notin dom(\rho'_{11})$

for some ρ'_{11} and ρ_{12}.

Let $\rho_{11} = \ell' : D; \rho'_{11}$. Then, it suffices to show the followings.

- $\rho_1 = \ell' : D; \rho'_1 \equiv \ell' : D; \rho'_{11} \odot \rho_{12}$. By (a) and (Eq_Cons).
- $\ell' : D; \rho'_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \sim \ell' : C; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$. By (b) and (C_Cons) with $D \sim C$.
- Supposing $(\ell' : C; \rho'_{21}) \odot \rho_{22}$ ends with $\star$ for some ρ_2, we have to show

$$\ell' : D; \rho'_{11} \odot (\ell : A; \cdot) \odot \rho_3 \odot \rho_{12} \sim \ell' : C; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$$

  for any ρ_3 such that $dom(\rho_3) \cap dom(\ell' : C; \rho'_{21} \odot \rho_{22}) = \emptyset$. By (c) and (C_Cons) with $D \sim C$.
- $\ell \notin dom(\ell' : D; \rho'_{11})$. By (d) and (2).

Case (C_ConsL): We have $\rho_1 = \ell'' : D; \rho'_1$ and, by inversion,

- $\ell'' \notin dom(\ell' : C; \rho'_{21} \odot \rho_{22})$,
- $\ell' : C; \rho'_{21} \odot \rho_{22}$ ends with $\star$, and
- $\rho'_1 \sim \ell' : C; \rho'_{21} \odot \rho_{22}$

for some ℓ'', D, ρ'_1, and ρ_2.

By the IH, there exist some ρ'_{11} and ρ'_{12} such that

(a) $\rho'_1 \equiv \rho'_{11} \odot \rho'_{12}$,
(b) $\rho'_{11} \odot (\ell : A; \cdot) \odot \rho'_{12} \sim \ell' : C; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$, and
(c) $\rho'_{11} \odot (\ell : A; \cdot) \odot \rho_3 \odot \rho'_{12} \sim \ell' : C; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$ for any ρ_3 such that $dom(\rho_3) \cap dom(\ell' : C; \rho'_{21} \odot \rho_{22}) = \emptyset$ if $\ell' : C; \rho'_{21} \odot \rho_{22}$ ends with $\star$ for some ρ'_2, and
(d) $\ell \notin dom(\rho'_{11})$.

Suppose that $\ell'' = \ell$. By (d), $\ell'' \notin dom(\rho'_{11})$. Let $\rho_{11} = \rho'_{11}$ and $\rho_{12} = \ell'' : D; \rho'_{12}$. Then, it suffices to show the followings.

- $\ell'' : D; \rho'_1 \equiv \rho'_{11} \odot \ell'' : D; \rho'_{12}$. By (a) and (Eq_Cons), we have
$$\ell'' : D; \rho'_1 \equiv \ell'' : D; \cdot \odot \rho'_{11} \odot \rho'_{12}.$$
Since $\ell'' \notin \mathit{dom}\,(\rho'_{11})$, we have
$$\ell'' : D; \rho'_1 \equiv \rho'_{11} \odot \ell'' : D; \rho'_{12}.$$
- $\rho'_{11} \odot (\ell : A; \cdot) \odot \ell'' : D; \rho'_{12} \sim \ell' : C; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$. Since $\ell'' \notin \mathit{dom}\,(\ell' : C; \rho'_{21} \odot \rho_{22})$ and ρ_{22} ends with $\star$, we have
$$\rho'_{11} \odot (\ell : A; \cdot) \odot (\ell'' : D; \cdot) \odot \rho'_{12} \sim \ell' : C; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$$
by (c).
- Supposing $(\ell' : C; \rho'_{21}) \odot \rho_{22}$ ends with $\star$, we have to show
$$\rho'_{11} \odot (\ell : A; \cdot) \odot \rho_3 \odot \ell'' : D; \rho'_{12} \sim \ell' : C; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$$
for any ρ_3 such that $\mathit{dom}\,(\rho_3) \cap \mathit{dom}\,(\ell' : C; \rho'_{21} \odot \rho_{22}) = \emptyset$. Since $\ell'' \notin \mathit{dom}\,(\ell' : C; \rho'_{21} \odot \rho_{22})$, we have
$$\rho'_{11} \odot (\ell : A; \cdot) \odot \rho_3 \odot (\ell'' : D; \cdot) \odot \rho'_{12} \sim \ell' : C; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}.$$
by (c).
- $\ell \notin \mathit{dom}\,(\rho'_{11})$. By (d).

Otherwise, suppose that $\ell'' \neq \ell$. Let $\rho_{11} = \ell'' : D; \rho'_{11}$ and $\rho_{12} = \rho'_{12}$. Then, it suffices to show the followings.

- $\ell'' : D; \rho'_1 \equiv \ell'' : D; \rho'_{11} \odot \rho'_{12}$. By (a) and (Eq_Cons).
- $\ell'' : D; \rho'_{11} \odot (\ell : A; \cdot) \odot \rho'_{12} \sim \ell' : C; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$. Since $\ell'' \notin \mathit{dom}\,(\ell' : C; \rho'_{21} \odot \rho_{22})$ and $\ell'' \neq \ell$, we have $\ell'' \notin \mathit{dom}\,(\ell' : C; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22})$. Since ρ_{22} ends with $\star$, we finish by (b) and (C_ConsL).
- Supposing $(\ell' : C; \rho'_{21}) \odot \rho_{22}$ ends with $\star$, we have to show
$$\ell'' : D; \rho'_{11} \odot (\ell : A; \cdot) \odot \rho_3 \odot \rho'_{12} \sim \ell' : C; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$$
for any ρ_3 such that $\mathit{dom}\,(\rho_3) \cap \mathit{dom}\,(\ell' : C; \rho'_{21} \odot \rho_{22}) = \emptyset$. By (c), we have
$$\rho'_{11} \odot (\ell : A; \cdot) \odot \rho_3 \odot \rho'_{12} \sim \ell' : C; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}.$$
Since $\ell'' \notin \mathit{dom}\,(\ell' : C; \rho'_{21} \odot \rho_{22})$ and $\ell'' \neq \ell$, we have $\ell'' \notin \mathit{dom}\,(\ell' : C; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22})$. Since ρ_{22} ends with $\star$, we finish by (C_ConsL).
- $\ell \notin \mathit{dom}\,(\ell'' : D; \rho'_{11})$. By (d) and $\ell'' \neq \ell$.

Case (C_ConsR): By inversion, we have

- $\ell' \notin \mathit{dom}\,(\rho_1)$,
- ρ_1 ends with $\star$, and
- $\rho_1 \sim \rho'_{21} \odot \rho_{22}$.

By the IH, there exist some ρ_{11} and ρ_{12} such that

(a) $\rho_1 \equiv \rho_{11} \odot \rho_{12}$,
(b) $\rho_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \sim \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$,
(c) $\rho_{11} \odot (\ell : A; \cdot) \odot \rho_3 \odot \rho_{12} \sim \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$ for any ρ_3 such that $\mathit{dom}\,(\rho_3) \cap \mathit{dom}\,(\rho'_{21} \odot \rho_{22}) = \emptyset$ if $\rho'_{21} \odot \rho_{22}$ ends with $\star$ for some ρ_2, and
(d) $\ell \notin \mathit{dom}\,(\rho_{11})$.

First, we show
$$\ell' \notin \mathit{dom}\,(\rho_{11} \odot (\ell : A; \cdot) \odot \rho_{12}). \tag{3}$$
Since $\ell \notin \mathit{dom}\,(\rho_{21})$ from the assumption and $\rho_{21} = \ell' : C; \rho'_{21}$, $\ell \neq \ell'$. Since $\ell' \notin \mathit{dom}\,(\rho_1)$ and $\rho_1 \equiv \rho_{11} \odot \rho_{12}$, we have $\ell' \notin \mathit{dom}\,(\rho_{11} \odot (\ell : A; \cdot) \odot \rho_{12})$.

It suffices to show the followings.

- $\rho_1 \equiv \rho_{11} \odot \rho_{12}$. By (a).

- $\rho_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \sim \ell' : C; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$. Since ρ_1 ends with $\star$ and $\rho_1 \equiv \rho_{11} \odot \rho_{12}$, ρ_{12} ends with $\star$. Thus, we have
$$\rho_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \sim \ell' : C; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$$
by (b), (3), and (C_ConsR).
- Supposing $(\ell' : C; \rho'_{21}) \odot \rho_{22}$ ends with $\star$ for some ρ_2, we have to show
$$\rho_{11} \odot (\ell : A; \cdot) \odot \rho_3 \odot \rho_{12} \sim \ell' : C; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$$
for any ρ_3 such that $dom\,(\rho_3) \cap dom\,(\ell' : C; \rho'_{21} \odot \rho_{22}) = \emptyset$. By (c), we have
$$\rho_{11} \odot (\ell : A; \cdot) \odot \rho_3 \odot \rho_{12} \sim \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}.$$
By (3), $\ell' \notin dom\,(\rho_{11} \odot (\ell : A; \cdot) \odot \rho_3 \odot \rho_{12})$. Since ρ_{12} ends with $\star$, we have
$$\rho_{11} \odot (\ell : A; \cdot) \odot \rho_3 \odot \rho_{12} \sim \ell' : C; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$$
by (C_ConsR).

Case (C_DynR), (C_Fun), (C_Poly), (C_PolyL), (C_PolyR), (C_Record), and (C_Variant): Note that the contradiction in the case of (C_PolyL) is proven by the definition of **QPoly**.

□

Lemma 16. *If $A \simeq B$, then $A \equiv C$ and $C \sim B$ for some C.*

Proof. By induction on the derivation of $A \simeq B$.

Case (CE_Refl): Obvious because $\equiv$ and $\sim$ are reflexive.

Case (CE_DynL): By $\star \equiv \star$ (Eq_Refl) and $\star \sim B$ (C_DynL).

Case (CE_DynR): By $A \equiv A$ (Eq_Refl) and $A \sim \star$ (C_DynR).

Case (CE_Fun): We have $A_1 \to A_2 \simeq B_1 \to B_2$ and, by inversion, $A_1 \simeq B_1$ and $A_2 \simeq B_2$. By the IHs,

- $A_1 \equiv C_1$,
- $C_1 \sim B_1$,
- $A_2 \equiv C_2$, and
- $C_2 \sim B_2$

for some C_1 and C_2. By (Eq_Fun), $A_1 \to A_2 \equiv C_1 \to C_2$. By (C_Fun), $C_1 \to C_2 \sim B_1 \to B_2$.

Case (CE_Poly): We have $\forall X{:}K.\, A' \simeq \forall X{:}K.\, B'$ and, by inversion, $A' \simeq B'$. By the IH, $A' \equiv C'$ and $C' \sim B'$ for some C'. By (Eq_Poly), $\forall X{:}K.\, A' \equiv \forall X{:}K.\, C'$. By (C_Poly), $\forall X{:}K.\, C' \sim \forall X{:}K.\, B'$.

Case (CE_PolyL): We have $\forall X{:}K.\, A' \simeq B$ and, by inversion, **QPoly** (B) and $X \notin ftv(B)$ and $A' \simeq B$. By the IH, $A' \equiv C$ and $C \sim B$ for some C. By (Eq_Poly), $\forall X{:}K.\, A' \equiv \forall X{:}K.\, C$. By (C_PolyL), $\forall X{:}K.\, C \sim B$.

Case (CE_PolyR): We have $A \simeq \forall X{:}K.\, B'$ and, by inversion, **QPoly** (A) and $X \notin ftv(A)$ and $A \simeq B'$. By the IH, $A \equiv C$ and $C \sim B'$ for some C. Since $A \equiv C$, we can find **QPoly** (C) by Lemma 1 and **QPoly** (A), and $X \notin ftv(C)$ by Lemma 2 and $X \notin ftv(A)$. Thus, by (C_PolyR), $C \sim \forall X{:}K.\, B'$.

Case (CE_Record): By the IH, (Eq_Record), and (C_Record).

Case (CE_Variant): By the IH, (Eq_Variant), and (C_Variant).

Case (CE_ConsL): We have $\ell : A'; \rho_1 \simeq B$ and, by inversion, $B \triangleright_\ell B', \rho_2$ and $A' \simeq B'$ and $\rho_1 \simeq \rho_2$. By the IHs,

- $A' \equiv C'$,
- $C' \sim B'$,

- $\rho_1 \equiv \rho$, and
- $\rho \sim \rho_2$

for some C' and ρ.

If $\ell \in \mathit{dom}(B)$, then $B = \rho_{21} \odot (\ell : B'; \cdot) \odot \rho_{22}$ for some ρ_{21} and ρ_{22} such that $\rho_2 = \rho_{21} \odot \rho_{22}$ and $\ell \notin \mathit{dom}(\rho_{21})$. Since $\rho \sim \rho_{21} \odot \rho_{22}$ and $C' \sim B'$, there exist some ρ_{11} and ρ_{12} such that

- $\rho \equiv \rho_{11} \odot \rho_{12}$,
- $\rho_{11} \odot (\ell : C'; \cdot) \odot \rho_{12} \sim \rho_{21} \odot (\ell : B'; \cdot) \odot \rho_{22}$, and
- $\ell \notin \mathit{dom}(\rho_{11})$.

Here, we have

$$\begin{array}{lll} & \ell : A'; \rho_1 & \\ \equiv & \ell : C'; \rho & \text{since } A' \equiv C' \text{ and } \rho_1 \equiv \rho \\ \equiv & \ell : C'; \rho_{11} \odot \rho_{12} & \text{since } \rho \equiv \rho_{11} \odot \rho_{12} \\ \equiv & \rho_{11} \odot (\ell : C'; \cdot) \odot \rho_{12} & \text{since } \ell \notin \mathit{dom}(\rho_{11}) \\ \sim & \rho_{21} \odot (\ell : B'; \cdot) \odot \rho_{22} & \\ = & B. & \end{array}$$

Otherwise, if $\ell \notin \mathit{dom}(B)$, it is found from $B \triangleright_\ell B', \rho_2$ that $B = \rho_2$ and B ends with $\star$. Since $\rho \sim \rho_2$, we have $\rho \sim B$. By (C_ConsL), $\ell : A'; \rho \sim B$. Here, we have

$$\ell : A'; \rho_1 \equiv \ell : A'; \rho \sim B.$$

Case (CE_ConsR): We have $A \simeq \ell : B'; \rho_2$ and, by inversion, $A \triangleright_\ell A', \rho_1$ and $A' \simeq B'$ and $\rho_1 \simeq \rho_2$. By the IHs,

- $A' \equiv C'$,
- $C' \sim B'$,
- $\rho_1 \equiv \rho$, and
- $\rho \sim \rho_2$

for some C' and ρ_2.

If $\ell \in \mathit{dom}(A)$, then $A = \rho_{11} \odot (\ell : A'; \cdot) \odot \rho_{12}$ for some ρ_{11} and ρ_{12} such that $\rho_1 = \rho_{11} \odot \rho_{12}$ and $\ell \notin \mathit{dom}(\rho_{11})$. Here, we have

$$\begin{array}{lll} & A & \\ = & \rho_{11} \odot \ell : A'; \cdot \odot \rho_{12} & \\ \equiv & \rho_{11} \odot \ell : C'; \cdot \odot \rho_{12} & \text{since } A' \equiv C' \\ \equiv & \ell : C'; \rho_{11} \odot \rho_{12} & \text{since } \ell \notin \mathit{dom}(\rho_{11}) \\ = & \ell : C'; \rho_1 & \\ \equiv & \ell : C'; \rho & \text{since } \rho_1 \equiv \rho \\ \sim & \ell : B'; \rho_2 & \text{by (C\_Cons) since } C' \sim B' \text{ and } \rho \sim \rho_2 \\ = & B. & \end{array}$$

□

Lemma 17. *If $\rho_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \simeq \rho_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$ and $\ell \notin \mathit{dom}(\rho_{11}) \cup \mathit{dom}(\rho_{21})$, then $A \simeq B$ and $\rho_{11} \odot \rho_{12} \simeq \rho_{21} \odot \rho_{22}$.*

Proof. By induction on the derivation of $\rho_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \simeq \rho_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$.

Case (CE_Refl): Obvious by (CE_Refl).

Case (CE_ConsL): By case analysis on ρ_{11}.

Case $\rho_{11} = \cdot$: We have $A \simeq B$ and $\rho_{12} \simeq \rho_{21} \odot \rho_{22}$ by inversion, and therefore we finish.

Case $\rho_{11} \neq \cdot$: We have $\rho_{11} = \ell' : A'; \rho'_{11}$. Since $\ell \notin \mathit{dom}(\rho_{11})$, it is found that $\ell \neq \ell'$.

Case $\ell' \in dom(\rho_{21})$: There exist some ρ_{211}, ρ_{212}, and B' such that

- $\rho_{21} = \rho_{211} \odot (\ell' : B'; \cdot) \odot \rho_{212}$,
- $\ell' \notin dom(\rho_{211})$,
- $A' \simeq B'$, and
- $\rho'_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \simeq \rho_{211} \odot \rho_{212} \odot (\ell : B; \cdot) \odot \rho_{22}$

by inversion. By the IH, $A \simeq B$ and $\rho'_{11} \odot \rho_{12} \simeq \rho_{211} \odot \rho_{212} \odot \rho_{22}$. By (CE_ConsL),

$$\rho_{11} \odot \rho_{12} = \ell' : A'; \rho'_{11} \odot \rho_{12} \simeq \rho_{211} \odot (\ell' : B'; \cdot) \odot \rho_{212} \odot \rho_{22} = \rho_{21} \odot \rho_{22}.$$

Case $\ell' \notin dom(\rho_{21})$ and $\ell' \in dom(\rho_{22})$: There exist some ρ_{221}, ρ_{222}, and B' such that

- $\rho_{22} = \rho_{221} \odot (\ell' : B'; \cdot) \odot \rho_{222}$,
- $\ell' \notin dom(\rho_{221})$,
- $A' \simeq B'$, and
- $\rho'_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \simeq \rho_{21} \odot (\ell : B; \cdot) \odot \rho_{221} \odot \rho_{222}$

by inversion. By the IH, $A \simeq B$ and $\rho'_{11} \odot \rho_{12} \simeq \rho_{21} \odot \rho_{221} \odot \rho_{222}$. By (CE_ConsL),

$$\rho_{11} \odot \rho_{12} = \ell' : A'; \rho'_{11} \odot \rho_{12} \simeq \rho_{21} \odot \rho_{221} \odot (\ell' : B'; \cdot) \odot \rho_{22} = \rho_{21} \odot \rho_{22}.$$

Case $\ell' \notin dom(\rho_{21} \odot \rho_{22})$: It is found that

- $\rho_{21} \odot \rho_{22}$ ends with $\star$ and
- $\rho'_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \simeq \rho_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$

by inversion. By the IH, $A \simeq B$ and $\rho'_{11} \odot \rho_{12} \simeq \rho_{21} \odot \rho_{22}$. By (CE_ConsL), $\ell' : A'; \rho'_{11} \odot \rho_{12} \simeq \rho_{21} \odot \rho_{22}$.

Case (CE_ConsR): Similar to the case for (CE_ConsL).

Case (CE_DynL), (CE_DynR), (CE_Fun), (CE_Poly), (CE_PolyL), (CE_PolyR), (CE_Record), and (CE_Variant): Contradictory.

□

Lemma 18. *If $A \simeq B$ and $\rho_{11} \odot \rho_{12} \simeq \rho_{21} \odot \rho_{22}$ and $\ell \notin dom(\rho_{11}) \cup dom(\rho_{21})$, then $\rho_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \simeq \rho_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$.*

Proof. By induction on the sum of the sizes of $\rho_{11} \odot \rho_{12}$ and $\rho_{21} \odot \rho_{22}$. Since $\rho_{11} \odot \rho_{12}$ is defined, there are only two cases on ρ_{11} to be considered.

Case $\rho_{11} = \cdot$: By (CE_ConsL).

Case $\rho_{11} = \ell' : A'; \rho'_{11}$: If $\rho_{21} = \cdot$, then $\rho_{11} \odot \rho_{21} = \rho_{22}$. By (CE_ConsR),

$$\rho_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \simeq \ell : B; \rho_{22} = \rho_{21} \odot (\ell : B; \cdot) \odot \rho_{22},$$

and so we finish.

In what follows, we suppose $\rho_{21} \neq \cdot$. By case analysis on the rule applied last to derive $\ell' : A'; \rho'_{11} \odot \rho_{12} \simeq \rho_{21} \odot \rho_{22}$.

Case (CE_Refl): Since $\rho_{21} \neq \cdot$, we can suppose that $\rho_{21} = \ell' : A'; \rho'_{21}$. Thus, $\rho'_{11} \odot \rho_{21} = \rho'_{21} \odot \rho_{22}$, and $\rho'_{11} \odot \rho_{21} \simeq \rho'_{21} \odot \rho_{22}$ by (CE_Refl). By the IH, $\rho'_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \simeq \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$. By (CE_Refl) and (CE_ConsL),

$$\rho_{11} \odot (\ell : A; \cdot) \odot \rho_{12} = \ell' : A'; \rho'_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \simeq \ell : A'; \rho'_{21} \odot (\ell : B; \cdot) \odot \rho_{22} = \rho_{21} \odot (\ell : B; \cdot) \odot \rho_{22}.$$

Case (CE_DynR): We have $\rho_{21} \odot \rho_{22} = \star$. By (CE_ConsR).

Case (CE_ConsL): By inversion, $\rho_{21} \odot \rho_{22} \rhd_{\ell'} B', \rho_2$ and $A' \simeq B'$ and $\rho'_{11} \odot \rho_{12} \simeq \rho_2$ for some B', and ρ_2.

Case $\ell' \in dom(\rho_{21})$: There exist some ρ_{211} and ρ_{212} such that

- $\rho_{21} = \rho_{211} \odot (\ell' : B'; \cdot) \odot \rho_{212}$,
- $\rho_2 = \rho_{211} \odot \rho_{212} \odot \rho_{22}$, and

- $\ell' \notin \mathit{dom}\,(\rho_{211})$.

Since $\rho'_{11} \odot \rho_{12} \simeq \rho_2$, we have $\rho'_{11} \odot \rho_{12} \simeq \rho_{211} \odot \rho_{212} \odot \rho_{22}$. By the IH, $\rho'_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \simeq \rho_{211} \odot \rho_{212} \odot (\ell : B; \cdot) \odot \rho_{22}$. By (CE_ConsL),

$$\ell' : A'; \rho'_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \simeq \rho_{211} \odot (\ell' : B'; \cdot) \odot \rho_{212} \odot (\ell : B; \cdot) \odot \rho_{22}.$$

Thus,

$$\rho_{11} \odot (\ell : A; \cdot) \odot \rho_{12} = \ell' : A'; \rho'_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \simeq \rho_{211} \odot (\ell' : B'; \cdot) \odot \rho_{212} \odot (\ell : B; \cdot) \odot \rho_{22} = \rho_{21} \odot (\ell : B; \cdot) \odot \rho_{22}.$$

Case $\ell' \notin \mathit{dom}\,(\rho_{21})$ and $\ell' \in \mathit{dom}\,(\rho_{22})$: There exist some ρ_{221} and ρ_{222} such that

- $\rho_{22} = \rho_{221} \odot (\ell' : B'; \cdot) \odot \rho_{222}$,
- $\rho_2 = \rho_{21} \odot \rho_{221} \odot \rho_{222}$, and
- $\ell' \notin \mathit{dom}\,(\rho_{221})$.

Since $\rho'_{11} \odot \rho_{12} \simeq \rho_2$, we have $\rho'_{11} \odot \rho_{12} \simeq \rho_{21} \odot \rho_{221} \odot \rho_{222}$. By the IH, $\rho'_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \simeq \rho_{21} \odot (\ell : B; \cdot) \odot \rho_{221} \odot \rho_{222}$. By (CE_ConsL),

$$\ell' : A'; \rho'_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \simeq \rho_{21} \odot (\ell : B; \cdot) \odot \rho_{221} \odot (\ell' : B'; \cdot) \odot \rho_{222}.$$

Thus,

$$\rho_{11} \odot (\ell : A; \cdot) \odot \rho_{12} = \ell' : A'; \rho'_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \simeq \rho_{21} \odot (\ell : B; \cdot) \odot \rho_{221} \odot (\ell' : B'; \cdot) \odot \rho_{222} = \rho_{21} \odot (\ell : B; \cdot) \odot \rho_{22}.$$

Case $\ell' \notin \mathit{dom}\,(\rho_{21})$ and $\ell' \notin \mathit{dom}\,(\rho_{22})$: It is found that

- $\rho_{21} \odot \rho_{22}$ ends with $\star$ and
- $\rho_2 = \rho_{21} \odot \rho_{22}$.

Since $\rho'_{11} \odot \rho_{12} \simeq \rho_2$, we have $\rho'_{11} \odot \rho_{12} \simeq \rho_{21} \odot \rho_{22}$. By the IH, $\rho'_{11} \odot (\ell : A; \cdot) \odot \rho_{12} \simeq \rho_{21} \odot (\ell : B; \cdot) \odot \rho_{22}$. By (CE_ConsL), we finish.

Case (CE_ConsR): Similar to the case for (CE_ConsL).

Case (CE_DynL), (CE_Fun), (CE_Poly), (CE_PolyL), (CE_PolyR), (CE_Record), and (CE_Variant): Contradictory. Note that the contradiction in the case of (C_PolyR) is proven by the definition of **QPoly**.

□

Lemma 19. *If $\rho_1 \simeq \rho_{21} \odot \rho_{22}$ and ρ_1 ends with $\star$ and $\ell \notin \mathit{dom}\,(\rho_1)$, then $\rho_1 \simeq \rho_{21} \odot (\ell : A; \cdot) \odot \rho_{22}$ for any A.*

Proof. By induction on the sizes of ρ_1 and ρ_{21}. If $\rho_{21} = \cdot$, then we finish by (CE_ConsR).

In what follows, since $\rho_{21} \odot \rho_{22}$ is defined, we can suppose that $\rho_{21} = \ell' : B; \rho'_{21}$ for some ℓ', B, and ρ'_{21}. By case analysis on the rule applied last to derive $\rho_1 \simeq \rho_{21} \odot \rho_{22}$.

Case (CE_Refl): We have $\rho_1 = \ell' : B; \rho'_1$ for some ρ'_1 such that $\rho'_1 = \rho'_{21} \odot \rho_{22}$. Since $\rho'_1 \simeq \rho'_{21} \odot \rho_{22}$ by (CE_Refl), we have $\rho' \simeq \rho'_{21} \odot (\ell : A; \cdot) \odot \rho_{22}$ by the IH. By (CE_ConsL), we finish.

Case (CE_DynL): By (CE_DynL).

Case (CE_ConsL): We have $\rho_1 = \ell'' : C; \rho'_1$ and, by inversion, $\rho_{21} \odot \rho_{22} \rhd_{\ell''} B', \rho'_2$ and $C \simeq B'$ and $\rho'_1 \simeq \rho'_2$ for some ℓ'', B', C, ρ'_1, and ρ'_2. Note that $\ell \neq \ell''$ since $\ell \notin \mathit{dom}\,(\rho_1)$.

Case $\ell'' \in \mathit{dom}\,(\rho_{21})$: There exist some ρ_{211} and ρ_{212} such that

- $\rho_{21} = \rho_{211} \odot (\ell'' : B'; \cdot) \odot \rho_{212}$,
- $\rho'_2 = \rho_{211} \odot \rho_{212} \odot \rho_{22}$, and
- $\ell'' \notin \mathit{dom}\,(\rho_{211})$.

Since $\rho'_1 \simeq \rho'_2$, we have $\rho'_1 \simeq \rho_{211} \odot \rho_{212} \odot \rho_{22}$. By the IH, $\rho'_1 \simeq \rho_{211} \odot \rho_{212} \odot (\ell : A; \cdot) \odot \rho_{22}$. Since $(\rho_{21} \odot (\ell : A; \cdot) \odot \rho_{22}) \rhd_{\ell''} B', \rho_{211} \odot \rho_{212} \odot (\ell : A; \cdot) \odot \rho_{22}$, we finish by (CE_ConsL).

Case $\ell'' \notin \mathit{dom}\,(\rho_{21})$ and $\ell'' \in \mathit{dom}\,(\rho_{22})$: There exist some ρ_{221} and ρ_{222} such that

- $\rho_{22} = \rho_{221} \odot (\ell'' : B'; \cdot) \odot \rho_{222}$,
- $\rho'_2 = \rho_{21} \odot \rho_{221} \odot \rho_{222}$, and
- $\ell'' \notin \mathit{dom}(\rho_{221})$.

Since $\rho'_1 \simeq \rho'_2$, we have $\rho'_1 \simeq \rho_{21} \odot \rho_{221} \odot \rho_{222}$. By the IH, $\rho'_1 \simeq \rho_{21} \odot (\ell : A; \cdot) \odot \rho_{221} \odot \rho_{222}$. Since $(\rho_{21} \odot (\ell : A; \cdot) \odot \rho_{22}) \triangleright_{\ell''} B', \rho_{21} \odot (\ell : A; \cdot) \odot \rho_{221} \odot \rho_{222}$, we finish by (CE_ConsL).

Case $\ell'' \notin \mathit{dom}(\rho_{21})$ and $\ell'' \notin \mathit{dom}(\rho_{22})$: We have $B' = \star$ and $\rho'_2 = \rho_{21} \odot \rho_{22}$ and $\rho_{21} \odot \rho_{22}$ ends with $\star$. Since $\rho'_1 \simeq \rho'_2$, we have $\rho'_1 \simeq \rho_{21} \odot \rho_{22}$. By the IH, $\rho'_1 \simeq \rho_{21} \odot (\ell : A; \cdot) \odot \rho_{22}$. By (CE_ConsL), we finish.

Case (CE_ConsR): Since $\rho_{21} = \ell' : B; \rho'_{21}$, by inversion $\rho_1 \triangleright_{\ell'} C, \rho'_1$ and $C \simeq B$ and $\rho'_1 \simeq \rho'_{21} \odot \rho_{22}$ for some C and ρ'_1. By the IH, $\rho'_1 \simeq \rho'_{21} \odot (\ell : A; \cdot) \odot \rho_{22}$. By (CE_ConsR), we finish.

Case (CE_DynR), (CE_Fun), (CE_Poly), (CE_PolyL), (CE_PolyR), (CE_Record), and (CE_Variant): Contradictory. Note that the contradiction in the case of (C_PolyL) is proven by the definition of **QPoly**.

□

Lemma 20. *If $A \equiv C$ and $C \equiv B$ and $A \simeq C$ and $C \simeq B$, then $A \simeq B$.*

Proof. By induction on $A \simeq C$.

Case (CE_Refl): Obvious.

Case (CE_DynL): By (CE_DynL).

Case (CE_DynR): We have $C = \star$. By Lemma 5 (1), $A = \star$. Thus, we finish by (CE_DynL).

Case (CE_Fun): We have $A = A_1 \rightarrow A_2$ and $C = C_1 \rightarrow C_2$ and, by inversion, $A_1 \simeq C_1$ and $A_2 \simeq C_2$ for some A_1, A_2, C_1, and C_2. Since $A \equiv C$, we have $A_1 \equiv C_1$ and $A_2 \equiv C_2$ by Lemma 5 (2). Again, by Lemma 5 (2), since $C \equiv B$, there exist some B_1 and B_2 such that $B = B_1 \rightarrow B_2$ and $C_1 \equiv B_1$ and $C_2 \equiv B_2$. Since $C \simeq B$, we have $C_1 \simeq B_1$ and $C_2 \simeq B_2$ by Lemma 10. Thus, by the IHs, $A_1 \simeq B_1$ and $A_2 \simeq B_2$. By (CE_Fun), $A_1 \rightarrow A_2 \simeq B_1 \rightarrow B_2$.

Case (CE_Poly): We have $A = \forall X{:}K.\, A'$ and $C = \forall X{:}K.\, C'$ and, by inversion, $A' \simeq C'$ for some X, K, A', and C'. Since $A \equiv C$, we have $A' \equiv C'$ by Lemma 5 (3). Again, by Lemma 5 (3), since $C \equiv B$, there exist some B' such that $B = \forall X{:}K.\, B'$ and $C' \equiv B'$. Since $C \simeq B$, we have $C' \simeq B'$ by Lemma 11. Thus, by the IH, $A' \simeq B'$. By (CE_Poly), $\forall X{:}K.\, A' \simeq \forall X{:}K.\, B'$.

Case (CE_PolyL): We have $A = \forall X{:}K.\, A'$ and, by inversion, $\mathbf{QPoly}(C)$ and $X \notin \mathit{ftv}(C)$, for some X, K, and A'. $\mathbf{QPoly}(C)$ is contradictory with the fact that $C = \forall X{:}K.\, C'$ for some C', which is implied by Lemma 5 (3) with $A \equiv C$ and $A = \forall X{:}K.\, A'$.

Case (CE_PolyR): We have $C = \forall X{:}K.\, C'$ and, by inversion, $\mathbf{QPoly}(A)$ and $X \notin \mathit{ftv}(A)$, for some X, K, and C'. $\mathbf{QPoly}(A)$ is contradictory with the fact that $A = \forall X{:}K.\, A'$ for some A', which is implied by Lemma 5 (3) with $A \equiv C$ and $C = \forall X{:}K.\, C'$.

Case (CE_Record): We have $A = [\rho_1]$ and $C = [\rho_3]$ and, by inversion, $\rho_1 \simeq \rho_3$ for some ρ_1 and ρ_3. Since $A \equiv C$, we have $\rho_1 \equiv \rho_3$ by Lemma 5 (4). Again, by Lemma 5 (4), since $C \equiv B$, there exists some ρ_2 such that $B = [\rho_2]$ and $\rho_3 \equiv \rho_2$. Since $C \simeq B$, we have $\rho_3 \simeq \rho_2$ by Lemma 13. By the IH, $\rho_1 \simeq \rho_2$. By (CE_Record), $[\rho_1] \simeq [\rho_2]$.

Case (CE_Variant): We have $A = \langle \rho_1 \rangle$ and $C = \langle \rho_3 \rangle$ and, by inversion, $\rho_1 \simeq \rho_3$ for some ρ_1 and ρ_3. Since $A \equiv C$, we have $\rho_1 \equiv \rho_3$ by Lemma 5 (5). Again, by Lemma 5 (5), since $C \equiv B$, there exists some ρ_2 such that $B = \langle \rho_2 \rangle$ and $\rho_3 \equiv \rho_2$. Since $C \simeq B$, we have $\rho_3 \simeq \rho_2$ by Lemma 14. By the IH, $\rho_1 \simeq \rho_2$. By (CE_Variant), $\langle \rho_1 \rangle \simeq \langle \rho_2 \rangle$.

Case (CE_ConsL): We have $A = \ell : A'; \rho_1$ and, by inversion, $C \triangleright_\ell C', \rho_3$ and $A' \simeq C'$ and $\rho_1 \simeq \rho_3$ for some ℓ, A', C', ρ_1, and ρ_3. Since $A \equiv C$, there exist ρ_{31} and ρ_{32} such that

- $C = \rho_{31} \odot (\ell : C'; \cdot) \odot \rho_{32}$,
- $A' \equiv C'$,

- $\rho_1 \equiv \rho_{31} \odot \rho_{32}$, and
- $\ell \notin \mathit{dom}\,(\rho_{31})$

by Lemma 5 (6). Again, by Lemma 5 (6), since $C \equiv B$, there exists some B', ρ_{21}, and ρ_{22} such that

- $B = \rho_{21} \odot (\ell : B'; \cdot) \odot \rho_{22}$,
- $C' \equiv B'$,
- $\rho_{31} \odot \rho_{32} \equiv \rho_{21} \odot \rho_{22}$, and
- $\ell \notin \mathit{dom}\,(\rho_{21})$.

Since $\rho_{31} \odot (\ell : C'; \cdot) \odot \rho_{32} = C \simeq B = \rho_{21} \odot (\ell : B'; \cdot) \odot \rho_{22}$ and $\ell \notin \mathit{dom}\,(\rho_{31}) \cup \mathit{dom}\,(\rho_{21})$, we have $C' \simeq B'$ and $\rho_{31} \odot \rho_{32} \simeq \rho_{21} \odot \rho_{22}$ by Lemma 17. Since $C \rhd_\ell C', \rho_3$, we have $\rho_3 = \rho_{31} \odot \rho_{32}$, so $\rho_1 \simeq \rho_{31} \odot \rho_{32}$. By the IHs, $A' \simeq B'$ and $\rho_1 \simeq \rho_{21} \odot \rho_{22}$. Since $(\rho_{21} \odot (\ell : B'; \cdot) \odot \rho_{22}) \rhd_\ell B', \rho_{21} \odot \rho_{22}$, we have $\ell : A'; \rho_1 \simeq \rho_{21} \odot (\ell : B'; \cdot) \odot \rho_{22} = B$ by (CE_ConsL).

(CE_ConsR): We have $C = \ell : C'; \rho_3$ and, by inversion, $A \rhd_\ell A', \rho_1$ and $A' \simeq C'$ and $\rho_1 \simeq \rho_3$ for some ℓ, A', C', ρ_1, and ρ_3. Since $A \equiv C$, there exist ρ_{11} and ρ_{12} such that

- $A = \rho_{11} \odot (\ell : A'; \cdot) \odot \rho_{12}$,
- $A' \equiv C'$,
- $\rho_{11} \odot \rho_{12} \equiv \rho_3$, and
- $\ell \notin \mathit{dom}\,(\rho_{11})$

by Lemma 5 (6). Again, by Lemma 5 (6), since $C \equiv B$, there exists some B', ρ_{21}, and ρ_{22} such that

- $B = \rho_{21} \odot (\ell : B'; \cdot) \odot \rho_{22}$,
- $C' \equiv B'$,
- $\rho_3 \equiv \rho_{21} \odot \rho_{22}$, and
- $\ell \notin \mathit{dom}\,(\rho_{21})$.

Since $\ell : C'; \rho_3 = C \simeq B = \rho_{21} \odot (\ell : B'; \cdot) \odot \rho_{22}$ and $\ell \notin \mathit{dom}\,(\rho_{21})$, we have $C' \simeq B'$ and $\rho_3 \simeq \rho_{21} \odot \rho_{22}$ by Lemma 17. Since $A \rhd_\ell A', \rho_1$, we have $\rho_1 = \rho_{11} \odot \rho_{12}$, so $\rho_{11} \odot \rho_{12} \simeq \rho_3$. By the IHs, $A' \simeq B'$ and $\rho_{11} \odot \rho_{12} \simeq \rho_{21} \odot \rho_{22}$. Since $\ell \notin \mathit{dom}\,(\rho_{11}) \cup \mathit{dom}\,(\rho_{21})$ and $A' \simeq B'$, we have $\rho_{11} \odot (\ell : A'; \cdot) \odot \rho_{12} \simeq \rho_{21} \odot (\ell : B'; \cdot) \odot \rho_{22}$.

□

Lemma 21. *If $A \equiv B$, then $A \simeq B$.*

Proof. By induction on the derivation of $A \equiv B$.

Case (Eq_Refl): By (CE_Refl).

Case (Eq_Trans): By inversion, $A \equiv C$ and $C \equiv B$ for some C. By the IHs, $A \simeq C$ and $C \simeq B$. We have $A \simeq B$ by Lemma 20.

Case (Eq_Sym): By inversion, $B \equiv A$. By the IH, $B \simeq A$. By Lemma 6, $A \simeq B$.

Case (Eq_Fun): By the IHs.

Case (Eq_Poly): By the IH.

Case (Eq_Record): By the IH.

Case (Eq_Variant): By the IH.

Case (Eq_Cons): By the IH, (CE_Refl), and (CE_Cons).

Case (Eq_Swap): By (CE_Refl) and (CE_ConsL).

□

Lemma 22. *If $A \equiv C$ and $C \sim B$, then $A \simeq B$.*

Proof. By induction on $C \sim B$.

Case (C_Refl): By Lemma 21.

Case (C_DynL): We have $C = \star$. By Lemma 5 (1), $A = \star$. Thus, we finish by (CE_DynL).

Case (C_DynR): We have $B = \star$. By (CE_DynR).

Case (C_Fun): We have $C = C_1 \rightarrow C_2$ and $B = B_1 \rightarrow B_2$ and, by inversion, $C_1 \sim B_1$ and $C_2 \sim B_2$ for some C_1, C_2, B_1, and B_2. Since $A \equiv C_1 \rightarrow C_2$, there exist some A_{11} and A_{12} such that $A = A_1 \rightarrow A_2$ and $A_1 \equiv C_1$ and $A_2 \equiv C_2$, by Lemma 5 (2). By the IHs, $A_1 \simeq B_1$ and $A_2 \simeq B_2$. By (CE_Fun), $A_1 \rightarrow A_2 \simeq B_1 \rightarrow B_2$.

Case (C_Poly): We have $C = \forall X{:}K.\, C'$ and $B = \forall X{:}K.\, B'$ and, by inversion, $C' \sim B'$ for some X, K, C', and B'. Since $A \equiv \forall X{:}K.\, C'$, there exists some A' such that $A = \forall X{:}K.\, A'$ and $A' \equiv C'$, by Lemma 5 (3). By the IH, $A' \simeq B'$. By (CE_Poly), we finish.

Case (C_PolyL): We have $C = \forall X{:}K.\, C'$ and, by inversion, $\mathbf{QPoly}\,(B)$ and $X \notin \mathit{ftv}(B)$ and $C' \sim B$. Since $A = \forall X{:}K.\, C'$, there exists some A' such that $A = \forall X{:}K.\, A'$ and $A' \equiv C'$, by Lemma 5 (3). By the IH, $A' \simeq B$. By (CE_PolyL), we finish.

Case (C_PolyR): We have $B = \forall X{:}K.\, B'$ and, by inversion, $\mathbf{QPoly}\,(C)$ and $X \notin \mathit{ftv}(C)$ and $C \sim B'$. By the IH, $A \simeq B'$. Since $A \equiv C$ and $\mathbf{QPoly}\,(C)$ and $X \notin \mathit{ftv}(C)$, we have $\mathbf{QPoly}\,(A)$ and $X \notin \mathit{ftv}(A)$ by Lemmas 1 and 2. Thus, by (CE_PolyR), we have $A \simeq \forall X{:}K.\, B'$.

Case (C_Record): We have $C = [\rho_3]$ and $B = [\rho_2]$ and, by inversion, $\rho_3 \sim \rho_2$ for some ρ_3 and ρ_2. Since $A \equiv C$, there exists some ρ_1 such that $A = [\rho_1]$ and $\rho_1 \equiv \rho_3$, by Lemma 5 (4). By the IH, $\rho_1 \simeq \rho_2$. Thus, by (CE_Record), we have $[\rho_1] \simeq [\rho_2]$.

Case (C_Variant): We have $C = \langle \rho_3 \rangle$ and $B = \langle \rho_2 \rangle$ and, by inversion, $\rho_3 \sim \rho_2$ for some ρ_3 and ρ_2. Since $A \equiv C$, there exists some ρ_1 such that $A = \langle \rho_1 \rangle$ and $\rho_1 \equiv \rho_3$, by Lemma 5 (5). By the IH, $\rho_1 \simeq \rho_2$. Thus, by (CE_Variant), we have $\langle \rho_1 \rangle \simeq \langle \rho_2 \rangle$.

Case (C_Cons): We have $C = \ell : C'; \rho_3$ and $B = \ell : B'; \rho_2$ and, by inversion, $C' \sim B'$ and $\rho_3 \sim \rho_2$ for some ℓ, C', B', ρ_3, and ρ_2. Since $A \equiv C$, there exist some A', ρ_{11}, ρ_{12} such that

- $A = \rho_{11} \odot (\ell : A'; \cdot) \odot \rho_{12}$,
- $A' \equiv C'$,
- $\rho_{11} \odot \rho_{12} \equiv \rho_3$, and
- $\ell \notin \mathit{dom}\,(\rho_{11})$

by Lemma 5 (6). By the IHs, $A' \simeq B'$ and $\rho_{11} \odot \rho_{12} \simeq \rho_2$. We have $A \rhd_\ell A', \rho_{11} \odot \rho_{12}$. Thus, by (CE_ConsR), $A \simeq \ell : B'; \rho_2$.

Case (C_ConsL): We have $C = \ell : C'; \rho_3$ and, by inversion, $\ell \notin \mathit{dom}\,(B)$ and B ends with $\star$ and $\rho_3 \sim B$ for some ℓ, C', and ρ_3. Since $A \equiv C$, there exist some A', ρ_{11}, ρ_{12} such that

- $A = \rho_{11} \odot (\ell : A'; \cdot) \odot \rho_{12}$,
- $A' \equiv C'$,
- $\rho_{11} \odot \rho_{12} \equiv \rho_3$, and
- $\ell \notin \mathit{dom}\,(\rho_{11})$

by Lemma 5 (6). By the IH, $\rho_{11} \odot \rho_{12} \simeq B$. By Lemmas 6 and 19, $\rho_{11} \odot (\ell : A'; \cdot) \odot \rho_{12} \simeq B$.

Case (C_ConsR): we have $B = \ell : B'; \rho_2$ and, by inversion, $\ell \notin dom(C)$ and C ends with $\star$ and $C \sim \rho_2$ for some ℓ, B', and ρ_2. By the IH, $A \simeq \rho_2$. Since $A \equiv C$ and $\ell \notin dom(C)$ and C ends with $\star$, we have $\ell \notin dom(A)$ and A ends with $\star$ by Lemmas 2 and 3. Thus, by (CE_ConsR), we have $A \simeq \ell : B'; \rho_2$.

□

Lemma 23. *If $A \simeq B$ and $B \equiv C$, then $A \simeq C$.*

Proof. By induction on the derivation of $A \simeq B$.

Case (CE_Refl): By Lemma 21.

Case (CE_DynL): By (CE_DynL).

Case (CE_DynR): We have $B = \star$. Since $B \equiv C$, we have $C = \star$ by Lemma 5 (1). By (CE_DynR).

Case (CE_Fun): We have $A = A_1 \rightarrow A_2$ and $B = B_1 \rightarrow B_2$ and, by inversion, $A_1 \simeq B_1$ and $A_2 \simeq B_2$ for some A_1, A_2, B_1, and B_2. Since $B \equiv C$, there exist some C_1 and C_2 such that $C = C_1 \rightarrow C_2$ and $B_1 \equiv C_1$ and $B_2 \equiv C_2$ by Lemma 5 (2). By the IHs, $A_1 \simeq C_1$ and $A_2 \simeq C_2$. Thus, $A_1 \rightarrow A_2 \simeq C_1 \rightarrow C_2$ by (CE_Fun).

Case (CE_Poly): We have $A = \forall X{:}K.\, A'$ and $B = \forall X{:}K.\, B'$ and, by inversion, $A' \simeq B'$ for some X, K, A', and B'. Since $B \equiv C$, there exist some C' such that $C = \forall X{:}K.\, C'$ and $B' \equiv C'$ by Lemma 5 (3). By the IH, $A' \simeq C'$. Thus, $\forall X{:}K.\, A' \simeq \forall X{:}K.\, C'$ by (CE_Poly).

Case (CE_PolyL): We have $A = \forall X{:}K.\, A'$ and, by inversion, $\mathbf{QPoly}(B)$ and $X \notin ftv(B)$ and $A' \simeq B$ for some X, K, and A'. By the IH, $A' \simeq C$. Since $B \equiv C$ and $\mathbf{QPoly}(B)$ and $X \notin ftv(B)$, we have $\mathbf{QPoly}(C)$ and $X \notin ftv(C)$ by Lemmas 1 and 2. Thus, $\forall X{:}K.\, A' \simeq C$ by (CE_PolyL).

Case (CE_PolyR): We have $B = \forall X{:}K.\, B'$ and, by inversion, $\mathbf{QPoly}(A)$ and $X \notin ftv(A)$ and $A \simeq B'$ for some X, K, and B'. By Lemma 5 (3), since $B \equiv C$, there exists some C' such that $C = \forall X{:}K.\, C'$ and $B' \equiv C'$. By the IH, $A \simeq C'$. By (CE_PolyR), $A \simeq \forall X{:}K.\, C'$.

Case (CE_Record): We have $A = [\rho_1]$ and $B = [\rho_2]$ and, by inversion, $\rho_1 \simeq \rho_2$ for some ρ_1 and ρ_2. By Lemma 5 (4), since $B \equiv C$, there exists some ρ_3 such that $C = [\rho_3]$ and $\rho_2 \equiv \rho_3$. By the IH, $\rho_1 \simeq \rho_3$. By (CE_Record), $[\rho_1] \simeq [\rho_3]$.

Case (CE_Variant): We have $A = \langle \rho_1 \rangle$ and $B = \langle \rho_2 \rangle$ and, by inversion, $\rho_1 \simeq \rho_2$ for some ρ_1 and ρ_2. By Lemma 5 (5), since $B \equiv C$, there exists some ρ_3 such that $C = \langle \rho_3 \rangle$ and $\rho_2 \equiv \rho_3$. By the IH, $\rho_1 \simeq \rho_3$. By (CE_Variant), $\langle \rho_1 \rangle \simeq \langle \rho_3 \rangle$.

Case (CE_ConsL): We have $A = \ell : A'; \rho_1$ and, by inversion, $B \triangleright_\ell B', \rho_2$ and $A' \simeq B'$ and $\rho_1 \simeq \rho_2$ for some ℓ, A', B', ρ_1, and ρ_2.

If $\ell \in dom(B)$, then there exist some ρ_{21} and ρ_{22} such that

- $B = \rho_{21} \odot (\ell : B'; \cdot) \odot \rho_{22}$,
- $\rho_2 = \rho_{21} \odot \rho_{22}$, and
- $\ell \notin dom(\rho_{21})$.

Since $B \equiv C$, there exist some C', ρ_{31}, and ρ_{32} such that

- $C = \rho_{31} \odot (\ell : C'; \cdot) \odot \rho_{32}$,
- $B' \equiv C'$,
- $\ell \notin dom(\rho_{31})$, and
- $\rho_{21} \odot \rho_{22} \equiv \rho_{31} \odot \rho_{32}$

by Lemma 5 (6). Since $\rho_1 \simeq \rho_2$ and $\rho_2 = \rho_{21} \odot \rho_{22} \equiv \rho_{31} \odot \rho_{32}$, we have $\rho_1 \simeq \rho_{31} \odot \rho_{32}$ by the IH. Besides, $A' \simeq B'$ and $B' \equiv C'$, we have $A' \simeq C'$ by the IH. Since $C \triangleright_\ell C', \rho_{31} \odot \rho_{32}$, we have $\ell : A'; \rho_1 \simeq C$ by (CE_ConsL).

Otherwise, if $\ell \notin dom(B)$, then $B' = \star$ and $\rho_2 = B$ and B ends with $\star$. Since $\rho_1 \simeq \rho_2$ and $\rho_2 = B \equiv C$, we have $\rho_1 \simeq C$ by the IH. Since $B \equiv C$, we can find $C \triangleright_\ell \star, C$ by Lemmas 4 and 3. Since $A' \simeq \star$ by (CE_DynR) and $\rho_1 \simeq C$, we have $\ell : A'; \rho_1 \simeq C$ by (CE_ConsL).

Case (CE_ConsR): We have $B = \ell : B'; \rho_2$ and, by inversion, $A \rhd_\ell A', \rho_1$ and $A' \simeq B'$ and $\rho_1 \simeq \rho_2$ for some ℓ, A', B', ρ_1, and ρ_2. Since $B \equiv C$, there exist some C', ρ_{31}, and ρ_{32} such that

- $C = \rho_{31} \odot (\ell : C'; \cdot) \odot \rho_{32}$,
- $B' \equiv C'$,
- $\ell \notin dom(\rho_{31})$, and
- $\rho_2 \equiv \rho_{31} \odot \rho_{32}$

by Lemma 5 (6).

If $\ell \in dom(A)$, then there exist some ρ_{11} and ρ_{12} such that

- $A = \rho_{11} \odot (\ell : A'; \cdot) \odot \rho_{12}$,
- $\rho_1 = \rho_{11} \odot \rho_{12}$, and
- $\ell \notin dom(\rho_{11})$.

Since $\rho_1 \simeq \rho_2$ and $\rho_2 \equiv \rho_{31} \odot \rho_{32}$, we have $\rho_1 \simeq \rho_{31} \odot \rho_{32}$ by the IH. Besides, $A' \simeq B'$ and $B' \equiv C'$, we have $A' \simeq C'$ by the IH. By Lemma 18,

$$A = \rho_{11} \odot (\ell : A'; \cdot) \odot \rho_{12} \simeq \rho_{31} \odot (\ell : C'; \cdot) \odot \rho_{32} = C.$$

Otherwise, if $\ell \notin dom(A)$, then $A' = \star$ and $\rho_1 = A$ and A ends with $\star$. Since $A = \rho_1 \simeq \rho_2$ and $\rho_2 \equiv \rho_{31} \odot \rho_{32}$, we have $A \simeq \rho_{31} \odot \rho_{32}$ by the IH. By Lemma 19, $A \simeq \rho_{31} \odot (\ell : C'; \cdot) \odot \rho_{32} = C$.

□

Theorem 1. *$A \simeq B$ if and only if $A \equiv A'$ and $A' \sim B'$ and $B' \equiv B$ for some A' and B'.*

Proof. First, we show the left-to-right direction. Suppose $A \simeq B$. By Lemma 16, there exists some A' such that $A \equiv A'$ and $A' \sim B$. Since $[B == B]$ by (Eq_Refl), we finish

Next, we show the right-to-left. Suppose that $A \equiv A'$ and $A' \sim B'$ and $B' \equiv B$. By Lemma 22, $A \simeq B'$. By Lemma 23, $A \simeq B$. □

2.2 Type Soundness

Lemma 24 (Weakening)**.** *Suppose that $\Sigma \vdash \Gamma_1, \Gamma_2$. Let Γ_3 be a typing context such that $dom(\Gamma_2) \cap dom(\Gamma_3) = \emptyset$.*

1. *If $\Sigma \vdash \Gamma_1, \Gamma_3$, then $\Sigma \vdash \Gamma_1, \Gamma_2, \Gamma_3$.*
2. *If $\Sigma; \Gamma_1, \Gamma_3 \vdash A : K$, then $\Sigma; \Gamma_1, \Gamma_2, \Gamma_3 \vdash A : K$.*
3. *If $\Sigma; \Gamma_1, \Gamma_3 \vdash e : A$, then $\Sigma; \Gamma_1, \Gamma_2, \Gamma_3 \vdash e : A$.*

Proof. Straightforward by mutual induction on the derivations. □

Lemma 25 (Weakening type names)**.** *Suppose that $\Sigma \subseteq \Sigma'$.*

1. *If $\Sigma \vdash B \prec^\Phi C$, then $\Sigma' \vdash B \prec^\Phi C$.*
2. *If $\Sigma \vdash \Gamma$, then $\Sigma' \vdash \Gamma$.*
3. *If $\Sigma; \Gamma \vdash B : K$, then $\Sigma'; \Gamma \vdash B : K$.*
4. *If $\Sigma; \Gamma \vdash e : B$, then $\Sigma'; \Gamma \vdash e : B$.*

Proof. Straightforward by mutual induction on the derivations. □

Lemma 26. *If* **QPoly** (A)*, then* **QPoly** $(A[B/X])$*.*

Proof. First, we show $A[B/X]$ is not a polymorphic type by case analysis on A.

Case $A = \star$, Y (where $X \neq Y$), α, ι, $A' \to B'$, $[\rho]$, $\langle\rho\rangle$, $\cdot$, and $\ell : C; \rho$: Obvious.

Case $A = X$: Since **QPoly** (A), A must contain the dynamic type; thus, contradictory.

Case $A = \forall Y{:}K.\, C$: Contradictory with **QPoly** (A).

Thus, it suffices to show that $A[B/X]$ contains the dynamic type, which is obvious since A contains the dynamic type (from **QPoly** (A)) and type substitution preserves that property. □

Lemma 27. *If $\rho_1 \triangleright_\ell A, \rho_2$, then $\rho_1[B/X] \triangleright_\ell A[B/X], \rho_2[B/X]$.*

Proof. By induction on ρ_1.

Case $\rho_1 = \ell' : C; \rho_1'$: If $\ell' = \ell$, then $A = C$ and $\rho_2 = \rho_1'$, and, therefore, the statement holds obviously.
Otherwise, if $\ell' \neq \ell$, then we have $\rho_1' \triangleright_\ell A, \rho_2'$ and $\rho_2 = \ell' : C; \rho_2'$. By the IH, $\rho_1'[B/X] \triangleright_\ell A[B/X], \rho_2'[B/X]$. Thus, $\ell' : C[B/X]; \rho_1'[B/X] \triangleright_\ell A[B/X], \ell' : C[B/X]; \rho_2'[B/X]$, which is what we have to prove.

Case $\rho_1 = \star$: Obvious.

□

Lemma 28 (Type substitution preserves consistency)**.** *If $A \simeq B$, then $A[C/X] \simeq B[C/X]$.*

Proof. By induction on the derivation of $A \simeq B$. We mention only the interesting cases below.

Case (CE_PolyL): We have $\forall Y{:}K.\, A' \simeq B$ and, by inversion, **QPoly** (B) and $Y \notin \mathit{ftv}(B)$ and $A' \simeq B$. Without loss of generality, we can suppose that $Y \notin \mathit{ftv}(C)$. Thus, $Y \notin \mathit{ftv}(B[C/X])$. By the IH, $A'[C/X] \simeq B[C/X]$. By Lemma 26, **QPoly** $(B[C/X])$. Thus, by (CE_PolyL), $\forall Y{:}K.\, A'[C/X] \simeq B[C/X]$

Case (CE_PolyR): Similar to the case for (CE_PolyL).

Case (CE_ConsL): We have $\ell : A'; \rho_1 \simeq B$ and, by inversion, $B \triangleright_\ell B', \rho_2$ and $A' \simeq B'$ and $\rho_1 \simeq \rho_2$. By the IHs, $A'[C/X] \simeq B'[C/X]$ and $\rho_1[C/X] \simeq \rho_2[C/X]$. By Lemma 27, $B[C/X] \triangleright_\ell B'[C/X], \rho_2[C/X]$. Thus, by (CE_ConsL), $\ell : A'[C/X]; \rho_1[C/X] \simeq B[C/X]$.

Case (CE_ConsR): Similar to the case for (CE_ConsL).

□

Lemma 29 (Type substitution)**.** *Suppose that $\Sigma; \Gamma_1 \vdash A : K$.*

1. *If $\Sigma \vdash \Gamma_1, X{:}K, \Gamma_2$, then $\Sigma \vdash \Gamma_1, \Gamma_2\,[A/X]$.*
2. *If $\Sigma; \Gamma_1, X{:}K, \Gamma_2 \vdash B : K'$, then $\Sigma; \Gamma_1, \Gamma_2\,[A/X] \vdash B[A/X] : K'$.*
3. *If $\Sigma; \Gamma_1, X{:}K, \Gamma_2 \vdash e : B$, then $\Sigma; \Gamma_1, \Gamma_2\,[A/X] \vdash e[A/X] : B[A/X]$.*

Proof. Straightforward by mutual induction on the derivations. Only the interesting case is (WF_TyVar). Suppose we have $\Sigma; \Gamma_1, X{:}K, \Gamma_2 \vdash Y : K'$. By inversion, $\Sigma \vdash \Gamma_1, X{:}K, \Gamma_2$ and $Y{:}K' \in \Gamma_1, X{:}K, \Gamma_2$. By the IH, $\Sigma \vdash \Gamma_1, \Gamma_2\,[A/X]$. If $X \neq Y$, then $Y{:}K' \in \Gamma_1, \Gamma_2\,[A/X]$ and, therefore, by (WF_TyVar), $\Sigma; \Gamma_1, \Gamma_2\,[A/X] \vdash Y : K'$. Otherwise, if $X = Y$, then we have to show $\Sigma; \Gamma_1, \Gamma_2\,[A/X] \vdash A : K$. Since $\Sigma; \Gamma_1 \vdash A : K$ and $\Sigma \vdash \Gamma_1, \Gamma_2\,[A/X]$, we have $\Sigma; \Gamma_1, \Gamma_2\,[A/X] \vdash A : K$ by Lemma 24 (2).

Note that the case for (T_Cast) uses Lemma 28 and that the case for (T_Conv) depends on the fact that e and B are closed. □

Lemma 30 (Type substitution on convertibility)**.** *Suppose that α does not occur in A.*

1. *$\Sigma, \alpha{:}K := B \vdash A[\alpha/X] \prec^{+\alpha} A[B/X]$.*
2. *$\Sigma, \alpha{:}K := B \vdash A[B/X] \prec^{-\alpha} A[\alpha/X]$.*

Proof. Let $\Sigma' = \Sigma, \alpha{:}K := B$. By induction on A.

Case $A = X$: We have $A[\alpha/X] = \alpha$ and $A[B/X] = B$.

First, we have to show $\Sigma' \vdash \alpha \prec^{+\alpha} B$, which is shwon by (Cv_Reveal).

Next, we have to show $\Sigma' \vdash B \prec^{-\alpha} \alpha$, which is shown by (Cv_Conceal).

Case $A = Y$ where $X \neq Y$: By (Cv_TyVar).

Case $A = \alpha$: Contradictory with the assumption that α does not occur in A.

Case $A = \alpha'$ where $\alpha \neq \alpha'$: By (Cv_TyName).

Case $A = \star$: By (Cv_Dyn).

Case $A = \iota$: By (Cv_Base).

Case $A = A_1 \rightarrow A_2$: By the IHs, we have

- $\Sigma' \vdash A_1[\alpha/X] \prec^{+\alpha} A_1[B/X]$,
- $\Sigma' \vdash A_2[\alpha/X] \prec^{+\alpha} A_2[B/X]$,
- $\Sigma' \vdash A_1[B/X] \prec^{-\alpha} A_1[\alpha/X]$, and
- $\Sigma' \vdash A_2[B/X] \prec^{-\alpha} A_2[\alpha/X]$.

By (Cv_Fun), $\Sigma' \vdash A_1[\alpha/X] \rightarrow A_2[\alpha/X] \prec^{+\alpha} A_1[B/X] \rightarrow A_2[B/X]$ and $\Sigma' \vdash A_1[B/X] \rightarrow A_2[B/X] \prec^{-\alpha} A_1[\alpha/X] \rightarrow A_2[\alpha/X]$.

Case $A = \forall X'{:}K.\, A'$: By the IH and (Cv_Poly).

Case $A = [\rho]$: By the IH and (Cv_Record).

Case $A = \langle\rho\rangle$: By the IH and (Cv_Variant).

Case $A = \cdot$: By (Cv_REmp).

Case $A = \ell : A'; \rho$: By the IHs and (Cv_Cons).

□

Lemma 31.

1. *If $\Sigma \vdash \Gamma_1, x{:}A, \Gamma_2$, then $\Sigma \vdash \Gamma_1, \Gamma_2$.*

2. *If $\Sigma; \Gamma_1, x{:}A, \Gamma_2 \vdash B : K$, then $\Sigma; \Gamma_1, \Gamma_2 \vdash B : K$.*

Proof. Straightforward by mutual induction on the derivations. □

Lemma 32 (Value substitution)**.** *If $\Sigma; \Gamma_1 \vdash v : A$ and $\Sigma; \Gamma_1, x{:}A, \Gamma_2 \vdash e : B$, then $\Sigma; \Gamma_1, \Gamma_2 \vdash e[v/x] : B$.*

Proof. By mutual induction on the derivations. The only interesting case is (T_Var).

Suppose that $\Sigma; \Gamma_1, x{:}A, \Gamma_2 \vdash y : B$. By inversion, $\Sigma \vdash \Gamma_1, x{:}A, \Gamma_2$ and $y{:}B \in \Gamma_1, x{:}A, \Gamma_2$. By Lemma 31, $\Sigma \vdash \Gamma_1, \Gamma_2$. If $x \neq y$, then $y{:}B \in \Gamma_1, \Gamma_2$. Thus, by (T_Var), $\Sigma; \Gamma_1, \Gamma_2 \vdash y : B$. Since $y[v/x] = y$, we finish. Otherwise, if $x = y$, then we have to show that $\Sigma; \Gamma_1, \Gamma_2 \vdash v : A$ (note that $y[v/x] = v$ and that $A = B$ since $y{:}B \in \Gamma_1, x{:}A, \Gamma_2$). Since $\Sigma; \Gamma_1 \vdash v : A$ and $\Sigma \vdash \Gamma_1, \Gamma_2$, we have $\Sigma; \Gamma_1, \Gamma_2 \vdash v : A$ by Lemma 24 (3).

The cases for (T_Const), (T_TApp), (T_REmp), (T_VInj), (T_VLift), (T_Cast), and (T_Conv) also use Lemma 31. □

Lemma 33 (Canonical forms)**.** *Suppose that $\Sigma; \emptyset \vdash v : A$.*

1. *If $A = \iota$, then $v = \kappa$ for some κ.*

2. *If $A = B \rightarrow C$, then $v = \lambda x{:}B.e$ for some x and e, or $v = \kappa$ for some κ such that $ty(\kappa) = B \rightarrow C$.*

3. *If $A = \forall X{:}K.\, B$, then $v = \Lambda X{:}K.e :: B$ for some e.*

4. If $A = [\cdot]$, then $v = \{\}$.

5. If $A = [\ell : B; \rho]$, then $v = \{\ell = v_1; v_2\}$ for some v_1 and v_2.

6. If $A = \langle \ell : B; \rho \rangle$, then $v = \ell\, v'$ or $v = \uparrow\langle \ell : B \rangle\, v'$ for some v'.

7. If $A = \star$, then $v = v' : G \stackrel{p}{\Rightarrow} \star$ for some v', G, and p.

8. If $A = [\star]$, then $v = v' : [\gamma] \stackrel{p}{\Rightarrow} [\star]$ for some v', γ, and p.

9. If $A = \langle \star \rangle$, then $v = v' : \langle \gamma \rangle \stackrel{p}{\Rightarrow} \langle \star \rangle$ for some v', γ, and p.

10. If $A = \alpha$, then $v = v' : B \stackrel{-\alpha}{\Rightarrow} \alpha$ for some v' and B.

11. If $A = [\alpha]$, then $v = v' : [\rho] \stackrel{-\alpha}{\Rightarrow} [\alpha]$ for some v' and ρ.

12. If $A = \langle \alpha \rangle$, then $v = v' : \langle \rho \rangle \stackrel{-\alpha}{\Rightarrow} \langle \alpha \rangle$ for some v' and ρ.

Proof. By case analysis on the typing rule applied to derive $\Sigma; \emptyset \vdash v : A$.

Case (T_Var), (T_App), (T_TApp), (T_RLet), (T_VCase), and (T_Blame): Contradictory.

Case (T_Const), (T_Lam), (T_TLam), (T_REmp), (T_RExt), (T_VInj), (T_VLift): Obvious.

Case (T_Cast): We have $\Sigma; \emptyset \vdash e : B \stackrel{p}{\Rightarrow} A : A$ for some e, B, and p. By inversion, $\Sigma; \emptyset \vdash A : \mathsf{T}$. We do case analysis on the rule applied last to derive $\Sigma; \emptyset \vdash A : \mathsf{T}$.

Case (WF_TyVar), (WF_REmp), and (WF_Cons): Contradictory.

Case (WF_TyName), (WF_Base), (WF_Fun), and (WF_Poly): Contradictory because there are no values of the form $e : B \stackrel{p}{\Rightarrow} A$ in these cases.

Case (WF_Dyn), (WF_Record), and (WF_Variant): Obvious because of the definition of values.

Case (T_Conv): We have $\Sigma; \emptyset \vdash e : B \stackrel{\Phi}{\Rightarrow} A : A$ for some e, B, and Φ. By inversion, $\Sigma; \emptyset \vdash A : \mathsf{T}$. We do case analysis on the rule applied last to derive $\Sigma; \emptyset \vdash A : \mathsf{T}$.

Case (WF_TyVar), (WF_REmp), and (WF_Cons): Contradictory.

Case (WF_Dyn), (WF_Base), (WF_Fun), and (WF_Poly): Contradictory because there are no values of the form $e : B \stackrel{\Phi}{\Rightarrow} A$ in these cases.

Case (WF_TyName), (WF_Record), and (WF_Variant): Obvious because of the definition of values.

□

Lemma 34. *If $\Sigma; \emptyset \vdash v : \langle \cdot \rangle$, contradictory.*

Proof. Straightforward by case analysis on the rule applied last to derive $\Sigma; \emptyset \vdash v : \langle \cdot \rangle$. □

Lemma 35 (Value inversion: constants)**.** *If $\Sigma; \emptyset \vdash \kappa : A$, then $A = ty(\kappa)$.*

Proof. Straightforward by case analysis on the derivation of $\Sigma; \emptyset \vdash \kappa : A$. □

Lemma 36 (Value inversion: constants)**.** *If $\Sigma; \emptyset \vdash \lambda x{:}A.e : A' \to B$, then $A = A'$ and $\Sigma; x{:}A \vdash e : B$.*

Proof. Straightforward by case analysis on the derivation of $\Sigma; \emptyset \vdash \lambda x{:}A.e : A' \to B$. □

Lemma 37 (Value inversion: constants)**.** *If $\Sigma; \emptyset \vdash \Lambda X{:}K.e :: A : \forall X'{:}K'.\, A'$, then $X = X'$ and $K = K'$ and $A = A'$ and $\Sigma; X{:}K \vdash e : A$.*

Proof. Straightforward by case analysis on the derivation of $\Sigma; \emptyset \vdash \Lambda X{:}K.e :: A : \forall X'{:}K'.\, A'$. □

Lemma 38 (Value inversion: record extensions)**.** *If* $\Sigma; \emptyset \vdash \{\ell = v_1; v_2\} : [\rho]$*, there exist some* A *and* ρ' *such that* $\rho = [\ell : A; \rho']$ *and* $\Sigma; \emptyset \vdash v_1 : A$ *and* $\Sigma; \emptyset \vdash v_2 : [\rho']$*.*

Proof. Straightforward by case analysis on the derivation of $\Sigma; \emptyset \vdash \{\ell = v_1; v_2\} : [\rho]$. □

Lemma 39 (Value inversion: variant injections)**.** *If* $\Sigma; \emptyset \vdash \ell\, v : \langle \ell : A; \rho \rangle$*, then* $\Sigma; \emptyset \vdash v : A$*.*

Proof. Straightforward by case analysis on the derivation of $\Sigma; \emptyset \vdash \ell\, v : \langle \ell : A; \rho \rangle$. □

Lemma 40 (Value inversion: variant lifts)**.** *If* $\Sigma; \emptyset \vdash \uparrow\langle \ell : A \rangle\, v : \langle \ell : B; \rho \rangle$*, then* $\Sigma; \emptyset \vdash v : \langle \rho \rangle$ *and* $A = B$*.*

Proof. Straightforward by case analysis on the derivation of $\Sigma; \emptyset \vdash \uparrow\langle \ell : A \rangle\, v : \langle \ell : A; \rho \rangle$. □

Lemma 41 (Value inversion: casts)**.** *If* $\Sigma; \emptyset \vdash v : A \overset{p}{\Rightarrow} B : B$*, then* $\Sigma; \emptyset \vdash v : A$ *and* $A \simeq B$*.*

Proof. Straightforward by case analysis on the derivation of $\Sigma; \emptyset \vdash v : A \overset{p}{\Rightarrow} B : B$. □

Lemma 42 (Value inversion: conversions)**.** *If* $\Sigma; \emptyset \vdash v : A \overset{-\alpha}{\Rightarrow} \alpha : \alpha$*, then* $\Sigma; \emptyset \vdash v : A$ *and* $\Sigma(\alpha) = A$*.*

Proof. Straightforward by case analysis on the derivation of $\Sigma; \emptyset \vdash v : A \overset{-\alpha}{\Rightarrow} \alpha : \alpha$. □

Lemma 43 (Value inversion: conversions with records)**.** *If* $\Sigma; \emptyset \vdash v : [\rho] \overset{-\alpha}{\Rightarrow} [\alpha] : \alpha$*, then* $\Sigma; \emptyset \vdash v : [\rho]$ *and* $\Sigma(\alpha) = \rho$*.*

Proof. Straightforward by case analysis on the derivation of $\Sigma; \emptyset \vdash v : [\rho] \overset{-\alpha}{\Rightarrow} [\alpha] : \alpha$. □

Lemma 44 (Value inversion: conversions with variants)**.** *If* $\Sigma; \emptyset \vdash v : \langle \rho \rangle \overset{-\alpha}{\Rightarrow} \langle \alpha \rangle : \alpha$*, then* $\Sigma; \emptyset \vdash v : \langle \rho \rangle$ *and* $\Sigma(\alpha) = \rho$*.*

Proof. Straightforward by case analysis on the derivation of $\Sigma; \emptyset \vdash v : \langle \rho \rangle \overset{-\alpha}{\Rightarrow} \langle \alpha \rangle : \alpha$. □

Lemma 45. *If* $\Sigma \mid e \longrightarrow \Sigma' \mid e'$ *or* $e = \mathsf{blame}\, p$*, then* $\Sigma \mid E[e] \longrightarrow \Sigma' \mid e''$ *for some* e''*.*

Proof. If $e = \mathsf{blame}\, p$, then we finish by (E_Blame). If $\Sigma \mid e \longrightarrow \Sigma' \mid e'$, we can prove the statement straightforwardly by case analysis on the evaluation rule applied to derive $\Sigma \mid e \longrightarrow \Sigma' \mid e'$. □

Lemma 46 (Unique ground type)**.** *If* $\Sigma; \emptyset \vdash A : \mathsf{T}$ *and* $A \neq \star$ *and* A *is not an universal type, then there exists an unique ground type* G *such that* $A \simeq G$*.*

Proof. By case analysis on A.

Case $A = X$: Contradictory with $\Sigma; \emptyset \vdash A : \mathsf{T}$

Case $A = \alpha$: Only ground type α is consistent with α.

Case $A = \star$: Contradictory with $A \neq \star$.

Case $A = \iota$: Only ground type ι is consistent with ι.

Case $A = B \to C$: Only ground type $\star \to \star$ is consistent with $B \to C$.

Case $A = \forall X{:}K.\, B$: Contradictory.

Case $A = [\rho]$: Only ground type $[\star]$ is consistent with $[\rho]$.

Case $A = \langle \rho \rangle$: Only ground type $\langle \star \rangle$ is consistent with $\langle \rho \rangle$.

Case $A = \cdot$: Contradictory with $\Sigma; \emptyset \vdash A : \mathsf{T}$.

Case $A = \ell : B; \rho$: Contradictory with $\Sigma; \emptyset \vdash A : \mathsf{T}$.

□

Lemma 47. *If $\Sigma; \emptyset \vdash \rho : \mathsf{R}$ and $\rho \neq \star$, then $grow(\rho)$ is defined and $grow(\rho)$ is a ground row type.*

Proof. Straightforward by case analysis on the derivation of $\Sigma; \emptyset \vdash \rho : \mathsf{R}$. □

Lemma 48. *If $grow(\rho)$ is defined, then $\rho \simeq grow(\rho)$.*

Proof. Obvious by definition of *grow*. □

Lemma 49. *If $grow(\rho)$ is defined and $\Sigma; \Gamma \vdash \rho : K$, then $\Sigma; \Gamma \vdash grow(\rho) : K$.*

Proof. Obvious by definition of *grow*. □

Lemma 50.

1. *If $\Sigma; \Gamma \vdash A : K$, then $\Sigma \vdash \Gamma$.*
2. *If $\Sigma; \Gamma \vdash e : A$, then $\Sigma \vdash \Gamma$ and $\Sigma; \Gamma \vdash A : \mathsf{T}$.*

Proof. Straightforward by induction on the typing derivations with Lemmas 24 and 29. □

Lemma 51 (Progress)**.** *If $\Sigma; \emptyset \vdash e : A$, then one of the followings holds:*

- *e is a value;*
- *$e = \mathsf{blame}\, p$ for some ℓ; or*
- *$\Sigma \mid e \longrightarrow \Sigma' \mid e'$ for some Σ' and e'.*

Proof. By induction on the derivation of $\Sigma; \emptyset \vdash e : A$.

Case (T_Var): Contradictory.

Case (T_Const), (T_Lam), (T_TLam), (T_REmp), and (T_Blame): Obvious.

Case (T_App): We have $\Sigma; \emptyset \vdash e_1\, e_2 : A$ and, by inversion, $\Sigma; \emptyset \vdash e_1 : B \rightarrow A$ and $\Sigma; \emptyset \vdash e_2 : B$. If $e_1 = \mathsf{blame}\, p$ for some p, or $\Sigma \mid e_1 \longrightarrow \Sigma' \mid e_1'$ for some Σ' and e_1', then we finish by Lemma 45.

In what follows, we can suppose that $e_1 = v_1$ for some v_1 by the IH. If $e_2 = \mathsf{blame}\, p$ for some p, or $\Sigma \mid e_2 \longrightarrow \Sigma' \mid e_2'$ for some Σ' and e_2', then we finish by Lemma 45.

In what follows, we can suppose that $e_2 = v_2$ for some v_2 by the IH. We have $\Sigma; \emptyset \vdash v_1 : B \rightarrow A$. Thus, by Lemma 33, there are two cases on v_1 to be considered.

Case $v_1 = \lambda x{:}B.e_1'$ for some x and e_1': By (R_Beta)/(E_Red).

Case $v_1 = \kappa_1$ and $ty(\kappa_1) = B \rightarrow A$ for some κ_1: By the assumption on constants, $B = \iota$ for some ι. Since $\Sigma; \emptyset \vdash v_2 : \iota$, we have $v_2 = \kappa_2$ for some κ_2. By the assumption on constants, $\zeta(\kappa_1, \kappa_2)$ is defined. Thus, we finish by (E_Const)/(R_Red).

Case (T_TApp): We have $\Sigma; \emptyset \vdash e_1\, B : C[B/X]$ and, by inversion, $\Sigma; \emptyset \vdash e_1 : \forall X{:}K.\, C$ and $\Sigma; \emptyset \vdash B : K$. If $e_1 = \mathsf{blame}\, p$ for some $[p]$, or $\Sigma \mid e_1 \longrightarrow \Sigma' \mid e_1'$ for some Σ' and e_1', then we finish by Lemma 45.

In what follows, we can suppose that $e_1 = v_1$ for some v_1 by the IH. We have $\Sigma; \emptyset \vdash v_1 : \forall X{:}K.\, C$. Thus, by Lemma 33, $v_1 = \Lambda X{:}K.e_1' :: C$ for some e_1'. By (E_TyBeta), we finish.

Case (T_RExt): We have $\Sigma; \emptyset \vdash \{\ell = e_1; e_2\} : [\ell : B; \rho]$ and, by inversion, $\Sigma; \emptyset \vdash e_1 : B$ and $\Sigma; \emptyset \vdash e_2 : [\rho]$. If $e_1 = \mathsf{blame}\, p$ for some p, or $\Sigma \mid e_1 \longrightarrow \Sigma' \mid e_1'$ for some Σ' and e_1', then we finish by Lemma 45.

In what follows, we can suppose that $e_1 = v_1$ for some v_1 by the IH. If $e_2 = \mathsf{blame}\, p$ for some p, or $\Sigma \mid e_2 \longrightarrow \Sigma' \mid e_2'$ for some Σ' and e_2', then we finish by Lemma 45.

In what follows, we can suppose that $e_2 = v_2$ for some v_2 by the IH. Then, we finish because $e = \{\ell = v_1; v_2\}$ is a value.

Case (T_RLet): We have $\Sigma; \emptyset \vdash \mathsf{let}\,\{\ell = x; y\} = e_1 \,\mathsf{in}\, e_2 : A$ and, by inversion, $\Sigma; \emptyset \vdash e_1 : [\ell{:}B; \rho]$ and $\Sigma; x{:}B, y{:}[\rho] \vdash e_2 : A$. If $e_1 = \mathsf{blame}\, p$ for some p, or $\Sigma \mid e_1 \longrightarrow \Sigma' \mid e_1'$ for some Σ' and e_1', then we finish by Lemma 45.

In what follows, we can suppose that $e_1 = v_1$ for some v_1 by the IH. Since $\Sigma; \emptyset \vdash v_1 : [\ell : B; \rho]$, we have $v_1 = \{\ell = v_1'; v_2'\}$ for some v_1' and v_2' by Lemma 33. Thus, we finish by (R_Record)/(E_Red).

Case (T_VInj): We have $\Sigma; \emptyset \vdash \ell\, e' : \langle \ell : B; \rho \rangle$ and, by inversion, $\Sigma; \emptyset \vdash e' : B$ and $\Sigma; \emptyset \vdash \rho : \mathsf{R}$. If $e' = \mathsf{blame}\, p$ for some p, or $\Sigma \mid e' \longrightarrow \Sigma' \mid e''$ for some Σ' and e'', then we finish by Lemma 45.

In what follows, we can suppose that $e' = v$ for some v by the IH. Then, we finish because $e = \ell\, v$ is a value.

Case (T_VLift): We have $\Sigma; \emptyset \vdash \uparrow \langle \ell : B \rangle\, e' : \langle \ell : B; \rho \rangle$ and, by inversion, $\Sigma; \emptyset \vdash e' : \langle \rho \rangle$ and $\Sigma; \emptyset \vdash B : \mathsf{T}$. If $e' = \mathsf{blame}\, p$ for some p, or $\Sigma \mid e' \longrightarrow \Sigma' \mid e''$ for some Σ' and e'', then we finish by Lemma 45.

In what follows, we can suppose that $e' = v$ for some v by the IH. Then, we finish because $e = \uparrow \langle \ell : B \rangle\, v$ is a value.

Case (T_VCase): We have $\Sigma; \emptyset \vdash \mathsf{case}\, e' \,\mathsf{with}\, \langle \ell\, x \rightarrow e_1; y \rightarrow e_2 \rangle : A$ and, by inversion, $\Sigma; \emptyset \vdash e' : \langle \ell : B; \rho \rangle$. If $e' = \mathsf{blame}\, p$ for some p, or $\Sigma \mid e' \longrightarrow \Sigma' \mid e''$ for some Σ' and e'', then we finish by Lemma 45.

In what follows, we can suppose that $e' = v$ for some v by the IH. We have $\Sigma; \emptyset \vdash v : \langle \ell : B; \rho \rangle$. Thus, by Lemma 33, there are two cases on v to be considered.

Case $v = \ell\, v'$ for some v': By (R_CaseL)(E_Red).

Case $v = \uparrow \langle \ell : B \rangle\, v'$ for some v': By (R_CaseR)(E_Red).

Case (T_Cast): We have $\Sigma; \emptyset \vdash e' : B \overset{p}{\Rightarrow} A : A$ and, by inversion, $\Sigma; \emptyset \vdash e' : B$ and $B \simeq A$ and $\Sigma; \emptyset \vdash A : \mathsf{T}$. If $e' = \mathsf{blame}\, q$ for some q, or $\Sigma \mid e' \longrightarrow \Sigma' \mid e''$ for some Σ' and e'', then we finish by Lemma 45.

In what follows, we can suppose that $e' = v$ for some v by the IH. By case analysis on $B \simeq A$.

Case (CE_Refl): We have $B = A$. By case analysis on A.

Case $A = X$: Contradictory with $\Sigma; \emptyset \vdash A : \mathsf{T}$.
Case $A = \alpha$: By (R_IdName)/(E_Red).
Case $A = \star$: By (R_IdDyn)/(E_Red).
Case $A = \iota$: By (R_IdBase)/(E_Red).
Case $A = A_1 \rightarrow A_2$: By (R_Wrap)/(E_Red).
Case $A = \forall X{:}K.\, A'$: By (R_Content)/(E_Red).
Case $A = [\rho]$: By case analysis on ρ. Note that $\Sigma; \emptyset \vdash \rho : \mathsf{R}$ since $\Sigma; \emptyset \vdash [\rho] : \mathsf{T}$.

Case $\rho = X$, ι, $A' \rightarrow B'$, $\forall X{:}K.\, A'$, $[\rho']$, and $\langle \rho' \rangle$: Contradictory with $\Sigma; \emptyset \vdash \rho : \mathsf{R}$.

Case $\rho = \alpha$: By (E_RIdName)/(E_Red).

Case $\rho = \star$: By Lemma 33, $v = v' : [\gamma'] \overset{q}{\Rightarrow} [\star]$ for some v', γ', and q. We have $[\gamma'] \simeq [\star]$ by (CE_DynR)/(CE_Record). Thus, we finish by (R_RFromDyn)/(E_Red).

Case $\rho = \cdot$: By (R_REmp)/(E_Red).

Case $\rho = \ell : C; \rho'$: By Lemma 33, $v = \{\ell = v_1; v_2\}$ for some v_1 and v_2. Thus, $v \rhd_\ell v_1, v_2$. Since $\ell : C; \rho' \rhd_\ell C, \rho'$, we finish by (R_Rev)/(E_Red).

Case $A = \langle \rho \rangle$: By case analysis on ρ. Note that $\Sigma; \emptyset \vdash \rho : \mathsf{R}$ since $\Sigma; \emptyset \vdash \langle \rho \rangle : \mathsf{T}$.

Case $\rho = X$, ι, $A' \rightarrow B'$, $\forall X{:}K.\, A'$, $[\rho']$, and $\langle \rho' \rangle$: Contradictory with $\Sigma; \emptyset \vdash \rho : \mathsf{R}$.

Case $\rho = \alpha$: By (E_VIdName)/(E_Red).

Case $\rho = \star$: By Lemma 33, $v = v' : \langle \gamma' \rangle \overset{q}{\Rightarrow} \langle \star \rangle$ for some v', γ', and q. We have $\langle \gamma' \rangle \simeq \langle \star \rangle$ by (CE_DynR)/(CE_Variant). Thus, we finish by (R_VFromDyn)/(E_Red).

Case $\rho = \cdot$: By Lemma 34.

Case $\rho = \ell : C; \rho'$: By Lemma 33, there are two cases to be considered.
If $v = \ell\, v'$ for some v', then we finish by (R_VRevInj)/(E_Red).
Otherwise, if $v = \uparrow \langle \ell : C \rangle\, v'$ for some v', then we finish by (R_VRevLift)(E_Red).

Case $A = \cdot$ and $\ell : B; \rho$: Contradictory with $\Sigma; \emptyset \vdash A : \mathsf{T}$.

Case (CE_DynL): We have $B = \star$. By Lemma 33, $v = v' : G \overset{q}{\Rightarrow} \star$ for some v', G, and q. By case analysis on A.

Case $A = H$: By (R_Ground)/(E_Red) or (R_Blame)/(E_Red).
Case $A = X$: Contradictory with $\Sigma; \emptyset \vdash A : \mathsf{T}$.
Case $A = \star$: By (R_IdDyn)/(E_Red).
Case $A = A_1 \rightarrow A_2$ ($A_1 \rightarrow A_2 \neq \star \rightarrow \star$): Since $A_1 \rightarrow A_2 \simeq \star \rightarrow \star$, we finish by (R_FromDyn)/(E_Red).
Case $A = \forall X{:}K.\, A'$: By (R_Gen)/(E_Red).
Case $A = [\rho]$ ($\rho \neq \star$): Since $[\rho] \simeq [\star]$, we finish by (R_FromDyn)/(E_Red).
Case $A = \langle\rho\rangle$ ($\rho \neq \star$): Since $\langle\rho\rangle \simeq \langle\star\rangle$, we finish by (R_FromDyn)/(E_Red).
Case $A = \cdot$: Contradictory with $\Sigma; \emptyset \vdash A : \mathsf{T}$.
Case $A = \ell : C; \rho$: Contradictory with $\Sigma; \emptyset \vdash A : \mathsf{T}$.

Case (CE_DynR): We have $A = \star$.
If $B = \star$, then we finish by (R_IdDyn)/(E_Red).
If $B = \forall X{:}K.\, B'$, then we finish by (R_Inst)/(E_Red).
Otherwise, by Lemma 46, there exists some G such that $B \simeq G$. If $B = G$, then $e = v : G \overset{p}{\Rightarrow} \star$ is a value. Otherwise, we finish by (R_ToDyn)/(E_Red).

Case (CE_Fun): By (R_Wrap)/(E_Red).

Case (CE_Poly): By (R_Content)/(E_Red).

Case (CE_PolyL): By (R_Inst)/(E_Red).

Case (CE_PolyR): By (R_Gen)/(E_Red).

Case (CE_Record): We have $A = [\rho_1]$ and $B = [\rho_2]$ and $\rho_2 \simeq \rho_1$ for some ρ_1 and ρ_2. Since $\Sigma; \emptyset \vdash [\rho_1] : \mathsf{T}$, we have $\Sigma; \emptyset \vdash \rho_1 : \mathsf{R}$. By Lemma 50, $\Sigma; \emptyset \vdash [\rho_2] : \mathsf{T}$, so $\Sigma; \emptyset \vdash \rho_2 : \mathsf{R}$.
If $\rho_2 = \star$, then we finish by Lemma 33, and (R_RFromDyn)/(E_Red) or (R_RBlame)/(E_Red).
In what follows, we suppose $\rho_2 \neq \star$. By case analysis on ρ_1.

Case $\rho_1 = \star$: Since $\rho_2 \neq \star$ and $\Sigma; \emptyset \vdash \rho_2 : \mathsf{R}$, $\mathit{grow}(\rho_2)$ is defined and is a ground row type by Lemma 47.
If $\mathit{grow}(\rho_2) = \rho_2$, then $v : [\rho_2] \overset{p}{\Rightarrow} [\star]$ is a value.
Otherwise, if $\mathit{grow}(\rho_2) \neq \rho_2$, we finish by (R_RToDyn)/(E_Red).
Case $\rho_1 = \alpha$: Since $\rho_2 \simeq \alpha$ and $\rho_2 \neq \star$, we have $\rho_2 = \alpha$ by Lemmas 6 and 7. We finish by (R_RIdName)/(E_Red).
Case $\rho_1 = \cdot$: Since $\rho_2 \simeq \cdot$ and $\rho_2 \neq \star$, we have $\rho_2 = \cdot$ by Lemmas 6 and 8. We finish by (R_REmp)/(E_Red).
Case $\rho_1 = \ell : C_1; \rho_1'$: By Lemmas 6 and 9, $\rho_2 \triangleright_\ell C_2, \rho_2'$ and $C_2 \simeq C_1$ and $\rho_2' \simeq \rho_1'$ for some C_2 and ρ_2'.
If $\ell \in \mathit{dom}(\rho_2)$, then there exist some ρ_{21} and ρ_{22} such that

- $\rho_2 = \rho_{21} \odot (\ell : C_2; \cdot) \odot \rho_{22}$,
- $\rho_2' = \rho_{21} \odot \rho_{22}$, and
- $\ell \notin \mathit{dom}(\rho_{21})$.

Since $\Sigma; \emptyset \vdash v : [\rho_2]$, there exist some v_1 and v_2 such that $v \triangleright_\ell v_1, v_2$ by Lemmas 33 and 38. Thus, we finish by (R_RRev)/(E_Red).
If $\ell \notin \mathit{dom}(\rho_2)$, then we finish by (R_RCon)/(E_Red).
Case $\rho_1 = X$, ι, $C \rightarrow D$, $\forall X{:}K.\, C$, $[\rho']$, and $\langle\rho'\rangle$: Contradictory with $\Sigma; \emptyset \vdash \rho_1 : \mathsf{R}$.

Case (CE_Variant): We have $A = \langle\rho_1\rangle$ and $B = \langle\rho_2\rangle$ and $\rho_2 \simeq \rho_1$ for some ρ_1 and ρ_2. Since $\Sigma; \emptyset \vdash \langle\rho_1\rangle : \mathsf{T}$, we have $\Sigma; \emptyset \vdash \rho_1 : \mathsf{R}$. By Lemma 50, $\Sigma; \emptyset \vdash \langle\rho_2\rangle : \mathsf{T}$, so $\Sigma; \emptyset \vdash \rho_2 : \mathsf{R}$.
By case analysis on ρ_2.

Case $\rho_2 = \star$: We finish by Lemma 33, and (R_VFromDyn)/(E_Red) or (R_VBlame)(E_Red).
Case $\rho_2 = \alpha$: Since $\rho_2 \simeq \rho_1$, we have $\rho_1 = \alpha$ or $\rho_1 = \star$ by Lemma 7.
If $\rho_1 = \star$, then $v : [\alpha] \overset{p}{\Rightarrow} [\star]$ is a value.
Otherwise, if $\rho_1 = \alpha$, then we finish by (R_VIdName)/(E_Red).
Case $\rho_2 = \cdot$: Contradictory by Lemma 34.

Case $\rho_2 = \ell : C_2; \rho'_2$: If $\ell \in \mathit{dom}(\rho_1)$, then we finish by Lemma 33, and (R_VRevInj)/(E_Red) or (R_VRevLift)/(E_Red).
Otherwise, suppose $\ell \notin \mathit{dom}(\rho_1)$. Since $\Sigma; \emptyset \vdash \rho_2 : \mathsf{R}$ and $\rho_2 \neq \star$, it is found that $\mathit{grow}(\rho_2)$ is defined. If $\rho_1 = \star$ and $\mathit{grow}(\rho_2) = \rho_2$, then $v : [\rho_2] \overset{p}{\Rightarrow} [\star]$ is a value. If $\rho_1 = \star$ and $\mathit{grow}(\rho_2) \neq \rho_2$, then we finish by (R_VToDyn)/(E_Red).
Otherwise, suppose $\rho_1 \neq \star$. Then, we finish by Lemma 33, and (R_VConInj)/(E_Red) or (R_VConLift)/(E_Red).

Case $\rho_2 = X, \iota, C \to D, \forall X{:}K.\, C, [\rho'],$ and $\langle \rho' \rangle$: Contradictory with $\Sigma; \emptyset \vdash \rho_2 : \mathsf{R}$.

Case (CE_ConsL): We have $B = \ell : C_2; \rho_2$ for some ℓ, C_2, and ρ_2. Since $\Sigma; \emptyset \vdash e' : B$, we have $\Sigma; \emptyset \vdash B : \mathsf{T}$ by Lemma 50. However, there is a contradiction that $\Sigma; \emptyset \vdash \ell : C_2; \rho_2 : \mathsf{T}$ does not hold.

Case (CE_ConsR): We have $A = \ell{:}C_1; \rho_1$ for some ℓ, C_1, and ρ_1. However, it is contradictory with $\Sigma; \emptyset \vdash A : \mathsf{T}$.

Case (T_Conv): We have $\Sigma; \emptyset \vdash e' : B \overset{\Phi}{\Rightarrow} A : A$ and, by inversion, $\Sigma; \emptyset \vdash e' : B$ and $\Sigma \vdash B \prec^{\Phi} A$ and $\Sigma; \emptyset \vdash A : \mathsf{T}$. If $e' = \mathsf{blame}\, q$ for some q, or $\Sigma \mid e' \longrightarrow \Sigma' \mid e''$ for some Σ' and e'', then we finish by Lemma 45.

In what follows, we can suppose that $e' = v$ for some v by the IH. By case analysis on $\Sigma \vdash B \prec^{\Phi} A$.

Case (Cv_Dyn): By (R_CIdDyn)/(E_Red).

Case (Cv_TyVar): Contradictory with $\Sigma; \emptyset \vdash A : \mathsf{T}$.

Case (Cv_TyName): By (R_CIdName)/(E_Red).

Case (Cv_Reveal): We have $B = \alpha$ and $\Phi = +\alpha$ and $\Sigma(\alpha) = A$ for some α. By Lemma 33, $v = v' : C \overset{-\alpha}{\Rightarrow} \alpha$ for some C. Since $\Sigma; \emptyset \vdash v : B$, we have $\Sigma(\alpha) = C$ by Lemma 42, so $A = C$. We finish by (R_CName)/(E_Red).

Case (Cv_Conceal): $v : B \overset{\Phi}{\Rightarrow} A$ is a value.

Case (Cv_Base): By (R_CIdBase)/(E_Red).

Case (Cv_Fun): By (R_CFun)/(E_Red).

Case (Cv_Poly): By (R_CForall)/(E_Red).

Case (Cv_Record): We have $B = [\rho_2]$ and $A = [\rho_1]$ and $\Sigma \vdash \rho_2 \prec^{\Phi} \rho_1$ for some ρ_1 and ρ_2. Since $\Sigma; \emptyset \vdash A : \mathsf{T}$, we have $\Sigma; \emptyset \vdash \rho_1 : \mathsf{R}$. By case analysis on $\Sigma \vdash \rho_2 \prec^{\Phi} \rho_1$.

Case (Cv_Dyn): By (R_CRIdDyn)/(E_Red).

Case (Cv_TyName): By (R_CRIdName)/(E_Red).

Case (Cv_Reveal): We have $\rho_2 = \alpha$ and $\Phi = +\alpha$ and $\Sigma(\alpha) = \rho_1$ for some α. By Lemma 33, $v = v' : [\rho'] \overset{-\alpha}{\Rightarrow} [\alpha]$ for some v' and ρ'. By Lemma 43, $\Sigma(\alpha) = \rho'$, so $\rho' = \rho$. We finish by (R_CRName)/(E_Red).

Case (Cv_Conceal): $v : B \overset{\Phi}{\Rightarrow} A$ is a value.

Case (Cv_REmp): By (R_CREmp)/(E_Red).

Case (Cv_Cons): By (R_CRExt)/(E_Red).

Case (Cv_TyVar), (Cv_Base), (Cv_Fun), (Cv_Poly), (Cv_Record), and (Cv_Variant): Contradictory with $\Sigma; \emptyset \vdash \rho_1 : \mathsf{R}$.

Case (Cv_Variant): We have $B = \langle \rho_2 \rangle$ and $A = \langle \rho_1 \rangle$ and $\Sigma \vdash \rho_2 \prec^{\Phi} \rho_1$ for some ρ_1 and ρ_2. Since $\Sigma; \emptyset \vdash A : \mathsf{T}$, we have $\Sigma; \emptyset \vdash \rho_1 : \mathsf{R}$. By case analysis on $\Sigma \vdash \rho_2 \prec^{\Phi} \rho_1$.

Case (Cv_Dyn): By (R_CVIdDyn)/(E_Red).

Case (Cv_TyName): By (R_CVIdName)/(E_Red).

Case (Cv_Reveal): We have $\rho_2 = \alpha$ and $\Phi = +\alpha$ and $\Sigma(\alpha) = \rho_1$ for some α. By Lemma 33, $v = v' : \langle \rho' \rangle \overset{-\alpha}{\Rightarrow} \langle \alpha \rangle$ for some v' and ρ'. By Lemma 44, $\Sigma(\alpha) = \rho'$, so $\rho' = \rho_1$. We finish by (R_CVName)/(E_Red).

Case (Cv_Conceal): $v : B \overset{\Phi}{\Rightarrow} A$ is a value.

Case (Cv_REmp): We have $\Sigma; \emptyset \vdash v : [\cdot]$, which is contradictory by Lemma 34.

Case (Cv_Cons): By (R_CVar)/(E_Red).

Case (Cv_TyVar), (Cv_Base), (Cv_Fun), (Cv_Poly), (Cv_Record), and (Cv_Variant): Contradictory with $\Sigma; \emptyset \vdash \rho_1 : \mathsf{R}$.

Case (Cv_REmp) and (Cv_Cons): Contradictory with $\Sigma; \emptyset \vdash A : \mathsf{T}$.

$\square$

Lemma 52. *If $\Sigma; \Gamma \vdash A : \mathsf{T}$ and $A \simeq G$, then $\Sigma; \Gamma \vdash G : \mathsf{T}$.*

Proof. By case analysis on G.

Case $G = \iota$, $\star \rightarrow \star$, $[\star]$, and $\langle \star \rangle$: Obvious.

Case $G = \alpha$: Since $A \simeq \alpha$, we have $A = \alpha$ or $A = \star$ by Lemmas 6 and 7. In either case, $\Sigma; \Gamma \vdash A : \mathsf{T}$.

$\square$

Lemma 53. *If $\Sigma; \emptyset \vdash v : [\rho]$ and $v \triangleright_\ell v_1, v_2$, then there exist some ρ_1, ρ_2, and A such that $\rho = \rho_1 \odot (\ell : A; \cdot) \odot \rho_2$ and $\ell \notin dom(\rho_1)$ and $\Sigma; \emptyset \vdash v_1 : A$ and $\Sigma; \emptyset \vdash v_2 : [\rho_1 \odot \rho_2]$.*

Proof. By induction on the derivation of $v \triangleright_\ell v_1, v_2$.

Case $\{\ell = v_1; v_2\} \triangleright_\ell v_1, v_2$: We have $v = \{\ell = v_1; v_2\}$. Since $\Sigma; \emptyset \vdash v : [\rho]$, there exist A and ρ' such that $\rho = \ell : A; [\rho']$ and $\Sigma; \emptyset \vdash v_1 : A$ and $\Sigma; \emptyset \vdash v_2 : [\rho']$.

Case $\{\ell' = v_1'; v_2'\} \triangleright_\ell v_1, \{\ell' = v_1'; v_2''\}$ where $\ell \neq \ell'$ and $v_2' \triangleright_\ell v_1, v_2''$: We have $v = \{\ell' = v_1'; v_2'\}$ and $v_2 = \{\ell' = v_1'; v_2''\}$. Since $\Sigma; \emptyset \vdash v : [\rho]$, there exist some B and ρ' such that $\Sigma; \emptyset \vdash v_1' : B$ and $\Sigma; \emptyset \vdash v_2' : [\rho']$ and $\rho = \ell : B; \rho'$. Since $\Sigma; \emptyset \vdash v_2' : [\rho']$ and $v_2' \triangleright_\ell v_1, v_2''$, there exist some ρ_1', ρ_2', and A such that

- $\rho' = \rho_1' \odot (\ell : A; \cdot) \odot \rho_2'$,
- $\ell \notin dom(\rho_1')$,
- $\Sigma; \emptyset \vdash v_1 : A$, and
- $\Sigma; \emptyset \vdash v_2'' : [\rho_1' \odot \rho_2']$

by the IH. Since $\Sigma; \emptyset \vdash v_1' : B$ and $\Sigma; \emptyset \vdash v_2'' : [\rho_1' \odot \rho_2']$, we have $\Sigma; \emptyset \vdash \{\ell' = v_1'; v_2''\} : [\ell' : B; (\rho_1' \odot \rho_2')]$ by (T_RExt).

$\square$

Lemma 54. *If $\Sigma; \Gamma \vdash e : \langle \rho \rangle$ and $\Sigma; \Gamma \vdash \rho' : \mathsf{R}$ and $\rho' \odot \rho$ is defined, then $\Sigma; \Gamma \vdash \uparrow \rho'\ e : \langle \rho' \odot \rho \rangle$.*

Proof. By induction on ρ'.

Case $\rho' = \cdot$: Trivial since $\uparrow \cdot\ e = e$.

Case $\rho' = \ell : A; \rho''$: We have $\uparrow \rho'\ e = \uparrow \langle \ell : A \rangle\, (\uparrow \rho''\ e)$. Since $\Sigma; \Gamma \vdash \rho' : \mathsf{R}$, we have $\Sigma; \Gamma \vdash A : \mathsf{T}$ and $\Sigma; \Gamma \vdash \rho'' : \mathsf{R}$. By the IH, $\Sigma; \Gamma \vdash \uparrow \rho''\ e : \langle \rho'' \odot \rho \rangle$. By (T_VLift), $\Sigma; \Gamma \vdash \uparrow \langle \ell : A \rangle\, (\uparrow \rho''\ e) : \langle \ell : A; \rho'' \odot \rho \rangle$.

Case otherwise: Contradictory with $\rho' \odot \rho$ is defined.

$\square$

Lemma 55. *If $\Sigma; \Gamma \vdash e : \langle \rho_1 \odot \rho_2 \rangle$ and $\Sigma; \Gamma \vdash A : \mathsf{T}$, then $\Sigma; \Gamma \vdash \downarrow^{\rho_1}_{\langle \ell : A \rangle} e : \langle \rho_1 \odot (\ell : A; \cdot) \odot \rho_2 \rangle$.*

Proof. By induction on ρ_1.

Case $\rho_1 = \ell' : B; \rho_1'$: We have $\downarrow^{\rho_1}_{\langle \ell : A \rangle} e = \mathsf{case}\, e\, \mathsf{with}\, \langle \ell'\, x \rightarrow \ell'\, x; y \rightarrow \uparrow \langle \ell' : B \rangle\, (\downarrow^{\rho_1'}_{\langle \ell : A \rangle} y) \rangle$. It suffices to show that

$$\Sigma; \Gamma \vdash \mathsf{case}\, e\, \mathsf{with}\, \langle \ell'\, x \rightarrow \ell'\, x; y \rightarrow \uparrow \langle \ell' : B \rangle\, (\downarrow^{\rho_1'}_{\langle \ell : A \rangle} y) \rangle : \langle \ell' : B; \rho_1' \odot (\ell : A; \cdot) \odot \rho_2 \rangle$$

Since $\Sigma; \Gamma, y{:}\langle \rho_1' \odot \rho_2 \rangle \vdash A : \mathsf{T}$ by Lemmas 50 and 24, we have

$$\Sigma; \Gamma, y{:}\langle \rho_1' \odot \rho_2 \rangle \vdash \downarrow^{\rho_1'}_{\langle \ell : A \rangle} y : \langle \rho_1' \odot (\ell : A; \cdot) \odot \rho_2 \rangle$$

by the IH. Thus, by (T_VLift),

$$\Sigma;\Gamma, y{:}\langle \rho'_1 \odot \rho_2\rangle \vdash \uparrow\langle \ell' : B\rangle\, (\downarrow^{\rho'_1}_{\langle \ell:A\rangle} y) : \langle \ell' : B; \rho'_1 \odot (\ell : A; \cdot) \odot \rho_2\rangle$$

(note that $\Sigma;\Gamma, y{:}\langle \rho'_1 \odot \rho_2\rangle \vdash B : \mathsf{T}$ by Lemmas 50 and 24). Since $\Sigma;\Gamma, x{:}B \vdash \ell'\, x : \langle \ell' : B; \rho'_1 \odot (\ell : A; \cdot) \odot \rho_2\rangle$ by (T_VInj) (note that $\Sigma;\Gamma, x{:}B \vdash \rho'_1 \odot (\ell : A; \cdot) \odot \rho_2 : \mathsf{R}$ by Lemma 24), and $\Sigma;\Gamma \vdash e : \langle \ell' : B; \rho'_1 \odot \rho_2\rangle$, we have

$$\Sigma;\Gamma \vdash \mathsf{case}\; e \;\mathsf{with}\; \langle \ell'\, x \to \ell'\, x; y \to \uparrow\langle \ell' : B\rangle\, (\downarrow^{\rho'_1}_{\langle \ell:A\rangle} y)\rangle : \langle \ell' : B; \rho'_1 \odot (\ell : A; \cdot) \odot \rho_2\rangle$$

by (T_VCase).

Case $\rho_1 = \cdot$: We have $\downarrow^{\rho_1}_{\langle \ell:A\rangle} e = \uparrow\langle \ell : A\rangle\, e$. It suffices to show that $\Sigma;\Gamma \vdash \uparrow\langle \ell : A\rangle\, e : \langle \ell : A; \rho_2\rangle$, which is shown by (T_VLift).

Case otherwise: Contradictory with the fact that $\rho_1 \odot \rho_2$ is defined.

□

Lemma 56 (Convertibility inversion: function types). *If $\Sigma \vdash A_1 \to B_1 \prec^{\Phi} A_2 \to B_2$, then $\Sigma \vdash A_2 \prec^{\overline{\Phi}} A_1$ and $\Sigma \vdash B_1 \prec^{\Phi} B_2$.*

Proof. Straightforward by case analysis on $\Sigma \vdash A_1 \to B_1 \prec^{\Phi} A_2 \to B_2$. □

Lemma 57 (Convertibility inversion: universal types). *If $\Sigma \vdash \forall X{:}K.\, A \prec^{\Phi} \forall X{:}K.\, B$, then $\Sigma \vdash A \prec^{\Phi} B$.*

Proof. Straightforward by case analysis on $\Sigma \vdash \forall X{:}K.\, A \prec^{\Phi} \forall X{:}K.\, B$. □

Lemma 58 (Convertibility inversion: record types). *If $\Sigma \vdash [\rho_1] \prec^{\Phi} [\rho_2]$, then $\Sigma \vdash \rho_1 \prec^{\Phi} \rho_2$.*

Proof. Straightforward by case analysis on $\Sigma \vdash [\rho_1] \prec^{\Phi} [\rho_2]$. □

Lemma 59 (Convertibility inversion: variant types). *If $\Sigma \vdash \langle\rho_1\rangle \prec^{\Phi} \langle\rho_2\rangle$, then $\Sigma \vdash \rho_1 \prec^{\Phi} \rho_2$.*

Proof. Straightforward by case analysis on $\Sigma \vdash \langle\rho_1\rangle \prec^{\Phi} \langle\rho_2\rangle$. □

Lemma 60 (Convertibility inversion: row cons). *If $\Sigma \vdash \ell : A; \rho_1 \prec^{\Phi} \ell : B; \rho_2$, then $\Sigma \vdash A \prec^{\Phi} B$ and $\Sigma \vdash \rho_1 \prec^{\Phi} \rho_2$.*

Proof. Straightforward by case analysis on $\Sigma \vdash \ell : A; \rho_1 \prec^{\Phi} \ell : B; \rho_2$. □

Lemma 61 (Subject reduction on reduction). *If $\Sigma;\emptyset \vdash e : A$ and $e \rightsquigarrow e'$, then $\Sigma;\emptyset \vdash e' : A$.*

Proof. By case analysis on the derivation of $\Sigma;\emptyset \vdash e : A$.

Case (T_Var), (T_Const), (T_Lam), (T_TLam), (T_REmp), (T_Blame): Contradictory; there are no reduction rules to apply.

Case (T_TApp), (T_RExt), (T_VInj), (T_VLift): Contradictory; there are no applicable reduction rules.

Case (T_App): We have $e = e_1\, e_2$ and, by inversion, $\Sigma;\emptyset \vdash e_1 : B \to A$ and $\Sigma;\emptyset \vdash e_2 : B$ for some e_1, e_2, and B. By case analysis on the reduction rules applicable to $e_1\, e_2$.

Case (R_Cons): We have $e_1 = \kappa_1$ and $e_2 = \kappa_2$ and $e' = \zeta(\kappa_1, \kappa_2)$ for some κ_1 and κ_2. By Lemma 35, $ty(\kappa_1) = B \to A$. By the assumptions about constants, $ty(\zeta(\kappa_1, \kappa_2)) = A$. Thus, $\Sigma;\emptyset \vdash \zeta(\kappa_1, \kappa_2) : A$ by (T_Const).

Case (R_Beta): By Lemma 33, $e_1 = \lambda x{:}B.e'_1$ and $e_2 = v_2$ and $e' = e'_1[v_2/x]$ for some x, e'_1, and v_2. By Lemma 36, $\Sigma; x{:}B \vdash e'_1 : A$. Since $\Sigma;\emptyset \vdash v_2 : B$, we have $\Sigma;\emptyset \vdash e'_1[v_2/x] : A$ by Lemma 32.

Case (T_RLet): We have $e = \mathsf{let}\,\{\ell = x; y\} = e_1 \;\mathsf{in}\; e_2$ and, by inversion, $\Sigma;\emptyset \vdash e_1 : [\ell : B; \rho]$ and $\Sigma; x{:}B, y{:}[\rho] \vdash e_2 : A$. The reduction rules applicable to e is only (R_Record). We can suppose that $e_1 = \{\ell = v_1; v_2\}$ and $e' = e_2[v_1/x, v_2/y]$. By Lemma 38, $\Sigma;\emptyset \vdash v_1 : B$ and $\Sigma;\emptyset \vdash v_2 : [\rho]$. Since $\Sigma; x{:}B, y{:}[\rho] \vdash e_2 : A$, we have $\Sigma;\emptyset \vdash e_2[v_1/x, v_2/y] : A$ by Lemma 32.

Case (T_VCase): We have $e = \mathsf{case}\, e_0\, \mathsf{with}\, \langle \ell\, x \rightarrow e_1; y \rightarrow e_2 \rangle$ and, by inversion, $\Sigma; \emptyset \vdash e_0 : \langle \ell : B; \rho \rangle$ and $\Sigma; x{:}B \vdash e_1 : A$ and $\Sigma; y{:}\langle \rho \rangle \vdash e_2 : A$ for some e_0, e_1, e_2, ℓ, x, y, B, and ρ. By case analysis on the reduction rules applicable to e.

Case (R_CaseL): We can suppose that $e_0 = \ell\, v$ and $e' = e_1[v/x]$ for some v. By Lemma 39, $\Sigma; \emptyset \vdash v : B$. Since $\Sigma; x{:}B \vdash e_1 : A$, we have $\Sigma; \emptyset \vdash e_1[v/x] : A$ by Lemma 32.

Case (R_CaseR): We can suppose that $e_0 = \uparrow \langle \ell : C \rangle\, v$ and $e' = e_2[v/y]$ for some C and v. By Lemma 40, $\Sigma; \emptyset \vdash v : \langle \rho \rangle$. Since $\Sigma; y{:}\langle \rho \rangle \vdash e_2 : A$, we have $\Sigma; \emptyset \vdash e_2[v/y] : A$ by Lemma 32.

Case (T_Cast): We have $e = e_0 : B \stackrel{p}{\Rightarrow} A$ and, by inversion, $\Sigma; \emptyset \vdash e_0 : B$ and $B \simeq A$ and $\Sigma; \emptyset \vdash A : \mathsf{T}$ for some e_0, B, and p. Besides, we have $\Sigma; \emptyset \vdash B : \mathsf{T}$ by Lemma 50. By case analysis on the reduction rules applicable to e.

Case (R_IdDyn), (R_IdBase), (R_IdName), (R_REmp), (R_RIdName), and (R_VIdName): We have $B = A$ and $e_0 = v$ and $e' = v$ for some v. Since $\Sigma; \emptyset \vdash e_0 : B$, we have $\Sigma; \emptyset \vdash v : A$, which is what we have to show.

Case (R_Blame), (R_RBlame), (R_VBlame): Obvious by (T_Blame) since $e' = \mathsf{blame}\, q$ for some q.

Case (R_ToDyn): We have $e_0 = v$ and $A = \star$ and $e' = v : B \stackrel{p}{\Rightarrow} G \stackrel{p}{\Rightarrow} \star$ for some v and G such that $B \simeq G$. Since $\Sigma; \emptyset \vdash v : B$ and $B \simeq G$, we have $\Sigma; \emptyset \vdash G : \mathsf{T}$ by Lemma 52. Thus, $\Sigma; \emptyset \vdash v : B \stackrel{p}{\Rightarrow} G \stackrel{p}{\Rightarrow} \star : \star$ by (T_Cast).

Case (R_FromDyn): We have $e_0 = v$ and $B = \star$ and $e' = v : \star \stackrel{p}{\Rightarrow} G \stackrel{p}{\Rightarrow} A$ for some v and G such that $A \simeq G$. Since $\Sigma; \emptyset \vdash A : \mathsf{T}$ and $A \simeq G$, we have $\Sigma; \emptyset \vdash G : \mathsf{T}$ by Lemma 52. Since $\Sigma; \emptyset \vdash v : \star$, we have $\Sigma; \emptyset \vdash v : \star \stackrel{p}{\Rightarrow} G \stackrel{p}{\Rightarrow} A : A$ by (T_Cast) (note that $G \simeq A$ by Lemma 6).

Case (R_Ground): We have $e_0 = v : G \stackrel{p}{\Rightarrow} \star$ and $B = \star$ and $A = G$ and $e' = v$ for some v and G. Since $\Sigma; \emptyset \vdash e_0 : B$, i.e., $\Sigma; \emptyset \vdash v : G \stackrel{p}{\Rightarrow} \star : \star$, we have $\Sigma; \emptyset \vdash v : G$ by Lemma 41. Thus, we have $\Sigma; \emptyset \vdash e' : A$.

Case (R_Wrap): We have $e_0 = v$ and $B = B_1 \rightarrow B_2$ and $A = A_1 \rightarrow A_2$ and $e' = \lambda x{:}A_1.v\, (x : A_1 \stackrel{\overline{p}}{\Rightarrow} B_1) : B_2 \stackrel{p}{\Rightarrow} A_2$. Since $B_1 \rightarrow B_2 \simeq A_1 \rightarrow A_2$, we have $A_1 \simeq B_1$ and $B_2 \simeq A_2$ by Lemmas 10 and 6. Besides, we have $\Sigma; \emptyset \vdash A_1 : \mathsf{T}$, $\Sigma; \emptyset \vdash A_2 : \mathsf{T}$, $\Sigma; \emptyset \vdash B_1 : \mathsf{T}$, and $\Sigma; \emptyset \vdash B_2 : \mathsf{T}$ since $\Sigma; \emptyset \vdash A_1 \rightarrow A_2 : \mathsf{T}$ and $\Sigma; \emptyset \vdash B_1 \rightarrow B_2 : \mathsf{T}$. Thus, since $\Sigma; x{:}A_1 \vdash v : B_1 \rightarrow B_2$ by Lemma 24, we have $\Sigma; \emptyset \vdash \lambda x{:}A_1.v\, (x : A_1 \stackrel{\overline{p}}{\Rightarrow} B_1) : B_2 \stackrel{p}{\Rightarrow} A_2 : A_1 \rightarrow A_2$.

Case (R_Content): We have $e_0 = v$ and $B = \forall X{:}K.\, B'$ and $A = \forall X{:}K.\, A'$ and $e' = \Lambda X{:}K.(v\, X : B' \stackrel{p}{\Rightarrow} A') :: A'$ for some v, X, K, A', and B'. Since $\forall X{:}K.\, B' \simeq \forall X{:}K.\, A'$, we have $B' \simeq A'$ by Lemma 11. Since $\Sigma; \emptyset \vdash \forall X{:}K.\, A' : \mathsf{T}$, we have $\Sigma; X{:}K \vdash A' : \mathsf{T}$. Thus, since $\Sigma; X{:}K \vdash v : \forall X{:}K.\, B'$ by Lemma 24, we have $\Sigma; \emptyset \vdash \Lambda X{:}K.(v\, X : B' \stackrel{p}{\Rightarrow} A') :: A' : \forall X{:}K.\, A'$.

Case (R_Inst): We have $e_0 = v$ and $B = \forall X{:}K.\, B'$ and $e' = (v\, \star) : B'[\star/X] \stackrel{p}{\Rightarrow} A$ for some v, X, K, and B'. Besides, **QPoly** (A).
Since $\Sigma; \emptyset \vdash v : \forall X{:}K.\, B'$, we have

$$\Sigma; \emptyset \vdash v\, \star : B'[\star/X].$$

Since **QPoly** (A) and $B \simeq A$, i.e., $\forall X{:}K.\, B' \simeq A$, we have $B' \simeq A$ and $[[X notin ftv(A)]]$. Thus, by Lemma 28, $B'[\star/X] \simeq A$. By (T_Cast),

$$\Sigma; \emptyset \vdash (v\, \star) : B'[\star/X] \stackrel{p}{\Rightarrow} A : A.$$

Case (R_Gen): We have $e_0 = v$ and $A = \forall X{:}K.\, A'$ and $e' = \Lambda X{:}K.(v : B \stackrel{p}{\Rightarrow} A') :: A'$ for some v, X, K, and A'. Besides, **QPoly** (B).
Since $\Sigma; \emptyset \vdash v : B$, we have

$$\Sigma; X{:}K \vdash v : B$$

by Lemma 24.
Since **QPoly** (B) and $B \simeq A$, i.e., $B \simeq \forall X{:}K.\, A'$, we have $B \simeq A'$ and $[[X notin ftv(B)]]$ by Lemmas 6 and 12. Furthermore, $\Sigma; \emptyset \vdash \forall X{:}K.\, A' : \mathsf{T}$, we have $\Sigma; X{:}K \vdash A' : \mathsf{T}$. Thus, we have $\Sigma; \emptyset \vdash \Lambda X{:}K.(v : B \stackrel{p}{\Rightarrow} A') :: A' : \forall X{:}K.\, A'$.

Case (R_RToDyn): We have $e_0 = v$ and $A = [\star]$ and $B = [\rho]$ and $e' = v : [\rho] \overset{p}{\Rightarrow} [grow(\rho)] \overset{p}{\Rightarrow} [\star]$ for some v and ρ such that $\rho \neq grow(\rho)$. By Lemma 48, $\rho \simeq grow(\rho)$, and therefore $[\rho] \simeq [grow(\rho)]$ by (CE_Record). Since $\Sigma; \emptyset \vdash v : [\rho]$ and $\Sigma; \emptyset \vdash [grow(\rho)] : \mathsf{T}$ by Lemma 49, we have $\Sigma; \emptyset \vdash v : [\rho] \overset{p}{\Rightarrow} [grow(\rho)] \overset{p}{\Rightarrow} [\star] : [\star]$.

Case (R_RFromDyn): We have $e_0 = v : [\gamma] \overset{q}{\Rightarrow} [\star]$ and $B = [\star]$ and $A = [\rho_1]$ and $e' = v : [\gamma] \overset{q}{\Rightarrow} [\rho_1]$ for some v, γ, ρ_1, and q such that $\gamma \simeq \rho_1$. Since $\gamma \simeq \rho_1$, we have $[\gamma] \simeq [\rho_1]$ by (CE_Record). Since $\Sigma; \emptyset \vdash e_0 : B$, i.e., $\Sigma; \emptyset \vdash v : [\gamma] \overset{q}{\Rightarrow} [\star] : [\star]$, we have $\Sigma; \emptyset \vdash v : [\gamma]$ by Lemma 41. Thus, we have $\Sigma; \emptyset \vdash v : [\gamma] \overset{q}{\Rightarrow} [\rho_1] : [\rho_1]$ by (T_Cast).

Case (R_RRev): We have

- $e_0 = v$,
- $A = [\ell : A'; \rho_1]$,
- $B = [\rho_2]$, and
- $e' = \{\ell = (v_1 : B' \overset{p}{\Rightarrow} A'); v_2 : [\rho_2'] \overset{p}{\Rightarrow} [\rho_1]\}$

for some v, ℓ, A', B', ρ_1, ρ_2, and ρ_2' such that $v \triangleright_\ell v_1, v_2$ and $\rho_2 \triangleright_\ell B', \rho_2'$. Since $\Sigma; \emptyset \vdash v : B$ $B = [\rho_2]$ and $v \triangleright_\ell v_1, v_2$, there exist some ρ_{21}, ρ_{22}, and B' such that

- $\rho_2 = \rho_{21} \odot (\ell : B'; \cdot) \odot \rho_{22}$,
- $\rho_2' = \rho_{21} \odot \rho_{22}$,
- $\ell \notin dom(\rho_{21})$,
- $\Sigma; \emptyset \vdash v_1 : B'$, and
- $\Sigma; \emptyset \vdash v_2 : [\rho_{21} \odot \rho_{22}]$.

Since $B \simeq A$, i.e., $[\rho_{21} \odot (\ell : B'; \cdot) \odot \rho_{22}] \simeq [\ell : A'; \rho_1]$, we have $B' \simeq A'$ and $\rho_{21} \odot \rho_{22} \simeq \rho_1$ by Lemmas 6, 13, and 9. Since $\Sigma; \emptyset \vdash A : \mathsf{T}$, i.e., $\Sigma; \emptyset \vdash [\ell : A'; \rho_1] : \mathsf{T}$, we have $\Sigma; \emptyset \vdash A' : \mathsf{T}$. Thus,

$$\Sigma; \emptyset \vdash v_1 : B' \overset{p}{\Rightarrow} A' : A'$$

by (T_Cast).

Since $\rho_{21} \odot \rho_{22} \simeq \rho_1$, i.e., $\rho_2' \simeq \rho_1$, we have $[\rho_2'] \simeq [\rho_1]$ by (CE_Record). Since $\Sigma; \emptyset \vdash v_2 : [\rho_2']$ (note that $\rho_2' = \rho_{21} \odot \rho_{22}$) and $\Sigma; \emptyset \vdash [\rho_1] : \mathsf{T}$ (from $\Sigma; \emptyset \vdash A : \mathsf{T}$), we have

$$\Sigma; \emptyset \vdash v_2 : [\rho_2'] \overset{p}{\Rightarrow} [\rho_1] : [\rho_1]$$

by (T_Cast).

Thus, by (T_RExt),

$$\Sigma; \emptyset \vdash \{\ell = (v_1 : B' \overset{p}{\Rightarrow} A'); v_2 : [\rho_2'] \overset{p}{\Rightarrow} [\rho_1]\} : [\ell : A'; \rho_1].$$

Case (R_RCon): We have

- $e_0 = v$,
- $A = [\ell : A'; \rho_1]$,
- $B = [\rho_2]$, and
- $e' = v : [\rho_2] \overset{p}{\Rightarrow} [\rho_2 \mathbin{@} \ell : A'] \overset{p}{\Rightarrow} [\ell : A'; \rho_1]$

for some v, ℓ, A', ρ_1, ρ_2 such that $\ell \notin dom(\rho_2)$ and $\rho_2 \neq \star$.

Since $B \simeq A$, there exist some B' and ρ_2' such that

- $\rho_2 \triangleright_\ell B', \rho_2'$,
- $B' \simeq A'$, and
- $\rho_2' \simeq \rho_1$

by Lemmas 13, 6, and 9. Since $\ell \notin dom(\rho_2)$ and $\rho_2 \triangleright_\ell B', \rho_2'$, it is found that ρ_2 ends with $\star$ and $B' = \star$ and $\rho_2' = \rho_2$. Thus, by Lemma 19, $\rho_2 \simeq \rho_2 \mathbin{@} \ell : A'$. Since $\Sigma; \emptyset \vdash \rho_2 : \mathsf{R}$ and $\Sigma; \emptyset \vdash A : \mathsf{T}$, we have $\Sigma; \emptyset \vdash [\rho_2 \mathbin{@} \ell : A'] : \mathsf{R}$. Thus,

$$\Sigma; \emptyset \vdash v : [\rho_2] \overset{p}{\Rightarrow} [\rho_2 \mathbin{@} \ell : A'] : [\rho_2 \mathbin{@} \ell : A']$$

by (T_Cast).

Since $\rho_2' \simeq \rho_1$ and $A' \simeq A'$ (CE_Refl) and $\ell \notin \mathit{dom}(\rho_2')$ (since $\ell \notin \mathit{dom}(\rho_2)$ and $\rho_2 = \rho_2'$), we have $\rho_2' \mathbin{@} \ell : A' \simeq \ell : A'; \rho_1$ by (CE_ConsR). Thus,

$$\Sigma; \emptyset \vdash v : [\rho_2] \stackrel{p}{\Rightarrow} [\rho_2 \mathbin{@} \ell : A'] \stackrel{p}{\Rightarrow} [\ell : A'; \rho_1] : [\ell : A'; \rho_1]$$

by (T_Cast).

Case (R_VToDyn): We have $e_0 = v$ and $A = \langle \star \rangle$ and $B = \langle \rho \rangle$ and $e' = v : \langle \rho \rangle \stackrel{p}{\Rightarrow} \langle \mathit{grow}(\rho) \rangle \stackrel{p}{\Rightarrow} \langle \star \rangle$ for some v and ρ $\rho \neq \mathit{grow}(\rho)$.

By Lemma 48, $\rho \simeq \mathit{grow}(\rho)$, and therefore $\langle \rho \rangle \simeq \langle \mathit{grow}(\rho) \rangle$ by (CE_Variant). Since $\Sigma; \emptyset \vdash v : \langle \rho \rangle$ and $\Sigma; \emptyset \vdash \langle \mathit{grow}(\rho) \rangle : \mathsf{T}$ by Lemma 49, we have $\Sigma; \emptyset \vdash v : \langle \rho \rangle \stackrel{p}{\Rightarrow} \langle \mathit{grow}(\rho) \rangle \stackrel{p}{\Rightarrow} \langle \star \rangle : \langle \star \rangle$.

Case (R_VFromDyn): We have $e_0 = v : \langle \gamma \rangle \stackrel{q}{\Rightarrow} \langle \star \rangle$ and $B = \langle \star \rangle$ and $A = \langle \rho_1 \rangle$ and $e' = v : \langle \gamma \rangle \stackrel{q}{\Rightarrow} \langle \rho_1 \rangle$ for some v, γ, ρ_1, and q such that $\gamma \simeq \rho_1$. Since $\gamma \simeq \rho_1$, we have $\langle \gamma \rangle \simeq \langle \rho_1 \rangle$ by (CE_Variant). Since $\Sigma; \emptyset \vdash e_0 : B$, i.e., $\Sigma; \emptyset \vdash v : \langle \gamma \rangle \stackrel{q}{\Rightarrow} \langle \star \rangle : \langle \star \rangle$, we have $\Sigma; \emptyset \vdash v : \langle \gamma \rangle$ by Lemma 41. Thus, we have $\Sigma; \emptyset \vdash v : [\gamma] \stackrel{q}{\Rightarrow} [\rho_1] : [\rho_1]$ by (T_Cast).

Case (R_VRevInj): We have $e_0 = \ell\, v$ and $A = \langle \rho_{11} \odot (\ell : A'; \cdot) \odot \rho_{12} \rangle$ and $B = \langle \ell : B'; \rho_2 \rangle$ and $e' = \uparrow \rho_{11}\, (\ell\, (v : B' \stackrel{p}{\Rightarrow} A'))$ for some ℓ, v, ρ_2, ρ_{11}, A', and B' such that $\ell \notin \mathit{dom}(\rho_{11})$.

Since $\Sigma; \emptyset \vdash \ell\, v : \langle \ell : B'; \rho_2 \rangle$, we have $\Sigma; \emptyset \vdash v : B'$ by Lemma 39. Since $\langle \ell : B'; \rho_2 \rangle \simeq \langle \rho_{11} \odot (\ell : A'; \cdot) \odot \rho_{12} \rangle$, we have $B' \simeq A'$ by Lemmas 14 and 9. Since $\Sigma; \emptyset \vdash \langle \rho_{11} \odot (\ell : A'; \cdot) \odot \rho_{12} \rangle : \mathsf{T}$, we have $\Sigma; \emptyset \vdash A' : \mathsf{T}$ and $\Sigma; \emptyset \vdash \rho_{12} : \mathsf{R}$.

$$\Sigma; \emptyset \vdash \ell\, (v : B' \stackrel{p}{\Rightarrow} A') : \langle \ell : A'; \rho_{12} \rangle$$

by (T_Cast) and (T_VInj).

Since $\Sigma; \emptyset \vdash \langle \rho_{11} \odot (\ell : A'; \cdot) \odot \rho_{12} \rangle : \mathsf{T}$, we have $\Sigma; \emptyset \vdash \rho_{11} : \mathsf{R}$. Thus, by Lemma 54,

$$\Sigma; \emptyset \vdash \uparrow \rho_{11}\, (\ell\, (v : B' \stackrel{p}{\Rightarrow} A')) : \langle \rho_{11} \odot (\ell : A'; \rho_{12}) \rangle.$$

Case (R_VRevLift): We have $e_0 = \uparrow \langle \ell : C \rangle\, v$ and $B = \langle \ell{:}C; \rho_2 \rangle$ and $A = \langle \rho_1 \rangle$ and $e' = \downarrow^{\rho_{11}}_{\langle \ell : C \rangle} (v : \langle \rho_2 \rangle \stackrel{p}{\Rightarrow} \langle \rho_{11} \odot \rho_{12} \rangle)$ for some ℓ, C, v, ρ_1, ρ_2, ρ_{11}, and ρ_{12} sch such that $\rho_1 = \rho_{11} \odot (\ell : C; \cdot) \odot \rho_{12}$ and $\ell \notin \mathit{dom}(\rho_{11})$.

Since $\Sigma; \emptyset \vdash e_0 : B$, i.e., $\Sigma; \emptyset \vdash \uparrow \langle \ell : C \rangle\, v : \langle \ell : C; \rho_2 \rangle$, we have $\Sigma; \emptyset \vdash v : \langle \rho_2 \rangle$ by Lemma 40. Since $B \simeq A$, i.e., $\langle \ell{:}C; \rho_2 \rangle \simeq \langle \rho_1 \rangle$, and $\rho_1 = \rho_{11} \odot (\ell{:}C; \cdot) \odot \rho_{12}$ and $\ell \notin \mathit{dom}(\rho_{11})$, we have $\rho_2 \simeq \rho_{11} \odot \rho_{12}$ by Lemmas 14 and 9. Thus, (CE_Variant), $\langle \rho_2 \rangle \simeq \langle \rho_{11} \odot \rho_{12} \rangle$. Since $\Sigma; \emptyset \vdash A : \mathsf{T}$, i.e., $\Sigma; \emptyset \vdash \langle \rho_1 \rangle : \mathsf{T}$, we have $\Sigma; \emptyset \vdash \langle \rho_{11} \odot \rho_{12} \rangle : \mathsf{T}$. Thus, by (T_Cast),

$$\Sigma; \emptyset \vdash v : \langle \rho_2 \rangle \stackrel{p}{\Rightarrow} \langle \rho_{11} \odot \rho_{12} \rangle : \langle \rho_{11} \odot \rho_{12} \rangle.$$

Since $\Sigma; \emptyset \vdash B : \mathsf{T}$, i.e., $\Sigma; \emptyset \vdash \langle \ell : C; \rho_2 \rangle : \mathsf{T}$, we have $\Sigma; \emptyset \vdash C : \mathsf{T}$. Thus, by Lemma 55,

$$\Sigma; \emptyset \vdash \downarrow^{\rho_{11}}_{\langle \ell : C \rangle} (v : \langle \rho_2 \rangle \stackrel{p}{\Rightarrow} \langle \rho_{11} \odot \rho_{12} \rangle) : \langle \rho_{11} \odot (\ell : C; \cdot) \odot \rho_{12} \rangle,$$

which is what we have to show.

Case (R_VConInj): We have $e_0 = \ell\, v$ and $B = \langle \ell : B'; \rho_2 \rangle$ and $A = \langle \rho_1 \rangle$ and $e' = \uparrow \rho_1\, (\ell\, v : \langle \ell : B'; \star \rangle \stackrel{p}{\Rightarrow} \langle \star \rangle)$ for some ℓ, v, B', ρ_1, and ρ_2 such that $\ell \notin \mathit{dom}(\rho_1)$ and $\rho_1 \neq \star$.

Since $B \simeq A$, i.e., $\langle \ell{:}B'; \rho_2 \rangle \simeq \langle \rho_1 \rangle$. By Lemma 14, $\ell{:}B'; \rho_2 \simeq \rho_1$. By 9, $\rho_1 \rhd_\ell A', \rho_1'$ for some A' and ρ_1'. Since $\ell \notin \mathit{dom}(\rho_1)$, ρ_1 ends with $\star$, that is, there exists some ρ_1'' such that $\rho_1'' \odot \star = \rho_1$. Since $\uparrow \rho_1\, e'' = \uparrow \rho_1''\, e''$ for any e'', it suffices to show that

$$\Sigma; \emptyset \vdash \uparrow \rho_1''\, (\ell\, v : \langle \ell : B'; \star \rangle \stackrel{p}{\Rightarrow} \langle \star \rangle) : \langle \rho_1'' \odot \star \rangle.$$

Since $\Sigma; \emptyset \vdash e_0 : B$, i.e., $\Sigma; \emptyset \vdash \ell\, v : \langle \ell : B'; \rho_2 \rangle$, we have $\Sigma; \emptyset \vdash v : B'$ by Lemma 39. Thus, by (T_VInj), $\Sigma; \emptyset \vdash \ell\, v : \langle \ell : B'; \star \rangle$. We have $\langle \ell : B'; \star \rangle \simeq \langle \star \rangle$ by (CE_Refl), (CE_ConsL), and (CE_Variant). Since $\Sigma; \emptyset \vdash \langle \star \rangle : \mathsf{R}$, we have

$$\Sigma; \emptyset \vdash \ell\, v : \langle \ell : B'; \star \rangle \stackrel{p}{\Rightarrow} \langle \star \rangle : \langle \star \rangle$$

by (T_Cast). Since $\Sigma; \emptyset \vdash A : \mathsf{T}$, i.e., $\Sigma; \emptyset \vdash [\rho_1] : \mathsf{T}$, we have $\Sigma; \emptyset \vdash \rho_1'' : \mathsf{R}$. Thus, by Lemma 54,

$$\Sigma; \emptyset \vdash \uparrow \rho_1''\, (\ell\, v : \langle \ell : B'; \star \rangle \stackrel{p}{\Rightarrow} \langle \star \rangle) : \langle \rho_1'' \odot \star \rangle.$$

Case (R_VConLift): We have $e_0 = \uparrow\langle \ell : B' \rangle\, v$ and $B = \langle \ell{:}B'; \rho_2 \rangle$ and $A = \langle \rho_1 \rangle$ and $e' = (\downarrow^{\rho_1}_{\langle \ell:B' \rangle}(v : \langle \rho_2 \rangle \stackrel{p}{\Rightarrow} \langle \rho_1 \rangle)) : \langle \rho_1 \mathbin{@} \ell : B' \rangle \stackrel{p}{\Rightarrow} \langle \rho_1 \rangle$ for some ℓ, v, B', ρ_1, and ρ_2 such that $\ell \notin \mathit{dom}\,(\rho_1)$ and $\rho_1 \neq \star$.

Since $[NS; gemp| - e0 : B]$, i.e., $\Sigma; \emptyset \vdash \uparrow\langle \ell : B' \rangle\, v : \langle \ell : B'; \rho_2 \rangle$, we have $\Sigma; \emptyset \vdash v : \langle \rho_2 \rangle$ by Lemma 40. Since $B \simeq A$, i.e., $\langle \ell : B'; \rho_2 \rangle \simeq \langle \rho_1 \rangle$, there exist some A' and ρ'_1 such that

- $\rho_1 \rhd_\ell A', \rho'_1$,
- $B' \simeq A'$, and
- $\rho_2 \simeq \rho'_1$

by Lemmas 14 and 9. Since $[lnotindom(r1)]$, it is found that

- ρ_1 ends with $\star$, i.e., $\rho_1 = \rho''_1 \odot \star$ for some ρ''_1,
- $A' = \star$, and
- $\rho'_1 = \rho_1$.

Thus, $\rho_2 \simeq \rho_1$, and therefore $\langle \rho_2 \rangle \simeq \langle \rho_1 \rangle$ by (CE_Variant). Since $\Sigma; \emptyset \vdash A : \mathsf{T}$, i.e., $\Sigma; \emptyset \vdash \langle \rho_1 \rangle : \mathsf{T}$, we have

$$\Sigma; \emptyset \vdash v : \langle \rho_2 \rangle \stackrel{p}{\Rightarrow} \langle \rho_1 \rangle : \langle \rho_1 \rangle$$

by (T_Cast). Since $\downarrow^{\rho_1}_{\langle \ell:B' \rangle} e'' = \downarrow^{\rho''_1}_{\langle \ell:B' \rangle} e''$ for any e'', and $\Sigma; \Gamma \vdash B' : \mathsf{T}$ from $\Sigma; \emptyset \vdash B : \mathsf{T}$, i.e., $\Sigma; \emptyset \vdash \langle \ell : B'; \rho_2 \rangle : \mathsf{T}$, we have

$$\Sigma; \emptyset \vdash \downarrow^{\rho_1}_{\langle \ell:B' \rangle}(v : \langle \rho_2 \rangle \stackrel{p}{\Rightarrow} \langle \rho_1 \rangle) : \langle \rho''_1 \odot (\ell : B'; \cdot) \odot \star \rangle.$$

Since $\rho''_1 \odot (\ell : B'; \cdot) \odot \star = \rho_1 \mathbin{@} \ell : B'$, we have

$$\Sigma; \emptyset \vdash \downarrow^{\rho_1}_{\langle \ell:B' \rangle}(v : \langle \rho_2 \rangle \stackrel{p}{\Rightarrow} \langle \rho_1 \rangle) : \langle \rho_1 \mathbin{@} \ell : B' \rangle.$$

Since $\rho_1 \simeq \rho_1$ by (CE_Refl), and ρ_1 ends with $\star$ and $\ell \notin \mathit{dom}\,(\rho_1)$, we have $\rho''_1 \odot (\ell : B'; \cdot) \odot \star \simeq \rho_1$, i.e., $\rho_1 \mathbin{@} \ell : B' \simeq \rho_1$ by Lemma 19. Thus, $\langle \rho_1 \mathbin{@} \ell : B' \rangle \simeq \langle \rho_1 \rangle$ by (CE_Variant). Since $\Sigma; \emptyset \vdash A : \mathsf{T}$, i.e., $\Sigma; \emptyset \vdash \langle \rho_1 \rangle : \mathsf{T}$, we have

$$\Sigma; \emptyset \vdash (\downarrow^{\rho_1}_{\langle \ell:B' \rangle}(v : \langle \rho_2 \rangle \stackrel{p}{\Rightarrow} \langle \rho_1 \rangle)) : \langle \rho_1 \mathbin{@} \ell : B' \rangle \stackrel{p}{\Rightarrow} \langle \rho_1 \rangle : \langle \rho_1 \rangle$$

by (T_Cast), which is what we have to show.

Case (T_Conv): We have $e = e_0 : B \stackrel{\Phi}{\Rightarrow} A$ and, by inversion, $\Sigma; \emptyset \vdash e_0 : B$ and $\Sigma; \emptyset \vdash A : \mathsf{T}$ and $\Sigma \vdash B \prec^\Phi A$ for some e_0, B, and Φ. Besides, we have $\Sigma; \emptyset \vdash B : \mathsf{T}$ by Lemma 50. By case analysis on the reduction rules applicable to e.

Case (R_CName), (R_CRName), and (R_CVName): We have $e_0 = v : A \stackrel{-\alpha}{\Rightarrow} B$ and $\Phi = +\alpha$ and $e' = v$ for some v and α. Since $\Sigma; \emptyset \vdash e_0 : B$, i.e., $\Sigma; \emptyset \vdash v : A \stackrel{-\alpha}{\Rightarrow} B : B$, we have $\Sigma; \emptyset \vdash v : A$ by Lemma 42, 43, or 44. This is what we have to show.

Case (R_CIdDyn), (R_CIdName), (R_CIdBase), (R_CREmp), (R_CRIdDyn), (R_CRIdName), (R_CVIdDyn), (R_CV
We have $e_0 = v$ and $e' = v$ and $A = B$ for some v. Since $\Sigma; \emptyset \vdash e_0 : B$, we finish.

Case (R_CFun): We have $e_0 = v$ and $B = B_1 \rightarrow B_2$ and $A = A_1 \rightarrow A_2$ and $e' = \lambda x{:}A_1.v\,(x : A_1 \stackrel{\overline{\Phi}}{\Rightarrow} B_1) : B_2 \stackrel{\Phi}{\Rightarrow} A_2$ for some v, A_1, A_2, B_1, B_2, and x. Since $\Sigma \vdash B \prec^\Phi A$, i.e., $\Sigma \vdash B_1 \rightarrow B_2 \prec^\Phi A_1 \rightarrow A_2$, we have $\Sigma \vdash A_1 \prec^{\overline{\Phi}} B_1$ and $\Sigma \vdash B_2 \prec^\Phi A_2$ by Lemma 56. Since $\Sigma; \emptyset \vdash A_1 \rightarrow A_2 : \mathsf{T}$ and $\Sigma; \emptyset \vdash B_1 \rightarrow B_2 : \mathsf{T}$, we have $\Sigma; \emptyset \vdash A_1 : \mathsf{T}$ and $\Sigma; \emptyset \vdash A_2 : \mathsf{T}$ and $\Sigma; \emptyset \vdash B_1 : \mathsf{T}$ and $\Sigma; \emptyset \vdash B_2 : \mathsf{T}$. Thus, we have $\Sigma; x{:}A_1 \vdash x : A_1 \stackrel{\overline{\Phi}}{\Rightarrow} B_1 : B_1$ by (T_Conv). Since $\Sigma; x{:}A_1 \vdash v : B_1 \rightarrow B_2$ by Lemma 24, we have

$$\Sigma; x{:}A_1 \vdash v\,(x : A_1 \stackrel{\overline{\Phi}}{\Rightarrow} B_1) : B_2 \stackrel{\Phi}{\Rightarrow} A_2 : A_2$$

by (T_App) and (T_Conv). Thus,

$$\Sigma; \emptyset \vdash \lambda x{:}A_1.v\,(x : A_1 \stackrel{\overline{\Phi}}{\Rightarrow} B_1) : B_2 \stackrel{\Phi}{\Rightarrow} A_2 : A_1 \rightarrow A_2$$

by (T_Lam).

Case (R_CForall): We have $e_0 = v$ and $B = \forall X{:}K.\, B'$ and $A = \forall X{:}K.\, A'$ and $e' = \Lambda X{:}K.(v\, x : B' \overset{\Phi}{\Rightarrow} A') :: A'$ for some v, X, K, A', B', and x. Since $\Sigma \vdash B \prec^\Phi A$, i.e., $\Sigma \vdash \forall X{:}K.\, B' \prec^\Phi \forall X{:}K.\, A'$, we have $\Sigma \vdash B' \prec^\Phi A'$ by Lemma 57. Since $\Sigma; \emptyset \vdash \forall X{:}K.\, A' : \mathsf{T}$ and $\Sigma; \emptyset \vdash \forall X{:}K.\, B' : \mathsf{T}$, we have $\Sigma; X{:}K \vdash A' : \mathsf{T}$ and $\Sigma; X{:}K \vdash B' : \mathsf{T}$. Since $\Sigma; X{:}K \vdash v : \forall X{:}K.\, B'$ by Lemma 24, we have

$$\Sigma; X{:}K \vdash v\, x : B' \overset{\Phi}{\Rightarrow} A' : A'$$

by (T_Var), (T_App), and (T_Conv). Thus,

$$\Sigma; \emptyset \vdash \Lambda X{:}K.(v\, x : B' \overset{\Phi}{\Rightarrow} A') :: A' : \forall X{:}K.\, A'$$

by (T_TLam).

Case (R_CRExt): We have $e_0 = v$ and $B = [\ell : B'; \rho_2]$ and $A = [\ell : A'; \rho_1]$ and $e' = \mathsf{let}\, \{\ell = x; y\} = v\, \mathsf{in}\, \{\ell = x : B' \overset{\Phi}{\Rightarrow} A'; y : [\rho_2] \overset{\Phi}{\Rightarrow} [\rho_1]\}$ for some v, ℓ, A', B', ρ_1, ρ_2, x, and y. Since $\Sigma \vdash B \prec^\Phi A$, i.e., $\Sigma \vdash [\ell : B'; \rho_2] \prec^\Phi [\ell : A'; \rho_1]$, we have $\Sigma \vdash B' \prec^\Phi A'$ and $\Sigma \vdash \rho_2 \prec^\Phi \rho_1$ by Lemmas 58 and 60, and $\Sigma \vdash [\rho_2] \prec^\Phi [\rho_1]$ by (Cv_Record). Since $\Sigma; \emptyset \vdash A : \mathsf{T}$, i.e., $\Sigma; \emptyset \vdash [\ell : A'; \rho_1] : \mathsf{T}$, we have $\Sigma; \emptyset \vdash A' : \mathsf{T}$ and $\Sigma; \emptyset \vdash \rho_1 : \mathsf{R}$, and therefore $\Sigma; \emptyset \vdash [\rho_1] : \mathsf{T}$. Thus,

$$\Sigma; x{:}B', y{:}[\rho_2] \vdash \{\ell = x : B' \overset{\Phi}{\Rightarrow} A'; y : [\rho_2] \overset{\Phi}{\Rightarrow} [\rho_1]\} : [\ell : A'; \rho_1]$$

by (T_Conv) and (T_RExt). Since $\Sigma; \emptyset \vdash e_0 : B$, i.e., $\Sigma; \emptyset \vdash v : [\ell : B'; \rho_2]$, we have

$$\Sigma; \emptyset \vdash \mathsf{let}\, \{\ell = x; y\} = v\, \mathsf{in}\, \{\ell = x : B' \overset{\Phi}{\Rightarrow} A'; y : [\rho_2] \overset{\Phi}{\Rightarrow} [\rho_1]\} : [\ell : A'; \rho_1],$$

which is what we have to prove.

Case (R_CVar): We have $e_0 = v$ and $B = \langle \ell : B'; \rho_2 \rangle$ and $A = \langle \ell : A'; \rho_1 \rangle$ and $e' = \mathsf{case}\, v\, \mathsf{with}\, \langle \ell\, x \to \ell\, (x : B' \overset{\Phi}{\Rightarrow} A'); y \to \uparrow \langle \ell : A' \rangle\, (y : \langle \rho_2 \rangle \overset{\Phi}{\Rightarrow} \langle \rho_1 \rangle) \rangle$ for some v, ℓ, A', B', ρ_1, $[r2]$, x, and y. Since $\Sigma \vdash B \prec^\Phi A$, i.e., $\Sigma \vdash \langle \ell : B'; \rho_2 \rangle \prec^\Phi \langle \ell : A'; \rho_1 \rangle$, we have $\Sigma \vdash B' \prec^\Phi A'$ and $\Sigma \vdash \rho_2 \prec^\Phi \rho_1$ by Lemmas 59 and 60, and $\Sigma \vdash \langle \rho_2 \rangle \prec^\Phi \langle \rho_1 \rangle$ by (Cv_Variant). Since $\Sigma; \emptyset \vdash A : \mathsf{T}$, i.e., $\Sigma; \emptyset \vdash \langle \ell : A'; \rho_1 \rangle : \mathsf{T}$, we have $\Sigma; \emptyset \vdash A' : \mathsf{T}$ and $\Sigma; \emptyset \vdash \rho_1 : \mathsf{R}$, and therefore $\Sigma; \emptyset \vdash \langle \rho_1 \rangle : \mathsf{T}$. Thus,

$$\Sigma; x{:}B' \vdash \ell\, (x : B' \overset{\Phi}{\Rightarrow} A') : \langle \ell : A'; \rho_1 \rangle$$

and

$$\Sigma; y{:}\langle \rho_2 \rangle \vdash \uparrow \langle \ell : A' \rangle\, (y : \langle \rho_2 \rangle \overset{\Phi}{\Rightarrow} \langle \rho_1 \rangle) : \langle \ell : A'; \rho_1 \rangle$$

by (T_Conv) and (T_VInj). Since $\Sigma; \emptyset \vdash e_0 : B$, i.e., $\Sigma; \emptyset \vdash v : \langle \ell : B'; \rho_2 \rangle$, we have

$$\Sigma; \emptyset \vdash \mathsf{case}\, v\, \mathsf{with}\, \langle \ell\, x \to \ell\, (x : B' \overset{\Phi}{\Rightarrow} A'); y \to \uparrow \langle \ell : A' \rangle\, (y : \langle \rho_2 \rangle \overset{\Phi}{\Rightarrow} \langle \rho_1 \rangle) \rangle : \langle \ell : A'; \rho_1 \rangle,$$

which is what we have to show.

□

Lemma 62. *If $\Sigma \mid e \longrightarrow \Sigma' \mid e'$, then $\Sigma \subseteq \Sigma'$.*

Proof. Obvious by case analysis on the evaluation rule applied to derive $\Sigma \mid e \longrightarrow \Sigma' \mid e'$. □

Lemma 63 (Subject reduction)**.** *If $\Sigma; \emptyset \vdash e : A$ and $\Sigma \mid e \longrightarrow \Sigma' \mid e'$, then $\Sigma'; \emptyset \vdash e' : A$.*

Proof. By induction on the derivation of $\Sigma; \emptyset \vdash e : A$.

Case (T_Var), (T_Const), (T_Lam), (T_TLam), (T_REmp), (T_Blame): Contradictory; there are no reduction rules to apply.

Case (T_App): We have $e = e_1\, e_2$ and, by inversion, $\Sigma;\emptyset \vdash e_1 : B \rightarrow A$ and $\Sigma;\emptyset \vdash e_2 : B$ for some e_1, e_2, and B.

If $\Sigma \mid e_1 \longrightarrow \Sigma' \mid e'_1$ for some e'_1, then we have $\Sigma';\emptyset \vdash e'_1 : B \rightarrow A$ by the IH, and therefore $\Sigma';\emptyset \vdash e'_1\, e_2 : A$ by Lemmas 62 and 25, and (T_App).

If $\Sigma \mid e_2 \longrightarrow \Sigma' \mid e'_2$ for some e'_2, then we have $\Sigma';\emptyset \vdash e'_2 : B$ by the IH, and therefore we have $\Sigma';\emptyset \vdash e_1\, e'_2 : A$ by Lemmas 62 and 25, and (T_App).

In what follows, we suppose that neither e_1 nor e_2 cannot be evaluated under Σ. By case analysis on the reduction rule applied to e.

Case (E_Red): We have $e_1\, e_2 = E[e'_1]$ and $e' = E[e'_2]$ for some E, e'_1, and e'_2 such that $e'_1 \rightsquigarrow e'_2$. Besides, $\Sigma' = \Sigma$. By case analysis on E.

Case $E = [\,]$: By Lemma 61.

Case $E = E'\, e_2$: Contradictory with the assumption that $e_1 = E'[e'_1]$ cannot be evaluated under Σ.

Case $E = v_1\, E'$: Contradictory with the assumption that $e_2 = E'[e'_1]$ cannot be evaluated under Σ.

Case otherwise: Contradictory with the assumption that $e_1\, e_2 = E[e'_1]$.

Case (E_Blame): By (T_Blame).

Case (E_TyBeta): Contradictory with the assumption that neither e_1 nor $[e2]$ cannot be evaluated under Σ.

Case (T_TApp): We have $e = e_1\, B$ and, by inversion, $\Sigma;\emptyset \vdash e_1 : \forall X{:}K.\, C$ and $\Sigma;\emptyset \vdash B : K$ and $A = C[B/X]$ for some e_1, X, K, B, and C.

If $\Sigma \mid e_1 \longrightarrow \Sigma' \mid e'_1$ for some e'_1, then we have $\Sigma';\emptyset \vdash e'_1 : \forall X{:}K.\, C$ by the IH, and therefore $\Sigma';\emptyset \vdash e'_1\, B : C[B/X]$ by Lemmas 62 and 25, and (T_TApp).

In what follows, we suppose that e_1 cannot be evaluated under Σ. By case analysis on the reduction rule applied to e.

Case (E_Red): We have $e_1\, B = E[e'_1]$ and $e' = E[e'_2]$ for some E, e'_1, and e'_2 such that $e'_1 \rightsquigarrow e'_2$. Besides, $\Sigma' = \Sigma$. By case analysis on E.

Case $E = [\,]$: By Lemma 61.

Case $E = E'\, B$: Contradictory with the assumption that $e_1 = E'[e'_1]$ cannot be evaluated under Σ.

Case otherwise: Contradictory with the assumption that $e_1\, B = E[e'_1]$.

Case (E_Blame): By (T_Blame).

Case (E_TyBeta): We have $e_1\, B = E[(\Lambda X'{:}K'.e'_0 :: C')\, B']$ and $e' = E[e'_0[\alpha/X'] : C'[\alpha/X'] \stackrel{\pm\alpha}{\Rightarrow} C'[B'/X]]$ and $\Sigma' = \Sigma, \alpha{:}K' := B'$ for some E, X', K', e'_0, B', C', and α. By case analysis on E.

Case $E = [\,]$: We have $e_1 = \Lambda X{:}K.e'_0 :: C$ by Lemma 33 (note that $X = X'$ and $K = K'$ and $C = C'$) and $B' = B$.
It suffices to show that

$$\Sigma, \alpha{:}K := B;\emptyset \vdash e'_0[\alpha/X] : C[\alpha/X] \stackrel{\pm\alpha}{\Rightarrow} C[B/X] : C[B/X].$$

Since $\Sigma;\emptyset \vdash e_1 : \forall X{:}K.\, C$, i.e., $\Sigma;\emptyset \vdash \Lambda X{:}K.e'_0 :: C : \forall X{:}K.\, C$, we have $\Sigma; X{:}K \vdash e'_0 : C$ by Lemma 37. Thus, $\Sigma, \alpha{:}K := B; X{:}K \vdash e'_0 : C$ by Lemma 25. Since $\Sigma, \alpha{:}K := B;\emptyset \vdash \alpha : K$ by (WF_TyName), we have

$$\Sigma, \alpha{:}K := B;\emptyset \vdash e'_0[\alpha/X] : C[\alpha/X]$$

by Lemma 29.
Since $\Sigma;\emptyset \vdash e_1 : \forall X{:}K.\, C$, we have $\Sigma;\emptyset \vdash \forall X{:}K.\, C : \mathsf{T}$ by Lemma 50. Thus, since α is a fresh type name for Σ, α does not occur in C. Therefore, we have

$$\Sigma, \alpha{:}K := B \vdash C[\alpha/X] \prec^{+\alpha} C[B/X]$$

by Lemma 30. Since $\Sigma;\emptyset \vdash e : A$, we have $\Sigma;\emptyset \vdash A : \mathsf{T}$ by Lemma 50, and therefore $\Sigma;\emptyset \vdash C[B/X] : \mathsf{T}$. Thus, by (T_Conv),

$$\Sigma, \alpha{:}K := B;\emptyset \vdash e'_0[\alpha/X] : C[\alpha/X] \stackrel{\pm\alpha}{\Rightarrow} C[B/X] : C[B/X].$$

Case $E = E'\,B$: Contradictory with the assumption that $e_1 = E'[(\Lambda X'{:}K'.e'_0 :: C')\,B']$ cannot be evaluated under Σ.

Case otherwise: Contradictory with the assumption that $e_1\,B = E[e'_1]$.

Case (T_RExt): We have $e = \{\ell = e_1; e_2\}$ and, by inversion, $\Sigma;\emptyset \vdash e_1 : B$ and $\Sigma;\emptyset \vdash e_2 : [\rho]$ and $A = [\ell : B; \rho]$ for some ℓ, e_1, e_2, B, and ρ.

If $\Sigma \mid e_1 \longrightarrow \Sigma' \mid e'_1$ for some e'_1, then we have $\Sigma';\emptyset \vdash e'_1 : B$ by the IH, and therefore $\Sigma';\emptyset \vdash \{\ell = e'_1; e_2\} : [\ell : B; \rho]$ by Lemmas 62 and 25, and (T_RExt).

If $\Sigma \mid e_2 \longrightarrow \Sigma' \mid e'_2$ for some e'_2, then we have $\Sigma';\emptyset \vdash e'_2 : [\rho]$ by the IH, and therefore we have $\Sigma';\emptyset \vdash \{\ell = e_1; e'_2\} : [\ell : B; \rho]$. by Lemmas 62 and 25, and (T_RExt).

In what follows, we suppose that neither e_1 nor e_2 cannot be evaluated under Σ. By case analysis on the reduction rule applied to e.

Case (E_Red): We have $\{\ell = e_1; e_2\} = E[e'_1]$ and $e' = E[e'_2]$ for some E, e'_1, and e'_2 such that $e'_1 \rightsquigarrow e'_2$. Besides, $\Sigma' = \Sigma$. By case analysis on E.

Case $E = [\,]$: By Lemma 61.

Case $E = \{\ell = E'; e_2\}$: Contradictory with the assumption that $e_1 = E'[e'_1]$ cannot be evaluated under Σ.

Case $E = \{\ell = v_1; E'\}$: Contradictory with the assumption that $e_2 = E'[e'_1]$ cannot be evaluated under Σ.

Case otherwise: Contradictory with the assumption that $\{\ell = e_1; e_2\} = E[e'_1]$.

Case (E_Blame): By (T_Blame).

Case (E_TyBeta): Contradictory with the assumption that neither e_1 nor $[e2]$ cannot be evaluated under Σ.

Case (T_RLet): We have $e = \mathsf{let}\,\{\ell = x; y\} = e_1 \,\mathsf{in}\, e_2$ and, by inversion, $\Sigma;\emptyset \vdash e_1 : [\ell : B; \rho]$ and $\Sigma; x{:}B, y{:}[\rho] \vdash e_2 : A$ for some ℓ, x, y, e_1, e_2, B, and ρ.

If $\Sigma \mid e_1 \longrightarrow \Sigma' \mid e'_1$ for some e'_1, then we have $\Sigma';\emptyset \vdash e'_1 : [\ell : B; \rho]$ by the IH, and therefore $\Sigma';\emptyset \vdash \mathsf{let}\,\{\ell = x; y\} = e'_1 \,\mathsf{in}\, e_2 : A$ by Lemmas 62 and 25, and (T_RLet).

In what follows, we suppose that e_1 cannot be evaluated under Σ. By case analysis on the reduction rule applied to e.

Case (E_Red): We have $\mathsf{let}\,\{\ell = x; y\} = e_1 \,\mathsf{in}\, e_2 = E[e'_1]$ and $e' = E[e'_2]$ for some E, e'_1, and e'_2 such that $e'_1 \rightsquigarrow e'_2$. Besides, $\Sigma' = \Sigma$. By case analysis on E.

Case $E = [\,]$: By Lemma 61.

Case $E = \mathsf{let}\,\{\ell = x; y\} = E' \,\mathsf{in}\, e_2$: Contradictory with the assumption that $e_1 = E'[e'_1]$ cannot be evaluated under Σ.

Case otherwise: Contradictory with the assumption that $\mathsf{let}\,\{\ell = x; y\} = e_1 \,\mathsf{in}\, e_2 = E[e'_1]$.

Case (E_Blame): By (T_Blame).

Case (E_TyBeta): Contradictory with the assumption that e_1 cannot be evaluated under Σ.

Case (T_VInj): We have $e = \ell\, e_0$ and, by inversion, $\Sigma;\emptyset \vdash e_0 : B$ and $A = \langle \ell : B; \rho \rangle$ for some ℓ, e_0, B, and ρ such that $\Sigma;\emptyset \vdash \rho : \mathsf{R}$.

If $\Sigma \mid e_0 \longrightarrow \Sigma' \mid e'_0$ for some e'_0, then we have $\Sigma';\emptyset \vdash e'_0 : B$ by the IH, and therefore $\Sigma';\emptyset \vdash \ell\, e'_0 : \langle \ell : B; \rho \rangle$ by Lemmas 62 and 25, and (T_VInj).

In what follows, we suppose that e_0 cannot be evaluated under Σ. By case analysis on the reduction rule applied to e.

Case (E_Red): We have $\ell\, e_0 = E[e'_1]$ and $e' = E[e'_2]$ for some E, e'_1, and e'_2 such that $e'_1 \rightsquigarrow e'_2$. Besides, $\Sigma' = \Sigma$. By case analysis on E.

Case $E = [\,]$: By Lemma 61.

Case $E = \ell\, E'$: Contradictory with the assumption that $e_0 = E'[e'_1]$ cannot be evaluated under Σ.

Case otherwise: Contradictory with the assumption that $\ell\, e_0 = E[e'_1]$.

Case (E_Blame): By (T_Blame).

Case (E_TyBeta): Contradictory with the assumption that e_0 cannot be evaluated under Σ.

Case (T_VLift): We have $e = \uparrow\langle \ell : B\rangle\, e_0$ and, by inversion, $\Sigma; \emptyset \vdash e_0 : \langle \rho \rangle$ and $A = \langle \ell : B; \rho\rangle$ for some ℓ, e_0, B, and ρ such that $\Sigma; \emptyset \vdash B : \mathsf{T}$.

If $\Sigma \mid e_0 \longrightarrow \Sigma' \mid e_0'$ for some e_0', then we have $\Sigma'; \emptyset \vdash e_0' : \langle \rho \rangle$ by the IH, and therefore $\Sigma'; \emptyset \vdash \uparrow\langle \ell : B\rangle\, e_0' : \langle \ell : B; \rho\rangle$ by Lemmas 62 and 25, and (T_VLift).

In what follows, we suppose that e_0 cannot be evaluated under Σ. By case analysis on the reduction rule applied to e.

Case (E_Red): We have $\uparrow\langle \ell : B\rangle\, e_0 = E[e_1']$ and $e' = E[e_2']$ for some E, e_1', and e_2' such that $e_1' \rightsquigarrow e_2'$. Besides, $\Sigma' = \Sigma$. By case analysis on E.

Case $E = [\,]$: By Lemma 61.

Case $E = \uparrow\langle \ell : B\rangle\, E'$: Contradictory with the assumption that $e_0 = E'[e_1']$ cannot be evaluated under Σ.

Case otherwise: Contradictory with the assumption that $\uparrow\langle \ell : B\rangle\, e_0 = E[e_1']$.

Case (E_Blame): By (T_Blame).

Case (E_TyBeta): Contradictory with the assumption that e_0 cannot be evaluated under Σ.

Case (T_VCase): We have $e = \mathsf{case}\, e_0\, \mathsf{with}\, \langle \ell\, x \rightarrow e_1; y \rightarrow e_2\rangle$ and, by inversion, $\Sigma; \emptyset \vdash e_0 : \langle \ell : B; \rho\rangle$ and $\Sigma; x{:}B \vdash e_1 : A$ and $\Sigma; y{:}\langle \rho\rangle \vdash e_2 : A$ for some ℓ, e_0, e_1, e_2, B, ρ, x, and y.

If $\Sigma \mid e_0 \longrightarrow \Sigma' \mid e_0'$ for some e_0', then we have $\Sigma'; \emptyset \vdash e_0' : \langle \ell : B; \rho\rangle$ by the IH, and therefore $\Sigma'; \emptyset \vdash \mathsf{case}\, e_0'\, \mathsf{with}\, \langle \ell\, x \rightarrow e_1; y \rightarrow e_2\rangle : A$ by Lemmas 62 and 25, and (T_VCase).

In what follows, we suppose that e_0 cannot be evaluated under Σ. By case analysis on the reduction rule applied to e.

Case (E_Red): We have $\mathsf{case}\, e_0\, \mathsf{with}\, \langle \ell\, x \rightarrow e_1; y \rightarrow e_2\rangle = E[e_1']$ and $e' = E[e_2']$ for some E, e_1', and e_2' such that $e_1' \rightsquigarrow e_2'$. Besides, $\Sigma' = \Sigma$. By case analysis on E.

Case $E = [\,]$: By Lemma 61.

Case $E = \mathsf{case}\, E'\, \mathsf{with}\, \langle \ell\, x \rightarrow e_1; y \rightarrow e_2\rangle$: Contradictory with the assumption that $e_0 = E'[e_1']$ cannot be evaluated under Σ.

Case otherwise: Contradictory with the assumption that $\mathsf{case}\, e_0\, \mathsf{with}\, \langle \ell\, x \rightarrow x; y \rightarrow e_2\rangle = E[e_1']$.

Case (E_Blame): By (T_Blame).

Case (E_TyBeta): Contradictory with the assumption that e_0 cannot be evaluated under Σ.

Case (T_Cast): We have $e = e_0 : B \stackrel{p}{\Rightarrow} A$ and, by inversion, $\Sigma; \emptyset \vdash e_0 : B$ and $B \simeq A$ and $\Sigma; \emptyset \vdash A : \mathsf{T}$ for some e_0, A, B, and p.

If $\Sigma \mid e_0 \longrightarrow \Sigma' \mid e_0'$ for some e_0', then we have $\Sigma'; \emptyset \vdash e_0' : B$ by the IH, and therefore $\Sigma'; \emptyset \vdash e_0' : B \stackrel{p}{\Rightarrow} A : A$ by Lemmas 62 and 25, and (T_Cast).

In what follows, we suppose that e_0 cannot be evaluated under Σ. By case analysis on the reduction rule applied to e.

Case (E_Red): We have $e_0 : B \stackrel{p}{\Rightarrow} A = E[e_1']$ and $e' = E[e_2']$ for some E, e_1', and e_2' such that $e_1' \rightsquigarrow e_2'$. Besides, $\Sigma' = \Sigma$. By case analysis on E.

Case $E = [\,]$: By Lemma 61.

Case $E = E' : B \stackrel{p}{\Rightarrow} A$: Contradictory with the assumption that $e_0 = E'[e_1']$ cannot be evaluated under Σ.

Case otherwise: Contradictory with the assumption that $e_0 : B \stackrel{p}{\Rightarrow} A = E[e_1']$.

Case (E_Blame): By (T_Blame).

Case (E_TyBeta): Contradictory with the assumption that e_0 cannot be evaluated under Σ.

Case (T_CONV): We have $e = e_0 : B \overset{\Phi}{\Rightarrow} A$ and, by inversion, $\Sigma; \emptyset \vdash e_0 : B$ and $\Sigma \vdash B \prec^{\Phi} A$ and $\Sigma; \emptyset \vdash A : \mathsf{T}$ for some e_0, A, B, and Φ.

If $\Sigma \mid e_0 \longrightarrow \Sigma' \mid e_0'$ for some e_0', then we have $\Sigma'; \emptyset \vdash e_0' : B$ by the IH, and therefore $\Sigma'; \emptyset \vdash e_0' : B \overset{\Phi}{\Rightarrow} A : A$ by Lemmas 62 and 25, and (T_CONV).

In what follows, we suppose that e_0 cannot be evaluated under Σ. By case analysis on the reduction rule applied to e.

Case (E_RED): We have $e_0 : B \overset{\Phi}{\Rightarrow} A = E[e_1']$ and $e' = E[e_2']$ for some E, e_1', and e_2' such that $e_1' \rightsquigarrow e_2'$. Besides, $\Sigma' = \Sigma$. By case analysis on E.

Case $E = [\,]$: By Lemma 61.

Case $E = E' : B \overset{\Phi}{\Rightarrow} A$: Contradictory with the assumption that $e_0 = E'[e_1']$ cannot be evaluated under Σ.

Case otherwise: Contradictory with the assumption that $e_0 : B \overset{\Phi}{\Rightarrow} A = E[e_1']$.

Case (E_BLAME): By (T_BLAME).

Case (E_TYBETA): Contradictory with the assumption that e_0 cannot be evaluated under Σ.

□

Theorem 2 (Type soundness)**.** *If $\emptyset; \emptyset \vdash e : A$ and $\emptyset \mid e \longrightarrow^* \Sigma' \mid e'$ and e' cannot be evaluated under Σ', then either e' is a value or $e' = \mathsf{blame}\, p$ for some p.*

Proof. By Lemmas 63 and 51. □

2.3 Type-preserving translation

Assumption 3. *We assume that $A \simeq A \oplus B$ and $B \simeq A \oplus B$ and that if $\Gamma \vdash A : \mathsf{T}$ and $\Gamma \vdash B : \mathsf{T}$, then $\Gamma \vdash A \oplus B : \mathsf{T}$.*

Lemma 64.

1. *If $A \triangleright B$, then $A \simeq B$. Furthermore, if $\Sigma; \Gamma \vdash A : K$, then $\Sigma; \Gamma \vdash B : K$.*
2. *If $A \triangleright [\rho]$ and $\rho \triangleright_\ell B, \rho'$, then $A \simeq [\ell : B; \rho']$. Furthermore, if $\Sigma; \Gamma \vdash A : \mathsf{T}$, then $\Sigma; \Gamma \vdash [\ell : B; \rho'] : \mathsf{T}$.*
3. *If $A \triangleright \langle \rho \rangle$ and $\rho \triangleright_\ell B, \rho'$, then $A \simeq \langle \ell : B; \rho' \rangle$. Furthermore, if $\Sigma; \Gamma \vdash A : \mathsf{T}$, then $\Sigma; \Gamma \vdash \langle \ell : B; \rho' \rangle : \mathsf{T}$.*

Proof.

1. Obvious by the definition of type matching.
2. If A is $\star$, it is trivial to show. Otherwise, $A = [\rho]$. If $\ell \in \mathit{dom}(\rho)$, then $\rho \equiv \ell : B; \rho'$. Thus, $\rho \simeq \ell : B; \rho'$ by Lemma 21. Thus, by (CE_RECORD), $[\rho] \simeq [\ell : B; \rho']$ Since $\Sigma; \Gamma \vdash \rho : \mathsf{R}$, we find that $\Sigma; \Gamma \vdash B : \mathsf{T}$ and $\Sigma; \Gamma \vdash \rho' : \mathsf{R}$. Thus, $\Sigma; \Gamma \vdash [\ell : B; \rho'] : \mathsf{T}$ by (WF_CONS) and (WF_RECORD).
3. Similarly to the case for record types.

□

Lemma 65.

1. *If $\vdash \Gamma$, then $\emptyset \vdash \Gamma$.*
2. *If $\Gamma \vdash A : K$, then $\emptyset; \Gamma \vdash A : K$.*

Proof. Straightforward by mutual induction on the derivations. □

Lemma 66. *If $\Gamma \vdash M : A \hookrightarrow e$, then $\emptyset; \Gamma \vdash e : A$.*

Proof. By induction on the derivation of $\Gamma \vdash M : A \hookrightarrow e$. The proof is straightforward by using the assumption about $\oplus$ stated in this section and Lemmas 64, 65, and 50. □

Lemma 67. *If $\Gamma \vdash M : A$, then $\Gamma \vdash M : A \hookrightarrow e$ for some e.*

Proof. Straightforward by induction on the typing derivation. □

Theorem 3. *If $\Gamma \vdash M : A$, then there exists some e such that $\Gamma \vdash M : A \hookrightarrow e$ and $\emptyset; \Gamma \vdash e : A$.*

Proof. By Lemmas 67 and 66. □

2.4 Conservativity over typing

In this section, we write Γ^s, A^s, ρ^s, M^s for typing contexts, types, rows, and terms where $\star$ and any type name do not appear.

Definition 29. *We write $\Gamma_1 \equiv \Gamma_2$ if and only if (1) $\Gamma_1 = \emptyset$ and $\Gamma_2 = \emptyset$; (2) $\Gamma_1 = \Gamma'_1, x{:}A$ and $\Gamma_2 = \Gamma'_2, x{:}B$ and $\Gamma'_1 \equiv \Gamma'_2$ and $A \equiv B$; or (3) $\Gamma_1 = \Gamma'_1, X{:}K$ and $\Gamma_2 = \Gamma'_2, X{:}K$ and $\Gamma'_1 \equiv \Gamma'_2$.*

Assumption 4. *We assume that $A^s \oplus B^s$ is defined if and only if $A^s \equiv B^s$, and if $A^s \equiv B^s$, then $A^s \oplus B^s \equiv A^s$.*

Assumption 5. *We assume that, if $A_1 \equiv A_2$ and $B_1 \equiv B_2$, then $A_1 \oplus B_1 \equiv A_2 \oplus B_2$.*

Lemma 68. *Suppose that $\Gamma \equiv \Gamma'$.*

1. *If $\vdash \Gamma$, then $\vdash \Gamma'$.*
2. *If $\Gamma \vdash A : K$, then $\Gamma' \vdash A' : K$ for any A' such that $A \equiv A'$.*
3. *If $\Gamma \vdash M : A$, then $\Gamma' \vdash M : A'$ for some A' such that $A \equiv A'$.*

We mention only the interesting cases.

Case (WFg_Cons): *We are given $\Gamma \vdash \ell : B; \rho : \mathsf{R}$ and, by inversion, $\Gamma \vdash B : \mathsf{T}$ and $\Gamma \vdash \rho : \mathsf{R}$.*

We suppose that some ρ' such that $\ell : B; \rho \equiv \rho'$ is given. Since $\ell : B; \rho \equiv \rho'$, there exists some B'' and ρ'' such that $\rho' \rhd_\ell B'', \rho''$ and $B \equiv B''$ and $\rho \equiv \rho''$. By the IHs, $\Gamma' \vdash B'' : \mathsf{T}$ and $\Gamma' \vdash \rho'' : \mathsf{R}$. Thus, $\Gamma' \vdash \ell : B''; \rho'' : \mathsf{R}$ by (WFg_Cons). We can show that $\Gamma' \vdash \rho' : \mathsf{R}$ by the fact that $\rho' \rhd_\ell B'', \rho''$.

Case (Tg_App): *We are given $\Gamma \vdash M_1\, M_2 : A$ and, by inversion, $\Gamma \vdash M_1 : A_1$ and $\Gamma \vdash M_2 : A_2$ and $A_1 \rhd A_{11} \to A$ and $A_2 \simeq A_{11}$.*

If $A = \star$, it is easy to show.

Otherwise, we can suppose that $A = A_{11} \to A$. By the IHs with Lemma 5, $\Gamma' \vdash M_1 : A'_{11} \to A'$ and $\Gamma' \vdash M_2 : A'_2$ for some A', A'_{11}, A'_2 such that $A \equiv A'$, $A_{11} \equiv A'_{11}$, and $A_2 \equiv A'_2$. By Theorem 1, $A'_2 \simeq A'_{11}$. Thus, we finish by (Tg_App).

Case (Tg_TApp): *This case uses the fact that $A \equiv B$, then $A[C/X] \equiv B[C/X]$.*

Case (Tg_VCase): *This cases uses the second assumption about $\oplus$ stated in this section.*

Lemma 69. *If $A^s \simeq B^s$, then $A^s \equiv B^s$.*

Proof. By Lemma 16, there exists some C^s such that $A^s \equiv C^s$ and $C^s \sim B^s$. Then, it is easy to show that $C^s = B^s$ by induction on the derivation of $C^s \sim B^s$. □

Lemma 70.

1. *If $\vdash \Gamma^s$, then $\vdash^s \Gamma^s$.*
2. *If $\Gamma^s \vdash A^s : K$, then $\Gamma^s \vdash^s A^s : K$.*
3. *If $\Gamma^s \vdash M^s : A$, then $\Gamma^s \vdash^s M^s : A$.*

xo

Proof. By mutual induction on the derivations.
Below are important facts to show this lemma.

1. If $\Gamma^s \vdash^s M^s : A$, then $\star$ and any type name do not appear in A.
2. If $A^s \rhd B^s$, then $A^s = B^s$.
3. If $\rho^s_1 \rhd_\ell A^s, \rho^s_2$, then $\rho^s_1 \equiv \ell : A^s; \rho^s_2$.

The case for (TG_APP) is interesting, so we mention only that case. We are given $\Gamma^s \vdash M_1^s\, M_2^s : A$ and, by inversion, $\Gamma^s \vdash M_1^s : B$ and $\Gamma^s \vdash M_2^s : C$ and $B \triangleright B_1 \to A$ and $C \simeq B_1$. By the IHs, $\Gamma^s \vdash^s M_1^s : B$ and $\Gamma^s \vdash^s M_2^s : C$. Thus, we can find $\star$ and any type name do not appear in B nor C. Thus, $B = B_1 \to A$. Since $C \simeq B_1$, we find $C \equiv B_1$ by Lemma 69. Thus, by (TS_EQUIV), $\Gamma^s \vdash^s M_2^s : B_1$. By (TS_APP), we have $\Gamma^s \vdash^s M_1^s\, M_2^s : A$.

The first assumption about $\oplus$ stated in this section is used in the case for (TG_VCASE). □

Lemma 71.

1. *If* $\vdash^s \Gamma^s$, *then* $\vdash \Gamma^s$.
2. *If* $\Gamma^s \vdash^s A^s : K$, *then* $\Gamma^s \vdash A^s : K$.
3. *If* $\Gamma^s \vdash^s M^s : A^s$, *then* $\Gamma^s \vdash M^s : B^s$ *for some* B^s *such that* $A^s \equiv B^s$.

Proof. By mutual induction on the derivations. We mention only the interesting cases.

Case (TS_APP): We are given $\Gamma^s \vdash^s M_1^s\, M_2^s : A^s$ and, by inversion, $\Gamma^s \vdash^s M_1^s : B^s \to A^s$ and $\Gamma^s \vdash^s M_2^s : B^s$. By the IHs, $\Gamma^s \vdash M_1^s : B_1^s \to A_1^s$ and $\Gamma^s \vdash M_2^s : B_2^s$ and $B^s \to A^s \equiv B_1^s \to A_1^s$ and $B^s \equiv B_2^s$ for some B_1^s, B_2^s, and A_1^s.

We have $B_1^s \to A_1^s \triangleright B_1^s \to A_1^s$. By Lemma 5 (2), we have $B^s \equiv B_1^s$ and $A^s \equiv A_1^s$. Thus, $B_2^s \equiv B_1^s$. By Lemma 22, $B_2^s \simeq B_1^s$. Thus, by (TG_APP), $\Gamma^s \vdash M_1^s\, M_2^s : A_1^s$.

Case (TS_TAPP): Similar to the case of (TS_APP); we use the fact that, if $A \equiv B$, then $A[C/X] \equiv B[C/X]$.

Case (TS_RLET): We are give $\Gamma^s \vdash^s \mathsf{let}\, \{\ell = x; y\} = M_1^s\, \mathsf{in}\, M_2^s : A^s$ and, by inversion, $\Gamma^s \vdash^s M_1^s : [\ell : B^s; \rho^s]$ and $\Gamma^s, x{:}B^s, y{:}[\rho^s] \vdash^s M_2^s : A^s$.

By the IHs with Lemma 5, $\Gamma^s \vdash M_1^s : [\ell : B_0^s; \rho_0^s]$ and $\Gamma^s, x{:}B^s, y{:}[\rho^s] \vdash M_2^s : A_0^s$ for some ρ_0^s, A_0^s, and B_0^s such that $\rho^s \equiv \rho_0^s$ and $A^s \equiv A_0^s$ and $B^s \equiv B_0^s$.

Since $\Gamma^s, x{:}B^s, y{:}[\rho^s] \equiv \Gamma^s, x{:}B_0^s, y{:}[\rho_0^s]$, we have $\Gamma^s, x{:}B_0^s, y{:}[\rho_0^s] \vdash M_2^s : A_1^s$ for some A_1^s such that $A_0^s \equiv A_1^s$ by Lemma 68. Since $A^s \equiv A_1^s$, we finish by (T_RLET).

Case (TS_VCASE): Similar to the case of (TS_RLET). This case also uses the first assumption about $\oplus$ stated in this section.

□

Theorem 4. 1. *If* $\Gamma^s \vdash M^s : A^s$, *then* $\Gamma^s \vdash^s M^s : A^s$.

2. *If* $\Gamma^s \vdash^s M^s : A^s$, *then* $\Gamma^s \vdash M^s : B^s$ *for some* B^s *such that* $A^s \equiv B^s$.

Proof. By Lemmas 70 and 71. □