Supplementary Material for
"Space-Efficient Polymorphic Gradual Typing, Mostly Parametric"

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A Definition: Polymorphic Coercion Calculus $\lambda\mathbf{C}^{\forall}_{mp}$

A.1 Syntax

Base types	ι		Blame labels p, q Type variables X, Y, Z
Types	A, B, C	::=	$\iota \mid \star \mid A \to B \mid \forall X.A \mid X \mid \alpha$
Ground types	G, H	::=	$\iota \mid \star \to \star \mid \forall X.\star \mid X \mid \alpha$
Coercions	c, d	::=	$\mathsf{id}_A \mid G! \mid G?^p \mid \alpha^- \mid \alpha^+ \mid c \to d \mid c\,;d \mid \forall X.c \mid \bot^p_{A \rightsquigarrow B}$
No-op coercions	c^I	::=	$\mathsf{id}_A \mid \alpha^- \mid \alpha^+ \mid c^I \to d^I \mid \forall X.c^I \mid c^I\,; d^I$
Terms	M	::=	$k \mid x \mid \lambda x\!:\!A.M \mid M\,M \mid \Lambda X.(M : A) \mid M\,A \mid M\langle c\rangle \mid \mathsf{blame}\,p$
Values	V	::=	$k \mid \lambda x\!:\!A.M \mid \Lambda X.(M : A) \mid V\langle G!\rangle \mid V\langle \alpha^-\rangle \mid V\langle c \to d\rangle \mid V\langle \forall X.c\rangle$
Evaluation frames	E	::=	$\Box\,M \mid V\,\Box \mid \Box\,A \mid \Box\langle c\rangle$
Evaluation contexts	F	::=	$\Box \mid F\,M \mid V\,F \mid F\,A \mid F\langle c\rangle$
Contexts	$\mathcal{C}_C$	::=	$\Box \mid \lambda x\!:\!A.\mathcal{C}_C \mid \mathcal{C}_C\,M \mid M\,\mathcal{C}_C \mid \Lambda X.(\mathcal{C}_C : A) \mid \mathcal{C}_C\,A \mid \mathcal{C}_C\,\langle c\rangle$
Type environments	Γ	::=	$\emptyset \mid \Gamma, x : A \mid \Gamma, X$
Stores	Σ	::=	$\emptyset \mid \Sigma, \alpha := \mathbb{A}$

Definition A.1 (Non-Dynamic Types)**.** We use metavariables $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ to denote types that are not the dynamic type $\star$.

Definition A.2 (Free Type Variables and Type Substitution)**.** We define the notion of free type variables in a standard way. The notation $\mathrm{ftv}(A)$ denotes the set of type variables occurring free in type A. We write $A[X := B]$ for capture-avoiding substitution of type B for free type variable X in type A. We write $M[X := \alpha]$ and $c[X := \alpha]$ for substitution of type name α for free type variable X in term M and coercion c, respectively. We write $M[X := \star]$ and $c[X := \star]$ for substitution of type $\star$ for free type variable X in term M and coercion c, respectively. It is defined in a standard manner, as substitution of type names, except for the case that coercion c is a projection or injection; in such a case, the substitution of $\star$ is defined as follows:

$$G?^p[X := \star] \stackrel{\text{def}}{=} \begin{cases} \mathsf{id}_\star & (\text{if } G = X) \\ G?^p & (\text{if } G \neq X) \end{cases}$$

$$G![X := \star] \stackrel{\text{def}}{=} \begin{cases} \mathsf{id}_\star & (\text{if } G = X) \\ G! & (\text{if } G \neq X) \end{cases}$$

Note that $V[X := \star]$ is not a value in general because, if $V = V'\langle X!\rangle$, then $V[X := \star] = V'[X := \star]\langle \mathsf{id}_\star\rangle$, which is not a value. However, if V is closed, it contains no coercion of the form $X!$ or $X?^p$ for any type variable X free in V, and, therefore, $V[X := \star]$ is still a value.

Definition A.3 (Types and denotations of constants)**.** We assume a meta-level function ty that assigns a first-order type of the form $\iota_1 \to \iota_2 \to \cdots \to \iota_n (n \geq 1)$ to every constant, and a meta-level partial function δ that maps pairs of constants to constants. We also suppose δ to respect function ty in the sense that, for any constant k_1 and k_2, if $ty(k_1) = \iota \to A$ and $ty(k_2) = \iota$, then $\delta(k_1, k_2)$ is defined and $ty(\delta(k_1, k_2)) = A$.

Definition A.4 (Notation for Type Environments and Stores)**.** We write $\mathrm{dom}(\Gamma)$ for the set of variables and type variables bound by type environment Γ, and $\mathrm{dom}(\Sigma)$ for the set of type names bound by store Σ. We use the notations $\Gamma_1 \# \Gamma_2$ and $\Sigma_1 \# \Sigma_2$ to denote that their domains are disjoint, that is, $\mathrm{dom}(\Gamma_1) \cap \mathrm{dom}(\Gamma_2) = \emptyset$ and $\mathrm{dom}(\Sigma_1) \cap \mathrm{dom}(\Sigma_2) = \emptyset$, respectively. We write $\Sigma_1 \supseteq \Sigma_2$ if and only if $\alpha := \mathbb{A} \in \Sigma_2$ implies $\alpha := \mathbb{A} \in \Sigma_1$ for any α and $\mathbb{A}$. Let Δ contains only type variables. We write $\Gamma \setminus \Delta$ for the environment obtained by removing all type variables in Δ from Γ.

A.2 Dynamic semantics

Definition A.5 (Reduction)**.** The reduction relation $\longrightarrow$ is a binary relation over pairs of a store and a term. It is the smallest relation satisfying the rules shown in Section A.2.2 (where stores Σ are omitted if they are not important). We use the notation $\longrightarrow^*$ to denote the reflexive, transitive closure of $\longrightarrow$, and $\Sigma \triangleright M \Uparrow$ to denote that term M diverges under store Σ, that is, for any Σ' and M' such that $\Sigma \triangleright M \longrightarrow^* \Sigma' \triangleright M'$, there exist some Σ'' and M'' such that $\Sigma' \triangleright M' \longrightarrow \Sigma'' \triangleright M''$. We may write $\longrightarrow_C$ and $\longrightarrow_C^*$ to emphasize the reduction in $\lambda C^{\forall}_{mp}$.

A.2.1 Coercion generation function $\boxed{coerce^+_\alpha(A) = c, coerce^-_\alpha(A) = c}$

$$
\begin{array}{rcll}
coerce^+_\alpha(\iota) & = & \mathsf{id}_\iota & \\
coerce^+_\alpha(\star) & = & \mathsf{id}_\star & \\
coerce^+_\alpha(A \to B) & = & coerce^-_\alpha(A) \to coerce^+_\alpha(B) & \\
coerce^+_\alpha(\forall X.A) & = & \forall X.(coerce^+_\alpha(A)) & \\
coerce^+_\alpha(X) & = & \mathsf{id}_X & \\
coerce^+_\alpha(\beta) & = & \mathsf{id}_\beta & \text{if } \beta \neq \alpha \\
coerce^+_\alpha(\alpha) & = & \alpha^+ &
\end{array}
$$

$$
\begin{array}{rcll}
coerce^-_\alpha(\iota) & = & \mathsf{id}_\iota & \\
coerce^-_\alpha(\star) & = & \mathsf{id}_\star & \\
coerce^-_\alpha(A \to B) & = & coerce^+_\alpha(A) \to coerce^-_\alpha(B) & \\
coerce^-_\alpha(\forall X.A) & = & \forall X.(coerce^-_\alpha(A)) & \\
coerce^-_\alpha(X) & = & \mathsf{id}_X & \\
coerce^-_\alpha(\beta) & = & \mathsf{id}_\beta & \text{if } \beta \neq \alpha \\
coerce^-_\alpha(\alpha) & = & \alpha^- &
\end{array}
$$

A.2.2 Reduction $\boxed{\Sigma \triangleright M \longrightarrow \Sigma' \triangleright M'}$

$$
\begin{array}{rcllr}
k_1\ k_2 & \longrightarrow & \delta(k_1, k_2) & & (\text{R\_DELTA\_C}) \\
(\lambda x : A.M)\ V & \longrightarrow & M[x := V] & & (\text{R\_BETA\_C}) \\
V\langle \mathsf{id}_A \rangle & \longrightarrow & V & & (\text{R\_ID\_C}) \\
(V\langle c \to d \rangle)\ V' & \longrightarrow & (V\ (V'\langle c \rangle))\langle d \rangle & & (\text{R\_WRAP\_C}) \\
V\langle G! \rangle\ \langle G?^p \rangle & \longrightarrow & V & & (\text{R\_COLLAPSE\_C}) \\
V\langle G! \rangle\ \langle H?^p \rangle & \longrightarrow & \mathsf{blame}\ p & \text{if } G \neq H & (\text{R\_CONFLICT\_C}) \\
V\langle \alpha^- \rangle\ \langle \alpha^+ \rangle & \longrightarrow & V & & (\text{R\_REMOVE\_C}) \\
V\langle c\ ;\ d \rangle & \longrightarrow & V\langle c \rangle\ \langle d \rangle & & (\text{R\_SPLIT\_C}) \\
V\langle \bot^p_{A \rightsquigarrow B} \rangle & \longrightarrow & \mathsf{blame}\ p & & (\text{R\_FAIL\_C}) \\
(\Lambda X.(M : A_0))\overline{\langle \forall X.c \rangle}\ \star & \longrightarrow & (M\overline{\langle c \rangle})[X := \star] & & (\text{R\_TYBETADYN\_C})
\end{array}
$$

$$
\frac{\Sigma \vdash \overline{\langle \forall X.c \rangle} : \forall X.A_0 \rightsquigarrow \forall X.A_n \quad \alpha \notin \mathrm{dom}(\Sigma)}{\begin{array}{l}\Sigma \triangleright (\Lambda X.(M : A_0))\overline{\langle \forall X.c \rangle}\ \mathbb{B} \\ \qquad \longrightarrow \Sigma, \alpha := \mathbb{B} \triangleright (M\overline{\langle c \rangle})[X := \alpha]\langle coerce^+_\alpha(A_n[X := \alpha]) \rangle\end{array}}\ (\text{R\_TYBETA\_C})
$$

$$
E[\mathsf{blame}\ p] \longrightarrow \mathsf{blame}\ p\ \ (\text{R\_BLAME\_C}) \qquad \frac{\Sigma \triangleright M \longrightarrow \Sigma' \triangleright M'}{\Sigma \triangleright E[M] \longrightarrow \Sigma' \triangleright E[M']}\ (\text{R\_CTX\_C})
$$

A.3 Type system

This section defines typing rules in $\lambda C^{\forall}_{mp}$. We may write $\Sigma \mid \Gamma \vdash_C M : A$ for typing judgment $\Sigma \mid \Gamma \vdash M : A$ in $\lambda C^{\forall}_{mp}$ for clarity.

A.3.1 Type well-formedness $\boxed{\Sigma \mid \Gamma \vdash A}$

$$\Sigma \mid \Gamma \vdash \iota \ (\textsc{Tw\_Base}) \qquad \Sigma \mid \Gamma \vdash \star \ (\textsc{Tw\_Star}) \qquad \frac{\Sigma \mid \Gamma \vdash A \qquad \Sigma \mid \Gamma \vdash B}{\Sigma \mid \Gamma \vdash A \to B} \ (\textsc{Tw\_Arrow})$$

$$\frac{\alpha \in \mathrm{dom}(\Sigma)}{\Sigma \mid \Gamma \vdash \alpha} \ (\textsc{Tw\_Name}) \qquad \frac{X \in \Gamma}{\Sigma \mid \Gamma \vdash X} \ (\textsc{Tw\_Var}) \qquad \frac{\Sigma \mid \Gamma, X \vdash A}{\Sigma \mid \Gamma \vdash \forall X.A} \ (\textsc{Tw\_Poly})$$

A.3.2 Store well-formedness $\boxed{\vdash \Sigma}$

$$\vdash \emptyset \ (\textsc{Sw\_Empty}) \qquad \frac{\vdash \Sigma \qquad \Sigma \mid \emptyset \vdash \mathbb{A} \qquad \alpha \notin \mathrm{dom}(\Sigma)}{\vdash \Sigma, \alpha := \mathbb{A}} \ (\textsc{Sw\_Binding})$$

A.3.3 Type environment well-formedness $\boxed{\Sigma \vdash \Gamma}$

$$\Sigma \vdash \emptyset \ (\textsc{Tew\_Empty}) \qquad \frac{\Sigma \vdash \Gamma \qquad \Sigma \mid \Gamma \vdash A \qquad x \notin \mathrm{dom}(\Gamma)}{\Sigma \vdash \Gamma, x : A} \ (\textsc{Tew\_Var})$$

$$\frac{\Sigma \vdash \Gamma \qquad X \notin \mathrm{dom}(\Gamma)}{\Sigma \vdash \Gamma, X} \ (\textsc{Tew\_Tyvar})$$

A.3.4 Coercion typing $\boxed{\Sigma \mid \Gamma \vdash c : A \rightsquigarrow B}$

$$\frac{\vdash \Sigma \qquad \Sigma \vdash \Gamma \qquad \Sigma \mid \Gamma \vdash A}{\Sigma \mid \Gamma \vdash \mathsf{id}_A : A \rightsquigarrow A} \ (\textsc{Ct\_Id\_C}) \qquad \frac{\vdash \Sigma \qquad \Sigma \vdash \Gamma \qquad \Sigma \mid \Gamma \vdash A \qquad \Sigma \mid \Gamma \vdash B}{\Sigma \mid \Gamma \vdash \bot^p_{A \rightsquigarrow B} : A \rightsquigarrow B} \ (\textsc{Ct\_Fail\_C})$$

$$\frac{\vdash \Sigma \qquad \Sigma \vdash \Gamma \qquad \Sigma \mid \Gamma \vdash G}{\Sigma \mid \Gamma \vdash G! : G \rightsquigarrow \star} \ (\textsc{Ct\_Inj\_C}) \qquad \frac{\vdash \Sigma \qquad \Sigma \vdash \Gamma \qquad \Sigma \mid \Gamma \vdash G}{\Sigma \mid \Gamma \vdash G?^p : \star \rightsquigarrow G} \ (\textsc{Ct\_Proj\_C})$$

$$\frac{\vdash \Sigma \qquad \Sigma \vdash \Gamma \qquad \alpha := \mathbb{A} \in \Sigma}{\Sigma \mid \Gamma \vdash \alpha^- : \mathbb{A} \rightsquigarrow \alpha} \ (\textsc{Ct\_Conceal\_C}) \qquad \frac{\vdash \Sigma \qquad \Sigma \vdash \Gamma \qquad \alpha := \mathbb{A} \in \Sigma}{\Sigma \mid \Gamma \vdash \alpha^+ : \alpha \rightsquigarrow \mathbb{A}} \ (\textsc{Ct\_Reveal\_C})$$

$$\frac{\Sigma \mid \Gamma \vdash c : A' \rightsquigarrow A \qquad \Sigma \mid \Gamma \vdash d : B \rightsquigarrow B'}{\Sigma \mid \Gamma \vdash c \to d : (A \to B) \rightsquigarrow (A' \to B')} \ (\textsc{Ct\_Arrow\_C}) \qquad \frac{\Sigma \mid \Gamma, X \vdash c : A \rightsquigarrow B}{\Sigma \mid \Gamma \vdash \forall X.c : \forall X.A \rightsquigarrow \forall X.B} \ (\textsc{Ct\_All\_C})$$

$$\frac{\Sigma \mid \Gamma \vdash c : A \rightsquigarrow B \qquad \Sigma \mid \Gamma \vdash d : B \rightsquigarrow C}{\Sigma \mid \Gamma \vdash c\,;d : A \rightsquigarrow C} \ (\textsc{Ct\_Seq\_C})$$

A.3.5 Coercion sequence typing $\boxed{\Sigma \vdash \overline{\langle c \rangle} : A \rightsquigarrow B}$

$$\frac{}{\Sigma \vdash \emptyset : A \rightsquigarrow A} \ (\textsc{Ct\_Nil\_C}) \qquad \frac{\Sigma \vdash \overline{\langle c' \rangle} : A \rightsquigarrow B \qquad \Sigma \mid \emptyset \vdash c : B \rightsquigarrow C}{\Sigma \vdash \overline{\langle c' \rangle}, \langle c \rangle : A \rightsquigarrow C} \ (\textsc{Ct\_Cons\_C})$$

A.3.6 Term typing $\boxed{\Sigma \mid \Gamma \vdash M : A}$

$$\frac{\vdash \Sigma \qquad \Sigma \vdash \Gamma \qquad ty(k) = A}{\Sigma \mid \Gamma \vdash k : A} \ (\textsc{T\_Const\_C}) \qquad \frac{\vdash \Sigma \qquad \Sigma \vdash \Gamma \qquad x : A \in \Gamma}{\Sigma \mid \Gamma \vdash x : A} \ (\textsc{T\_Var\_C})$$

$$\frac{\Sigma \mid \Gamma, x : A \vdash M : B}{\Sigma \mid \Gamma \vdash \lambda x{:}A.M : A \to B} \ (\textsc{T\_Abs\_C}) \qquad \frac{\Sigma \mid \Gamma \vdash M_1 : A \to B \qquad \Sigma \mid \Gamma \vdash M_2 : A}{\Sigma \mid \Gamma \vdash M_1\, M_2 : B} \ (\textsc{T\_App\_C})$$

$$\frac{\Sigma \mid \Gamma, X \vdash M : A}{\Sigma \mid \Gamma \vdash \Lambda X.(M : A) : \forall X.A} \text{ (T\_TYABS\_C)} \qquad \frac{\Sigma \mid \Gamma \vdash M : \forall X.B \qquad \Sigma \mid \Gamma \vdash A}{\Sigma \mid \Gamma \vdash M\,A : B[X := A]} \text{ (T\_TYAPP\_C)}$$

$$\frac{\vdash \Sigma \qquad \Sigma \vdash \Gamma \qquad \Sigma \mid \Gamma \vdash A}{\Sigma \mid \Gamma \vdash \mathsf{blame}\, p : A} \text{ (T\_BLAME\_C)} \qquad \frac{\Sigma \mid \Gamma \vdash M : A \qquad \Sigma \mid \Gamma \vdash c : A \rightsquigarrow B}{\Sigma \mid \Gamma \vdash M\langle c \rangle : B} \text{ (T\_CRC\_C)}$$

A.4 Context typing $\boxed{\Sigma \vdash_C \mathcal{C}_C : (\Gamma \vdash A) \Rightarrow (\Gamma' \vdash B)}$

$$\Sigma \vdash_C \square : (\Gamma \vdash A) \Rightarrow (\Gamma \vdash A) \text{ (CTXT\_HOLE\_C)}$$

$$\frac{\Sigma \vdash_C \mathcal{C}_C : (\Gamma \vdash A) \Rightarrow (\Gamma', x : A' \vdash B)}{\Sigma \vdash_C \lambda x{:}A'.\mathcal{C}_C : (\Gamma \vdash A) \Rightarrow (\Gamma' \vdash A' \to B)} \text{ (CTXT\_ABS\_C)}$$

$$\frac{\Sigma \vdash_C \mathcal{C}_C : (\Gamma \vdash A) \Rightarrow (\Gamma' \vdash B \to C) \qquad \Sigma \mid \Gamma' \vdash_C M : B}{\Sigma \vdash_C \mathcal{C}_C\, M : (\Gamma \vdash A) \Rightarrow (\Gamma' \vdash C)} \text{ (CTXT\_APP1\_C)}$$

$$\frac{\Sigma \mid \Gamma' \vdash_C M : B \to C \qquad \Sigma \vdash_C \mathcal{C}_C : (\Gamma \vdash A) \Rightarrow (\Gamma' \vdash B)}{\Sigma \vdash_C M\, \mathcal{C}_C : (\Gamma \vdash A) \Rightarrow (\Gamma' \vdash C)} \text{ (CTXT\_APP2\_C)}$$

$$\frac{\Sigma \vdash_C \mathcal{C}_C : (\Gamma \vdash A) \Rightarrow (\Gamma', X \vdash A')}{\Sigma \vdash_C \Lambda X.(\mathcal{C}_C : A') : (\Gamma \vdash A) \Rightarrow (\Gamma' \vdash \forall X.A')} \text{ (CTXT\_TYABS\_C)}$$

$$\frac{\Sigma \vdash_C \mathcal{C}_C : (\Gamma \vdash A) \Rightarrow (\Gamma' \vdash \forall X.B) \qquad \Sigma \mid \Gamma' \vdash A'}{\Sigma \vdash_C \mathcal{C}_C\, A' : (\Gamma \vdash A) \Rightarrow (\Gamma' \vdash B[X := A'])} \text{ (CTXT\_TYAPP\_C)}$$

$$\frac{\Sigma \vdash_C \mathcal{C}_C : (\Gamma \vdash A) \Rightarrow (\Gamma' \vdash B) \qquad \Sigma \mid \Gamma' \vdash_C c : B \rightsquigarrow C}{\Sigma \vdash_C \mathcal{C}_C\,\langle c \rangle : (\Gamma \vdash A) \Rightarrow (\Gamma' \vdash C)} \text{ (CTXT\_CRC\_C)}$$

A.5 Logical relation

Definition A.6 (Mappings)**.** We use metavariable $\alpha^\star$ to denote type names or the dynamic type $\star$. Metavariables ρ, κ, and θ range over finite mappings from type variables to type names or $\star$ (i.e., types ranged over by $\alpha^\star$), ones from type names to relations in $\bigcup_{n \geq 0} \mathrm{Rel}_n$, and ones from variables to pairs of values, respectively. For $X \notin \mathrm{dom}(\rho)$, $\alpha \notin \mathrm{dom}(\kappa)$, and $x \notin \mathrm{dom}(\theta)$, we write $\rho\{X \mapsto \alpha^\star\}$, $\kappa\{\alpha \mapsto R\}$, and $\theta\{x \mapsto (V_1, V_2)\}$ for the mapping that is the same as ρ, κ, and θ except that X, α, and x is mapped to $\alpha^\star$, R, and (V_1, V_2), respectively. We write θ^1 and θ^2 for the substitutions that map variable x to values V_1 and V_2, respectively, if θ maps x to (V_1, V_2).

Figure 2 defines the logical relation, and Figure 1 defines the auxiliary definitions for it.

A.6 Contextual equivalence

Definition A.7 (Contextual equivalence for terms)**.** Terms M_1 and M_2 of type A are *contextually equivalent* under store Σ and type environment Γ, written $\Sigma \mid \Gamma \vdash_C M_1 \overset{\text{ctx}}{=} M_2 : A$, if $\Sigma \mid \Gamma \vdash_C M_1 : A$ and $\Sigma \mid \Gamma \vdash_C M_2 : A$ and, for any context $\mathcal{C}_C$ and type B, $\Sigma \vdash_C \mathcal{C}_C : (\Gamma \vdash A) \Rightarrow (\emptyset \vdash B)$ implies one of the followings:

- $\Sigma \triangleright \mathcal{C}_C[M_1] \longrightarrow_C^* \Sigma_1 \triangleright V_1$ and $\Sigma \triangleright \mathcal{C}_C[M_2] \longrightarrow_C^* \Sigma_2 \triangleright V_2$ for some values V_1 and V_2, and stores Σ_1 and Σ_2,
- $\Sigma \triangleright \mathcal{C}_C[M_1] \longrightarrow_C^* \Sigma_1 \triangleright \mathsf{blame}\, p$ and $\Sigma \triangleright \mathcal{C}_C[M_2] \longrightarrow_C^* \Sigma_2 \triangleright \mathsf{blame}\, p$ for some label p, and stores Σ_1 and Σ_2, or
- $\Sigma \triangleright \mathcal{C}_C[M_1] \Uparrow$ and $\Sigma \triangleright \mathcal{C}_C[M_2] \Uparrow$.

$$
\begin{array}{lcl}
\text{Atom}_n\,[\![A_1, A_2]\!] & \stackrel{\text{def}}{=} & \{(W, M_1, M_2) \mid W \in \text{World}_n \,\wedge\, W.\Sigma_1 \mid \emptyset \vdash M_1 : A_1 \,\wedge\, W.\Sigma_2 \mid \emptyset \vdash M_2 : A_2\} \\
\text{Atom}^{\text{val}}_n\,[\![A_1, A_2]\!] & \stackrel{\text{def}}{=} & \{(W, V_1, V_2) \mid (W, V_1, V_2) \in \text{Atom}_n\,[\![A_1, A_2]\!]\} \\
\text{Atom}\,[\![A]\!]\,\rho & \stackrel{\text{def}}{=} & \bigcup_{n\geq 0} \text{Atom}_n\,[\![\rho(A), \rho(A)]\!] \\
\text{Atom}^{\text{val}}\,[\![A]\!]\,\rho & \stackrel{\text{def}}{=} & \bigcup_{n\geq 0} \text{Atom}^{\text{val}}_n\,[\![\rho(A), \rho(A)]\!] \\
\text{Rel}_n\,[\![A_1, A_2]\!] & \stackrel{\text{def}}{=} & \{R \subseteq \text{Atom}^{\text{val}}_n\,[\![A_1, A_2]\!] \mid \\
 & & \quad \forall\,(W_1, V_1, V_2) \in R.\ \forall\, W_2 \sqsupseteq W_1.\ (W_2, V_1, V_2) \in R\,\} \\
\text{Rel}_n & \stackrel{\text{def}}{=} & \bigcup_{A_1, A_2} \text{Rel}_n\,[\![A_1, A_2]\!] \\
\text{World}_n & \stackrel{\text{def}}{=} & \{(m, \Sigma_1, \Sigma_2, \kappa) \in \mathit{Nat} \times \mathit{TNStore} \times \mathit{TNStore} \times (\mathit{TyName} \rightharpoonup \text{Rel}_m) \mid \\
 & & \quad m < n \,\wedge \vdash \Sigma_1 \,\wedge \vdash \Sigma_2 \,\wedge\, \forall\,\alpha \in \text{dom}(\kappa).\ \kappa(\alpha) \in \text{Rel}_m\,[\![\Sigma_1(\alpha), \Sigma_2(\alpha)]\!]\} \\
\text{World} & \stackrel{\text{def}}{=} & \bigcup_{n\geq 0} \text{World}_n \\
 & & \\
\lfloor R \rfloor_n & \stackrel{\text{def}}{=} & \{(W, M_1, M_2) \in R \mid W.n < n\} \\
\lfloor \kappa \rfloor_n & \stackrel{\text{def}}{=} & \{\alpha \mapsto \lfloor \kappa(\alpha) \rfloor_n \mid \alpha \in \text{dom}(\kappa)\} \\
W_1 \sqsupseteq W_2 & \stackrel{\text{def}}{=} & W_1.n \leq W_2.n \,\wedge\, W_1.\Sigma_1 \supseteq W_2.\Sigma_1 \,\wedge\, W_1.\Sigma_2 \supseteq W_2.\Sigma_2 \,\wedge\, W_1.\kappa \sqsupseteq \lfloor W_2.\kappa \rfloor_{W_1.n} \,\wedge \\
 & & W_1, W_2 \in \text{World} \\
W_1 \sqsupseteq_n W_2 & \stackrel{\text{def}}{=} & W_1.n = W_2.n - n \,\wedge\, W_1 \sqsupseteq W_2 \\
\kappa_1 \sqsupseteq \kappa_2 & \stackrel{\text{def}}{=} & \forall\,\alpha \in \text{dom}(\kappa_2).\ \kappa_1(\alpha) = \kappa_2(\alpha) \\
\blacktriangleright(n+1, \Sigma_1, \Sigma_2, \kappa) & \stackrel{\text{def}}{=} & (n, \Sigma_1, \Sigma_2, \lfloor \kappa \rfloor_n) \\
\blacktriangleright R & \stackrel{\text{def}}{=} & \{(W, M_1, M_2) \mid W.n > 0 \implies (\blacktriangleright W, M_1, M_2) \in R\} \\
W \boxplus (\alpha, \mathbb{A}_1, \mathbb{A}_2, R) & \stackrel{\text{def}}{=} & (W.n, (W.\Sigma_1, \alpha := \mathbb{A}_1), (W.\Sigma_2, \alpha := \mathbb{A}_2), W.\kappa\{\alpha \mapsto R\})
\end{array}
$$

Figure 1: Auxiliary definitions for logical relation.

Definition A.8 (Contextual equivalence for coercion sequences)**.** Coercion sequences $\overline{\langle c \rangle}$ and $\overline{\langle d \rangle}$ from type A to type B are *contextually equivalent* under store Σ and type environment Γ, written $\Sigma \mid \Gamma \vdash \overline{\langle c \rangle} \stackrel{\text{ctx}}{=} \overline{\langle d \rangle} : A \rightsquigarrow B$, if $\Sigma \mid \Gamma \vdash_C \lambda x : A.x\overline{\langle c \rangle} \stackrel{\text{ctx}}{=} \lambda x : A.x\overline{\langle d \rangle} : A \to B$.

$$
\begin{array}{lcl}
\mathcal{V}\llbracket \iota \rrbracket \rho & \stackrel{\text{def}}{=} & \{(W, V_1, V_2) \in \mathrm{Atom}^{\mathrm{val}}\llbracket \iota \rrbracket \rho \mid \exists k.\ V_1 = k \wedge V_2 = k\} \\
\mathcal{V}\llbracket A \to B \rrbracket \rho & \stackrel{\text{def}}{=} & \{(W, V_1, V_2) \in \mathrm{Atom}^{\mathrm{val}}\llbracket A \to B \rrbracket \rho \mid \forall W' \sqsupseteq W.\ \forall V_1', V_2'. \\
 & & \quad (W', V_1', V_2') \in \mathcal{V}\llbracket A \rrbracket \rho \Longrightarrow (W', V_1\, V_1', V_2\, V_2') \in \mathcal{E}\llbracket B \rrbracket \rho\} \\
\mathcal{V}\llbracket \forall X. A \rrbracket \rho & \stackrel{\text{def}}{=} & \{(W, V_1, V_2) \in \mathrm{Atom}^{\mathrm{val}}\llbracket \forall X. A \rrbracket \rho \mid \\
 & & \quad (\forall W' \sqsupseteq W.\ \forall \mathbb{B}_1, \mathbb{B}_2, R, M_1, M_2, \alpha. \\
 & & \qquad W'.\Sigma_1 \mid \emptyset \vdash \mathbb{B}_1 \wedge W'.\Sigma_2 \mid \emptyset \vdash \mathbb{B}_2 \wedge R \in \mathrm{Rel}_{W'.n}\llbracket \mathbb{B}_1, \mathbb{B}_2 \rrbracket \wedge \\
 & & \qquad W'.\Sigma_1 \triangleright V_1\, \mathbb{B}_1 \longrightarrow W'.\Sigma_1, \alpha := \mathbb{B}_1 \triangleright M_1 \langle \mathit{coerce}^{+}_{\alpha}(\rho(A)[X := \alpha]) \rangle \wedge \\
 & & \qquad W'.\Sigma_2 \triangleright V_2\, \mathbb{B}_2 \longrightarrow W'.\Sigma_2, \alpha := \mathbb{B}_2 \triangleright M_2 \langle \mathit{coerce}^{+}_{\alpha}(\rho(A)[X := \alpha]) \rangle \\
 & & \qquad \Longrightarrow (W' \boxplus (\alpha, \mathbb{B}_1, \mathbb{B}_2, R), M_1, M_2) \in \blacktriangleright \mathcal{E}\llbracket A \rrbracket \rho\{X \mapsto \alpha\}) \wedge \\
 & & \quad \forall W' \sqsupseteq W.\ (W', V_1 \star, V_2 \star) \in \mathcal{E}\llbracket A \rrbracket \rho\{X \mapsto \star\}\} \\
\mathcal{V}\llbracket X \rrbracket \rho & \stackrel{\text{def}}{=} & \mathcal{V}\llbracket \rho(X) \rrbracket \rho \\
\mathcal{V}\llbracket \alpha \rrbracket \rho & \stackrel{\text{def}}{=} & \{(W, V_1\langle \alpha^- \rangle, V_2\langle \alpha^- \rangle) \in \mathrm{Atom}^{\mathrm{val}}\llbracket \alpha \rrbracket \emptyset \mid (W, V_1, V_2) \in \blacktriangleright(W.\kappa(\alpha))\} \\
\mathcal{V}\llbracket \star \rrbracket \rho & \stackrel{\text{def}}{=} & \{(W, V_1\langle G! \rangle, V_2\langle G! \rangle) \in \mathrm{Atom}^{\mathrm{val}}\llbracket \star \rrbracket \emptyset \mid (W, V_1, V_2) \in \blacktriangleright \mathcal{V}\llbracket G \rrbracket \emptyset\} \\
\mathcal{E}\llbracket A \rrbracket \rho & \stackrel{\text{def}}{=} & \{(W, M_1, M_2) \in \mathrm{Atom}\llbracket A \rrbracket \rho \mid \forall n < W.n. \\
 & & \quad (\forall \Sigma_1, V_1.\ W.\Sigma_1 \triangleright M_1 \longrightarrow^n \Sigma_1 \triangleright V_1 \Longrightarrow \\
 & & \qquad \exists W', V_2.\ W.\Sigma_2 \triangleright M_2 \longrightarrow^* W'.\Sigma_2 \triangleright V_2 \wedge W' \sqsupseteq_n W \wedge \\
 & & \qquad\qquad W'.\Sigma_1 = \Sigma_1 \wedge (W', V_1, V_2) \in \mathcal{V}\llbracket A \rrbracket \rho) \wedge \\
 & & \quad (\forall \Sigma_1, p.\ W.\Sigma_1 \triangleright M_1 \longrightarrow^n \Sigma_1 \triangleright \mathsf{blame}\, p \Longrightarrow \exists \Sigma_2.\ W.\Sigma_2 \triangleright M_2 \longrightarrow^* \Sigma_2 \triangleright \mathsf{blame}\, p)\} \\
\mathcal{G}\llbracket \emptyset \rrbracket & \stackrel{\text{def}}{=} & \{(W, \emptyset, \emptyset) \mid W \in \mathrm{World}\} \\
\mathcal{G}\llbracket \Gamma, x : A \rrbracket & \stackrel{\text{def}}{=} & \{(W, \theta\{x \mapsto (V_1, V_2)\}, \rho) \mid (W, \theta, \rho) \in \mathcal{G}\llbracket \Gamma \rrbracket \wedge (W, V_1, V_2) \in \mathcal{V}\llbracket A \rrbracket \rho\} \\
\mathcal{G}\llbracket \Gamma, X \rrbracket & \stackrel{\text{def}}{=} & \{(W, \theta, \rho\{X \mapsto \alpha\}) \mid (W, \theta, \rho) \in \mathcal{G}\llbracket \Gamma \rrbracket \wedge \alpha \in \mathrm{dom}(W.\kappa)\} \cup \\
 & & \{(W, \theta, \rho\{X \mapsto \star\}) \mid (W, \theta, \rho) \in \mathcal{G}\llbracket \Gamma \rrbracket\} \\
\mathcal{S}\llbracket \emptyset \rrbracket & \stackrel{\text{def}}{=} & \mathrm{World} \\
\mathcal{S}\llbracket \Sigma, \alpha := \mathbb{A} \rrbracket & \stackrel{\text{def}}{=} & \{W \in \mathcal{S}\llbracket \Sigma \rrbracket \mid W.\Sigma_1(\alpha) = \mathbb{A} \wedge W.\Sigma_2(\alpha) = \mathbb{A} \wedge W.\kappa(\alpha) = \lfloor \mathcal{V}\llbracket \mathbb{A} \rrbracket \emptyset \rfloor_{W.n}\} \\
\Sigma \mid \Gamma \vdash M_1 \preceq M_2 : A & \stackrel{\text{def}}{=} & \Sigma \mid \Gamma \vdash M_1 : A \wedge \Sigma \mid \Gamma \vdash M_2 : A \wedge \\
 & & \forall W, \theta, \rho.\ W \in \mathcal{S}\llbracket \Sigma \rrbracket \wedge (W, \theta, \rho) \in \mathcal{G}\llbracket \Gamma \rrbracket \Longrightarrow (W, \rho(\theta^1(M_1)), \rho(\theta^2(M_2))) \in \mathcal{E}\llbracket A \rrbracket \rho \\
\Sigma \mid \Gamma \vdash M_1 \approx M_2 : A & \stackrel{\text{def}}{=} & \Sigma \mid \Gamma \vdash M_1 \preceq M_2 : A \wedge \Sigma \mid \Gamma \vdash M_2 \preceq M_1 : A
\end{array}
$$

Figure 2: Logical relation.

B Definition: Space-Efficient Polymorphic Coercion Calculus $\lambda\mathrm{S}^{\forall}_{mp}$

B.1 Syntax

$$
\begin{array}{rrcl}
\text{Types} & A, B, C & ::= & \iota \mid \star \mid A \to B \mid \forall X.A \mid X \mid \alpha \\
\text{Ground types} & G, H & ::= & \iota \mid \star \to \star \mid \forall X.\star \mid X \mid \alpha \\
\text{Space-efficient coercions} & s, t & ::= & G?^p \,;\, b \mid b \\
\text{Possibly blaming coercions} & b & ::= & \bot^p \mid i \\
\text{Intermediate coercions} & i, j & ::= & g \,;\, G! \mid g \\
\text{Ground coercions} & g, h & ::= & \mathsf{id} \mid s \to t \mid \forall X.s \,,, t \\
\text{Terms} & M & ::= & k \mid x \mid \lambda x\!:\!A.M \mid M\,M \mid \Lambda X.M \mid M\,A \mid M\langle s\rangle \mid \mathsf{blame}\,p \\
\text{Values} & V & ::= & U \mid U\langle g \,;\, G!\rangle \mid U\langle s \to t\rangle \mid U\langle \forall X.s \,,, t\rangle \\
\text{Uncoerced values} & U & ::= & k \mid \lambda x\!:\!A.M \mid \Lambda X.M \\
\text{Evaluation frames} & E & ::= & \square\, M \mid V\, \square \mid \square\, A \\
\text{Contexts} & \mathcal{C}_S & ::= & \square \mid \lambda x\!:\!A.\mathcal{C}_S \mid \mathcal{C}_S\, M \mid M\, \mathcal{C}_S \mid \Lambda X.\mathcal{C}_S \mid \mathcal{C}_S\, A \mid \mathcal{C}_S\,\langle s\rangle \\
\text{Type environments} & \Gamma & ::= & \emptyset \mid \Gamma, x : A \mid \Gamma, X \\
\text{Stores} & \Sigma & ::= & \emptyset \mid \Sigma, \alpha := \mathbb{A}
\end{array}
$$

Definition B.1 (Free Type Variables and Type Substitution)**.** We define the notion of free type variables as well as type substitution $A[X := B]$ and $M[X := \alpha]$ as those in $\lambda\mathrm{C}^{\forall}_{mp}$. The notation $M[X := \star]$ and $s[X := \star]$ denote the term and space-efficient coercion obtained by substituting type $\star$ for free type variable X in term M and coercion s, respectively. The term $M[X := \star]$ is defined in a standard manner, and space-efficient coercion $s[X := \star]$ is defined as follows (only the interesting cases are shown):

$$
(G?^p \,;\, b)[X := \star] \stackrel{\text{def}}{=} \begin{cases} b[X := \star] & (\text{if } G = X) \\ G?^p \,;\, (b[X := \star]) & (\text{if } G \neq X) \end{cases}
$$

$$
(g \,;\, G!)[X := \star] \stackrel{\text{def}}{=} \begin{cases} g[X := \star] & (\text{if } G = X) \\ (g[X := \star]) \,;\, G! & (\text{if } G \neq X) \end{cases}
$$

Note that $V[X := \star]$ is a value if value V is closed.

We write $A[\alpha := B]$ for the type obtained by replacing type name α in type A with type B. The notation $\Sigma(A)$ denotes the type obtained by replacing type names in A with the corresponding types associated by store Σ. Formally, it is defined as follows:

$$
\begin{array}{lcl}
\emptyset(A) & \stackrel{\text{def}}{=} & A \\
(\Sigma, \alpha := \mathbb{B})(A) & \stackrel{\text{def}}{=} & \Sigma(A[\alpha := \mathbb{B}])\ .
\end{array}
$$

We also write $\Sigma(\Gamma)$ for the type environment obtained by mapping every binding $x : A$ in type environment Γ to $x : \Sigma(A)$.

B.2 Dynamic semantics

Definition B.2 (Reduction)**.** The reduction relation $\longrightarrow$ is the smallest relations satisfying the rules in Section B.2.2. We use the notation $\longrightarrow^*$ to denote the reflexive, transitive closure of $\longrightarrow$, and $\Sigma \triangleright M \Uparrow$ to denote that term M diverges under store Σ. We may write $\longrightarrow_S$ and $\longrightarrow_S^*$ to emphasize the reduction in $\lambda\mathrm{S}^{\forall}_{mp}$.

B.2.1 Coercion composition $\boxed{s \fatsemi t = s'}$

$$
\begin{array}{rcll}
(G?^p \,;\, b) \fatsemi t & = & G?^p \,;\, (b \fatsemi t) & \\
\bot^p \fatsemi t & = & \bot^p & \\
i \fatsemi \bot^{p'} & = & \bot^{p'} & \\
i \fatsemi (h \,;\, H!) & = & (i \fatsemi h) \,;\, H! & \\
i \fatsemi \mathsf{id} & = & i & \\
(g \,;\, G!) \fatsemi (G?^p \,;\, b) & = & g \fatsemi b & \\
(g \,;\, G!) \fatsemi (H?^p \,;\, b) & = & \bot^p & \text{if } G \neq H \\
\mathsf{id} \fatsemi t & = & t & \text{if } t \neq \bot^{p'} \wedge t \neq (h \,;\, H!) \wedge t \neq \mathsf{id} \\
(s \to t) \fatsemi (s' \to t') & = & (s' \fatsemi s) \to (t \fatsemi t') & \\
(\forall X.s_1 \,,,\, s_2) \fatsemi (\forall X.t_1 \,,,\, t_2) & = & \forall X.(s_1 \fatsemi t_1) \,,,\, (s_2 \fatsemi t_2) &
\end{array}
$$

B.2.2 Reduction $\boxed{\Sigma \triangleright M \longrightarrow \Sigma' \triangleright M'}$

$$
\begin{array}{rcll}
k_1\, k_2 & \longrightarrow & \delta(k_1, k_2) & (\text{R\_Delta\_S}) \\
(\lambda x : A.M)\; V & \longrightarrow & M[x := V] & (\text{R\_Beta\_S}) \\
U\langle \mathsf{id} \rangle & \longrightarrow & U & (\text{R\_Id\_S}) \\
(\,U\langle s \to t \rangle)\; V & \longrightarrow & (\,U\;(\,V\langle s \rangle))\langle t \rangle & (\text{R\_Wrap\_S}) \\
U\langle \bot^p \rangle & \longrightarrow & \mathsf{blame}\; p & (\text{R\_Fail\_S}) \\
M\langle s \rangle\, \langle t \rangle & \longrightarrow & M\langle s \fatsemi t \rangle & (\text{R\_Merge\_S}) \\
(\Lambda X.M)\, \star & \longrightarrow & M[X := \star] & (\text{R\_TybetaDyn\_S}) \\
(\Lambda X.M)\langle \forall X.s \,,,\, t \rangle\, \star & \longrightarrow & (M[X := \star])\langle t \rangle & (\text{R\_TybetaDynC\_S}) \\
\Sigma \triangleright (\Lambda X.M)\, \mathbb{A} & \longrightarrow & \Sigma, \alpha := \mathbb{A} \triangleright M[X := \alpha] & (\text{R\_Tybeta\_S}) \\
 & & \quad \text{where } \alpha \notin \mathrm{dom}(\Sigma) & \\
\Sigma \triangleright (\Lambda X.M)\langle \forall X.s \,,,\, t \rangle\, \mathbb{A} & \longrightarrow & \Sigma, \alpha := \mathbb{A} \triangleright (M\langle s \rangle)[X := \alpha] & (\text{R\_TybetaC\_S}) \\
 & & \quad \text{where } \alpha \notin \mathrm{dom}(\Sigma) &
\end{array}
$$

$$E[\mathsf{blame}\; p] \longrightarrow \mathsf{blame}\; p \;\; (\text{R\_BlameE\_S}) \qquad (\mathsf{blame}\; p)\langle s \rangle \longrightarrow \mathsf{blame}\; p \;\; (\text{R\_BlameC\_S})$$

$$\frac{\Sigma \triangleright M \longrightarrow \Sigma' \triangleright M'}{\Sigma \triangleright E[M] \longrightarrow \Sigma' \triangleright E[M']} \; (\text{R\_CtxE\_S}) \qquad \frac{\Sigma \triangleright M \longrightarrow \Sigma' \triangleright M'}{\Sigma \triangleright M\langle s \rangle \longrightarrow \Sigma' \triangleright M'\langle s \rangle} \; (\text{R\_CtxC\_S})$$

(M is not a coercion application.)

B.3 Type system

This section defines typing rules in $\lambda S^{\forall}_{mp}$. Note that the notations for type environments and stores ($\mathrm{dom}(\Gamma)$, $\mathrm{dom}(\Sigma)$, $\Gamma_1 \# \Gamma_2$, and $\Sigma_1 \# \Sigma_2$) as well as the well-formedness rules for types, stores, type environments are defined in the same way as those in $\lambda C^{\forall}_{mp}$. We may write $\Sigma \mid \Gamma \vdash_S M : A$ for typing judgment $\Sigma \mid \Gamma \vdash M : A$ in $\lambda S^{\forall}_{mp}$ for clarity.

B.3.1 Type well-formedness $\boxed{\Sigma \mid \Gamma \vdash A}$

$$\Sigma \mid \Gamma \vdash \iota \;\; (\text{Tw\_Base}) \qquad \Sigma \mid \Gamma \vdash \star \;\; (\text{Tw\_Star}) \qquad \frac{\Sigma \mid \Gamma \vdash A \qquad \Sigma \mid \Gamma \vdash B}{\Sigma \mid \Gamma \vdash A \to B} \; (\text{Tw\_Arrow})$$

$$\frac{\alpha \in \mathrm{dom}(\Sigma)}{\Sigma \mid \Gamma \vdash \alpha} \; (\text{Tw\_Name}) \qquad \frac{X \in \Gamma}{\Sigma \mid \Gamma \vdash X} \; (\text{Tw\_Var}) \qquad \frac{\Sigma \mid \Gamma, X \vdash A}{\Sigma \mid \Gamma \vdash \forall X.A} \; (\text{Tw\_Poly})$$

B.3.2 Store well-formedness $\boxed{\vdash \Sigma}$

$$\vdash \emptyset \ (\textsc{Sw\_Empty}) \qquad \frac{\vdash \Sigma \qquad \Sigma \mid \emptyset \vdash \mathbb{A} \qquad \alpha \notin \mathrm{dom}(\Sigma)}{\vdash \Sigma, \alpha := \mathbb{A}} \ (\textsc{Sw\_Binding})$$

B.3.3 Type environment well-formedness $\boxed{\Sigma \vdash \Gamma}$

$$\Sigma \vdash \emptyset \ (\textsc{Tew\_Empty}) \qquad \frac{\Sigma \vdash \Gamma \qquad \Sigma \mid \Gamma \vdash A \qquad x \notin \mathrm{dom}(\Gamma)}{\Sigma \vdash \Gamma, x : A} \ (\textsc{Tew\_Var})$$

$$\frac{\Sigma \vdash \Gamma \qquad X \notin \mathrm{dom}(\Gamma)}{\Sigma \vdash \Gamma, X} \ (\textsc{Tew\_Tyvar})$$

B.3.4 Coercion typing $\boxed{\Sigma \mid \Gamma \vdash s : A \rightsquigarrow B}$

$$\frac{\begin{array}{c}(A \neq A' \to B' \text{ and } A \neq \forall X.A') \\ \vdash \Sigma \qquad \emptyset \vdash \Gamma \qquad \Sigma \mid \Gamma \vdash A\end{array}}{\Sigma \mid \Gamma \vdash \mathsf{id} : \Sigma(A) \rightsquigarrow \Sigma(A)} \ (\textsc{Ct\_Id\_S}) \qquad \frac{\vdash \Sigma \qquad \emptyset \vdash \Gamma \qquad \emptyset \mid \Gamma \vdash A \qquad \emptyset \mid \Gamma \vdash B}{\Sigma \mid \Gamma \vdash \bot^p : A \rightsquigarrow B} \ (\textsc{Ct\_Fail\_S})$$

$$\frac{\Sigma \mid \Gamma \vdash g : A \rightsquigarrow \Sigma(G) \qquad \Sigma \mid \Gamma \vdash G}{\Sigma \mid \Gamma \vdash g \,;\, G! : A \rightsquigarrow \star} \ (\textsc{Ct\_Inj\_S}) \qquad \frac{\Sigma \mid \Gamma \vdash b : \Sigma(G) \rightsquigarrow A \qquad \Sigma \mid \Gamma \vdash G}{\Sigma \mid \Gamma \vdash G?^p \,;\, b : \star \rightsquigarrow A} \ (\textsc{Ct\_Proj\_S})$$

$$\frac{\Sigma \mid \Gamma \vdash s : A' \rightsquigarrow A \qquad \Sigma \mid \Gamma \vdash t : B \rightsquigarrow B'}{\Sigma \mid \Gamma \vdash s \to t : (A \to B) \rightsquigarrow (A' \to B')} \ (\textsc{Ct\_Arrow\_S})$$

$$\frac{\Sigma \mid \Gamma, X \vdash s : A \rightsquigarrow B \qquad \Sigma \mid \Gamma \vdash t : A[X := \star] \rightsquigarrow B[X := \star]}{\Sigma \mid \Gamma \vdash \forall X.s \,,, t : \forall X.A \rightsquigarrow \forall X.B} \ (\textsc{Ct\_All\_S})$$

B.3.5 Term typing $\boxed{\Sigma \mid \Gamma \vdash M : A}$

$$\frac{\vdash \Sigma \qquad \emptyset \vdash \Gamma \qquad ty(k) = A}{\Sigma \mid \Gamma \vdash k : A} \ (\textsc{T\_Const\_S}) \qquad \frac{\vdash \Sigma \qquad \emptyset \vdash \Gamma \qquad x : A \in \Gamma}{\Sigma \mid \Gamma \vdash x : A} \ (\textsc{T\_Var\_S})$$

$$\frac{\Sigma \mid \Gamma, x : \Sigma(A) \vdash M : B}{\Sigma \mid \Gamma \vdash \lambda x : A.M : \Sigma(A) \to B} \ (\textsc{T\_Abs\_S}) \qquad \frac{\Sigma \mid \Gamma \vdash M_1 : A \to B \qquad \Sigma \mid \Gamma \vdash M_2 : A}{\Sigma \mid \Gamma \vdash M_1\, M_2 : B} \ (\textsc{T\_App\_S})$$

$$\frac{\Sigma \mid \Gamma, X \vdash M : A}{\Sigma \mid \Gamma \vdash \Lambda X.M : \forall X.A} \ (\textsc{T\_Tyabs\_S}) \qquad \frac{\Sigma \mid \Gamma \vdash M : \forall X.B \qquad \Sigma \mid \Gamma \vdash A}{\Sigma \mid \Gamma \vdash M\, A : B[X := \Sigma(A)]} \ (\textsc{T\_Tyapp\_S})$$

$$\frac{\vdash \Sigma \qquad \emptyset \vdash \Gamma \qquad \emptyset \mid \Gamma \vdash A}{\Sigma \mid \Gamma \vdash \mathsf{blame}\, p : A} \ (\textsc{T\_Blame\_S}) \qquad \frac{\Sigma \mid \Gamma \vdash M : A \qquad \Sigma \mid \Gamma \vdash s : A \rightsquigarrow B}{\Sigma \mid \Gamma \vdash M\langle s \rangle : B} \ (\textsc{T\_Crc\_S})$$

C Definition: Translation from $\lambda\mathbf{C}^{\forall}_{mp}$ to $\lambda\mathbf{S}^{\forall}_{mp}$

C.1 Coercion translation $\boxed{|c|_\Gamma = s}$

$$
\begin{array}{rcl}
|\mathsf{id}_A|_\Gamma & = & \mathsf{id} \quad \text{if } A \text{ is a base type or } \star \text{ or a type name or type variable} \\
|\mathsf{id}_{A\to B}|_\Gamma & = & |\mathsf{id}_A|_\Gamma \to |\mathsf{id}_B|_\Gamma \\
|\mathsf{id}_{\forall X.A}|_\Gamma & = & \forall X.|\mathsf{id}_A|_{\Gamma,X} \,,, |\mathsf{id}_A|_\Gamma \\
|G!|_\Gamma & = & \begin{cases} \mathsf{id} & \text{if } G = X \notin \mathrm{dom}(\Gamma) \\ |\mathsf{id}_G|_\Gamma \,;\, G! & \text{otherwise} \end{cases} \\
|G?^p|_\Gamma & = & \begin{cases} \mathsf{id} & \text{if } G = X \notin \mathrm{dom}(\Gamma) \\ G?^p \,;\, |\mathsf{id}_G|_\Gamma & \text{otherwise} \end{cases} \\
|\alpha^-|_\Gamma & = & \mathsf{id} \\
|\alpha^+|_\Gamma & = & \mathsf{id} \\
|\bot^p_{A\leadsto B}|_\Gamma & = & \bot^p \\
|c \to d|_\Gamma & = & |c|_\Gamma \to |d|_\Gamma \\
|c \,;\, d|_\Gamma & = & |c|_\Gamma \,⨟\, |d|_\Gamma \\
|\forall X.c|_\Gamma & = & \forall X.|c|_{\Gamma,X} \,,, |c|_\Gamma
\end{array}
$$

C.2 Term translation $\boxed{|M|_\Gamma = M'}$

$$
\begin{array}{rcl}
|k|_\Gamma & = & k \\
|x|_\Gamma & = & x \\
|\lambda x : A.M|_\Gamma & = & \lambda x : A.|M|_{\Gamma,x:A} \\
|M_1\, M_2|_\Gamma & = & |M_1|_\Gamma\, |M_2|_\Gamma \\
|\Lambda X.(M : A)|_\Gamma & = & \Lambda X.|M|_{\Gamma,X} \\
|M\, A|_\Gamma & = & |M|_\Gamma\, A \\
|M\langle c\rangle|_\Gamma & = & |M|_\Gamma\langle |c|_\Gamma\rangle \\
|\mathsf{blame}\, p|_\Gamma & = & \mathsf{blame}\, p
\end{array}
$$

C.3 Bisimulation $\boxed{\Sigma \mid \Gamma \vdash M \approx M' : A}$

$$
\frac{\vdash \Sigma \qquad \Sigma \vdash \Gamma \qquad ty(k) = A}{\Sigma \mid \Gamma \vdash k \approx k : A} \;(\text{Bs\_Const})
$$

$$
\frac{\vdash \Sigma \qquad \Sigma \vdash \Gamma \qquad x : A \in \Gamma}{\Sigma \mid \Gamma \vdash x \approx x : A} \;(\text{Bs\_Var})
$$

$$
\frac{\Sigma \mid \Gamma, x : A \vdash M \approx M' : B'}{\Sigma \mid \Gamma \vdash \lambda x : A.M \approx \lambda x : A.M' : A \to B'} \;(\text{Bs\_Abs})
$$

$$
\frac{\Sigma \mid \Gamma \vdash M_1 \approx M_1' : A \to B \qquad \Sigma \mid \Gamma \vdash M_2 \approx M_2' : A}{\Sigma \mid \Gamma \vdash M_1\, M_2 \approx M_1'\, M_2' : B} \;(\text{Bs\_App})
$$

$$
\frac{\Sigma \mid \Gamma, X \vdash M \approx M' : A}{\Sigma \mid \Gamma \vdash \Lambda X.(M : A) \approx \Lambda X.M' : \forall X.A} \;(\text{Bs\_Tyabs})
$$

$$
\frac{\Sigma \mid \Gamma \vdash M \approx M' : \forall X.B \qquad \Sigma \mid \Gamma \vdash A}{\Sigma \mid \Gamma \vdash M\, A \approx M'\, A : B[X := A]} \;(\text{Bs\_Tyapp})
$$

$$\frac{\vdash \Sigma \qquad \Sigma \vdash \Gamma \qquad \Sigma \mid \Gamma \vdash A}{\Sigma \mid \Gamma \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A} \ (\textsc{Bs\_Blame})$$

$$\frac{\Sigma \mid \Gamma \vdash M \approx M' : B \qquad \Sigma \mid \Gamma \vdash_C \ c : B \rightsquigarrow A}{\Sigma \mid \Gamma \vdash M\langle c\rangle \approx M'\langle |c|_\Gamma\rangle : A} \ (\textsc{Bs\_Crc})$$

$$\frac{\Sigma \mid \Gamma \vdash M \approx M' : A \qquad \Sigma \mid \emptyset \vdash_C \ \mathsf{id}_A : A \rightsquigarrow A}{\Sigma \mid \Gamma \vdash M \approx M'\langle |\mathsf{id}_A|_\emptyset\rangle : A} \ (\textsc{Bs\_CrcId})$$

$$\frac{\Sigma \mid \Gamma \vdash M \approx M'\langle s\rangle : A \qquad \mathrm{ftv}(s) = \emptyset \qquad \Sigma \mid \emptyset \vdash_C \ c : A \rightsquigarrow B}{\Sigma \mid \Gamma \vdash M\langle c\rangle \approx M'\langle s \mathbin{;} |c|_\emptyset\rangle : B} \ (\textsc{Bs\_CrcMore})$$

$$\frac{\Sigma \mid \Gamma \vdash M \approx M' : B \qquad \Sigma \mid \emptyset \vdash_C \ c^I : B \rightsquigarrow A}{\Sigma \mid \Gamma \vdash M\langle c^I\rangle \approx M' : A} \ (\textsc{Bs\_CrcIdL})$$

D Auxiliary Lemmas

We first state various weakening and strengthening lemmas, which are common to both calculi. All of them are proved by straightforward induction.

Lemma D.1 (Weakening Type Environment Preserves Well-formedness)**.**

1. If $\Sigma \mid \Gamma_1, \Gamma_2 \vdash A$ and $\Gamma \# (\Gamma_1, \Gamma_2)$, then $\Sigma \mid \Gamma_1, \Gamma, \Gamma_2 \vdash A$.
2. If $\Sigma \vdash \Gamma_1, \Gamma_2$ and $\Sigma \vdash \Gamma_1, \Gamma$ and $\Gamma \# \Gamma_2$, then $\Sigma \vdash \Gamma_1, \Gamma, \Gamma_2$.

Proof. Straightforward by induction on $\Sigma \mid \Gamma_1, \Gamma_2 \vdash A$ and Γ_2, respectively. Note that the second case rests on the first case. □

Lemma D.2 (Weakening Stores Preserves Well-formedness)**.**

1. If $\Sigma \mid \Gamma \vdash A$ and $\Sigma' \supseteq \Sigma$, then $\Sigma' \mid \Gamma \vdash A$.
2. If $\Sigma \vdash \Gamma$ and $\Sigma' \supseteq \Sigma$, then $\Sigma' \vdash \Gamma$.

Proof. Straightforward by induction on $\Sigma \mid \Gamma \vdash A$ and $\Sigma \vdash \Gamma$, respectively. Note that the second case rests on the first case. □

Lemma D.3 (Strengthening Type Environment Preserves Well-formedness)**.**

1. If $\Sigma \mid \Gamma_1, x : B, \Gamma_2 \vdash A$, then $\Sigma \mid \Gamma_1, \Gamma_2 \vdash A$.
2. If $\Sigma \vdash \Gamma_1, x : B, \Gamma_2$, then $\Sigma \vdash \Gamma_1, \Gamma_2$.

Proof. Straightforward by induction on $\Sigma \mid \Gamma_1, x : B, \Gamma_2 \vdash A$ and Γ_2, respectively. Note that the second case rests on the first case. □

Lemma D.4 (Types in Type Environment and Store are Well-formed)**.**

1. If $\vdash \Sigma$ and $\alpha := \mathbb{A} \in \Sigma$, then $\Sigma \mid \emptyset \vdash \mathbb{A}$.
2. If $\Sigma \vdash \Gamma$ and $x : A \in \Gamma$, then $\Sigma \mid \Gamma \vdash A$.

Proof. Straightforward by induction on $\vdash \Sigma$ with Lemma D.2 (1) and on $\Sigma \vdash \Gamma$ with Lemma D.1 (1), respectively. □

Lemma D.5 (Type Substitution Preserves Well-formedness)**.**

1. If $\Sigma \mid \Gamma_1 \vdash A$ and $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash B$ and , then $\Sigma \mid \Gamma_1, \Gamma_2[X := A] \vdash B[X := A]$.
2. If $\Sigma \mid \Gamma_1 \vdash A$ and $\Sigma \vdash \Gamma_1, X, \Gamma_2$, then $\Sigma \vdash \Gamma_1, \Gamma_2[X := A]$.

Proof. Straightforward by induction on $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash B$ with Lemma D.1 (1) and on Γ_2 with the first case, respectively. □

Lemma D.6 (Types in Store are Well-formed)**.** If $\vdash \Sigma$ and $\alpha := \mathbb{A} \in \Sigma$, then $\Sigma \mid \emptyset \vdash \mathbb{A}$.

Proof. By straightforward induction on $\vdash \Sigma$ with Lemma D.2 (1). □

Lemma D.7 (Left Partitions of Well-formed Type Environment are Well-formed)**.** If $\Sigma \vdash \Gamma_1, \Gamma_2$, then $\Sigma \vdash \Gamma_1$.

Proof. By straightforward induction on Γ_2. □

Lemma D.8 (Type Well-formedness Tracks Free Type Variables)**.** If $\Sigma \mid \Gamma \vdash A$, then $\mathrm{ftv}(A) \subseteq \mathrm{dom}(\Gamma)$.

Proof. By straightforward induction on $\Sigma \mid \Gamma \vdash A$. □

Lemma D.9 (Type Substitution by Type Names Preserves Type Well-formedness)**.**

1. If $\vdash \Sigma, \alpha := \mathbb{B}$ and $\Sigma, \alpha := \mathbb{B} \mid \Gamma \vdash A$, then $\Sigma \mid \Gamma[\alpha := \mathbb{B}] \vdash A[\alpha := \mathbb{B}]$.

2. If $\vdash \Sigma, \alpha := \mathbb{B}$ and $\Sigma, \alpha := \mathbb{B} \vdash \Gamma$, then $\Sigma \vdash \Gamma[\alpha := \mathbb{B}]$.

Proof. (1) Straightforward by induction on the derivation of $\Sigma, \alpha := \mathbb{B} \mid \Gamma \vdash A$ with Lemma D.1 (1) and Lemma D.4 (1).

(2) Straightforward by induction on the derivation of $\Sigma, \alpha := \mathbb{B} \vdash \Gamma$ with Lemma D.9 (1). ☐

Lemma D.10 (Replacing Type Names Preserves Type Well-formedness)**.**

1. If $\vdash \Sigma$ and $\Sigma \mid \Gamma \vdash A$, then $\emptyset \mid \Gamma \vdash \Sigma(A)$.
2. If $\vdash \Sigma$ and $\Sigma \mid \Gamma \vdash A$, then $\Sigma \mid \Sigma(\Gamma) \vdash A$.
3. If $\vdash \Sigma$, then $\Sigma \mid \Gamma \vdash A$ iff $\emptyset \mid \Sigma(\Gamma) \vdash \Sigma(A)$.
4. If $\vdash \Sigma$ and $\Sigma \mid \Gamma \vdash A$, then $\Sigma \mid \Sigma(\Gamma) \vdash \Sigma(A)$.
5. If $\vdash \Sigma$, then $\Sigma \vdash \Gamma$ iff $\emptyset \vdash \Sigma(\Gamma)$.

Proof. (1) By lexicographic induction on the pair of the size of Σ and the derivation of $\Sigma \mid \Gamma \vdash A$. The cases except that A is a type name are easy to prove.

Assume that $A = \alpha$ for some α. From $\Sigma \mid \Gamma \vdash \alpha$, we have $\Sigma = (\Sigma_1, \alpha := \mathbb{B}, \Sigma_2)$ for some $\mathbb{B}, \Sigma_1, \Sigma_2$. Because $\Sigma(\alpha) = \Sigma_1(\mathbb{B})$, it suffices to show $\emptyset \mid \Gamma \vdash \Sigma_1(\mathbb{B})$. Because $\vdash \Sigma_1, \alpha := \mathbb{B}, \Sigma_2$, we have $\vdash \Sigma_1$ and $\Sigma_1 \mid \emptyset \vdash \mathbb{B}$. By the IH, $\emptyset \mid \emptyset \vdash \Sigma_1(\mathbb{B})$. By Lemma D.1 (1), we have $\emptyset \mid \Gamma \vdash \Sigma_1(\mathbb{B})$.

(2) Straightforward by induction on the derivation of $\Sigma \mid \Gamma \vdash A$.

(3) The "only if" direction is proved by lexicographic induction on the pair of the size of Σ and the derivation of $\Sigma \mid \Gamma \vdash A$ with Lemma D.9 (1). The "if" direction is by induction on A.

(4) By (3) and Lemma D.2.

(5) By straightforward induction on Γ.

☐

Corollary D.11 (Associated Types and Type Names in Stores Are Equal by Replacing Type Names)**.** If $\vdash \Sigma$ and $\alpha := \mathbb{A} \in \Sigma$, then $\Sigma(\alpha) = \Sigma(\mathbb{A})$.

Proof. Straightforward by the definition of $\Sigma(A)$. ☐

E Type Safety

E.1 $\lambda \mathbf{C}^{\forall}_{mp}$

Lemma E.1 (Uniqueness of Coercion Typing)**.** If $\Sigma \mid \Gamma \vdash c : A \rightsquigarrow B$ and $\Sigma \mid \Gamma \vdash c : A' \rightsquigarrow B'$, then $A = A'$ and $B = B'$.

Proof. By straightforward induction on $\Sigma \mid \Gamma \vdash c : A \rightsquigarrow B$. ☐

Lemma E.2 (Canonical forms)**.** If $\Sigma \mid \emptyset \vdash V : A$, then one of the followings holds:

- $V = k$ and $A = ty(k)$ for some k.
- $V = \lambda x{:}A'.M$ and $A = A' \to B$ for some A', B, x, M.
- $V = \Lambda X.(M : A')$ and $A = \forall X.A'$ for some X, A', M.
- $V = V'\langle G! \rangle$ and $A = \star$ for some G, V'.
- $V = V'\langle \alpha^- \rangle$ and $A = \alpha$ for some α, V'.
- $V = V'\langle c \to d \rangle$ and $A = A' \to B$ for some A', B, c, d, V'.
- $V = V'\langle \forall X.c \rangle$ and $A = \forall X.A'$ for some X, A', c, V'.

Proof. Straightforward by case analysis on V. ☐

Lemma E.3. If $\Sigma \mid \emptyset \vdash V : \forall X.A_n$, then $V = (\Lambda X.(M : A_0))\overline{\langle \forall X.c \rangle}$ and $\Sigma \mid \emptyset \vdash \Lambda X.(M : A_0) : \forall X.A_0$ and $\Sigma \vdash \overline{\langle \forall X.c \rangle} : \forall X.A_0 \rightsquigarrow \forall X.A_n$ for some A_0, M, $\overline{\langle c \rangle}$.

Proof. By straightforward induction on $\Sigma \mid \emptyset \vdash V : \forall X.A$. Note that the last rule applied in the typing derivation is either (T_TYABS_C) or (T_CRC_C) because V is a value. □

Theorem E.4 (Progress (Theorem **??** of the paper))**.** If $\Sigma \mid \emptyset \vdash M : A$, then one of the followings holds:

- $M = V$ for some V,
- $M = \mathsf{blame}\, p$ for some p, or
- $\Sigma \triangleright M \longrightarrow \Sigma' \triangleright M'$ for some Σ', M'.

Proof. By induction on $\Sigma \mid \emptyset \vdash M : A$ with case analysis on the last rule applied in the typing derivation.

Case (T_CONST_C), (T_ABS_C), (T_TYABS_C): $M = V$ for some V immediately.

Case (T_VAR_C): Contradictory.

Case (T_BLAME_C): Immediate.

Case (T_APP_C): We have

$$M = M_1\, M_2, \quad \Sigma \mid \emptyset \vdash M_1 : B \to A, \quad \Sigma \mid \emptyset \vdash M_2 : B \quad (\exists B, M_1, M_2)\ .$$

By the IH on $\Sigma \mid \emptyset \vdash M_1 : B \to A$, we have three subcases:

Case $M_1 = V_1 (\exists V_1)$: By the IH on $\Sigma \mid \emptyset \vdash M_2 : B$, we have three further subcases:

Case $M_2 = V_2 (\exists V_2)$: By $\Sigma \mid \emptyset \vdash V_1 : B \to A$ and Lemma E.2, we consider the following three cases.

Case $V_1 = k_1$ and $ty(k_1) = B \to A$ $(\exists k_1)$: By the definition of ty, there exists some ι such that $B = \iota$. Thus, by $\Sigma \mid \emptyset \vdash V_2 : \iota$ and Lemma E.2, $V_2 = k_2$ for some k_2 such that $ty(k_2) = \iota$. Then, by the definition of δ, $\delta(k_1, k_2)$ is defined. Therefore, (R_DELTA_C) implies $\Sigma \triangleright k_1\, k_2 \longrightarrow \Sigma \triangleright \delta(k_1, k_2)$.

Case $V_1 = \lambda x : A'.M' (\exists A', x, M')$: By (R_BETA_C), $\Sigma \triangleright (\lambda x : A'.M')\, V_2 \longrightarrow \Sigma \triangleright M'[x := V_2]$.

Case $V_1 = V_1' \langle c \to d \rangle (\exists c, d, V_1')$: By (R_WRAP_C), $\Sigma \triangleright (V_1' \langle c \to d \rangle)\, V_2 \longrightarrow \Sigma \triangleright (V_1' (V_2 \langle c \rangle)) \langle d \rangle$.

Case $M_2 = \mathsf{blame}\, p_2 (\exists p_2)$: By (R_BLAME_C), $\Sigma \triangleright V_1\, (\mathsf{blame}\, p_2) \longrightarrow \Sigma \triangleright \mathsf{blame}\, p_2$.

Case $\Sigma \triangleright M_2 \longrightarrow \Sigma_2 \triangleright M_2' (\exists \Sigma_2, M_2')$: By (R_CTX_C), $\Sigma \triangleright V_1\, M_2 \longrightarrow \Sigma_2 \triangleright V_1\, M_2'$.

Case $M_1 = \mathsf{blame}\, p_1 (\exists p_1)$: By (R_BLAME_C), $\Sigma \triangleright (\mathsf{blame}\, p_1)\, M_2 \longrightarrow \Sigma \triangleright \mathsf{blame}\, p_1$.

Case $\Sigma \triangleright M_1 \longrightarrow \Sigma_1 \triangleright M_1' (\exists \Sigma_1, M_1')$: By (R_CTX_C), $\Sigma \triangleright M_1\, M_2 \longrightarrow \Sigma_1 \triangleright M_1'\, M_2$.

Case (T_TYAPP_C): We have

$$M = M'\, A', \quad \Sigma \mid \emptyset \vdash M' : \forall X.B, \quad \Sigma \mid \emptyset \vdash A' \quad (\exists X, A', B, M')\ .$$

By the IH, we have three subcases:

Case $M' = V (\exists V)$: By Lemma E.3, there exists some A_0, M'', $\overline{\langle c \rangle}$ such that $V = (\Lambda X.(M'' : A_0))\overline{\langle \forall X.c \rangle}$ and $\Sigma \vdash \overline{\langle \forall X.c \rangle} : \forall X.A_0 \rightsquigarrow \forall X.B$. By case analysis on A'.

Case $A' = \star$: By (R_TYBETADYN_C), $\Sigma \triangleright (\Lambda X.(M'' : A_0))\overline{\langle \forall X.c \rangle}\, \star \longrightarrow \Sigma \triangleright (M'' \overline{\langle c \rangle})[X := \star]$.

Case $A' = \mathbb{C} (\exists \mathbb{C})$: By (R_TYBETA_C), $\Sigma \triangleright (\Lambda X.(M'' : A_0))\overline{\langle \forall X.c \rangle}\, \mathbb{C} \longrightarrow \Sigma, \alpha := \mathbb{C} \triangleright (M'' \overline{\langle c \rangle})[X := \alpha] \langle coerce^{+}_{\alpha}(B[X := \alpha]) \rangle$.

Case $M' = \mathsf{blame}\, p (\exists p)$: By (R_BLAME_C), $\Sigma \triangleright (\mathsf{blame}\, p)\, A' \longrightarrow \Sigma \triangleright \mathsf{blame}\, p$.

Case $\Sigma \triangleright M' \longrightarrow \Sigma' \triangleright M'' (\exists \Sigma', M'')$: By (R_CTX_C), $\Sigma \triangleright M'\, A' \longrightarrow \Sigma' \triangleright M''\, A'$.

Case (T_CRC_C): We have

$$M = M' \langle c \rangle, \quad \Sigma \mid \emptyset \vdash M' : B, \quad \Sigma \mid \emptyset \vdash c : B \rightsquigarrow A \quad (\exists B, c, M')\ .$$

By the IH, we have three subcases:

Case $M' = V(\exists V)$: We conduct case analysis on c.

Case $c = \mathsf{id}_{A'}(\exists A')$: By (R_ID_C), $\Sigma \rhd V\langle \mathsf{id}_{A'} \rangle \longrightarrow \Sigma \rhd V$.

Case $c = G?^p(\exists p, G)$: From $\Sigma \mid \emptyset \vdash G?^p : B \rightsquigarrow A$, we have $B = \star$. Thus, $\Sigma \mid \emptyset \vdash V : \star$ and, by Lemma E.2, $V = V'\langle H! \rangle$ for some H, V'. If $G = H$, then, by (R_COLLAPSE_C), we have $\Sigma \rhd V'\langle G! \rangle\langle G?^p \rangle \longrightarrow \Sigma \rhd V'$. If $G \neq H$, then, by (R_CONFLICT_C), $\Sigma \rhd V'\langle H! \rangle\langle G?^p \rangle \longrightarrow \Sigma \rhd \mathsf{blame}\, p$.

Case $c = \alpha^+(\exists \alpha)$: From $\Sigma \mid \emptyset \vdash \alpha^+ : B \rightsquigarrow A$, we have $B = \alpha$. Thus, $\Sigma \mid \emptyset \vdash V : \alpha$ and, by Lemma E.2, $V = V'\langle \alpha^- \rangle$ for some V'. By (R_REMOVE_C), $\Sigma \rhd V'\langle \alpha^- \rangle\langle \alpha^+ \rangle \longrightarrow \Sigma \rhd V'$.

Case $c = c' \,;\, d(\exists c', d)$: By (R_SPLIT_C), $\Sigma \rhd V\langle c' \,;\, d \rangle \longrightarrow \Sigma \rhd V\langle c' \rangle\langle d \rangle$.

Case $c = \perp^p_{A' \rightsquigarrow B'}(\exists p, A', B')$: By (R_FAIL_C), $\Sigma \rhd V\langle \perp^p_{A' \rightsquigarrow B'} \rangle \longrightarrow \Sigma \rhd \mathsf{blame}\, p$.

Otherwise: $V\langle c \rangle$ is a value.

Case $M' = \mathsf{blame}\, p(\exists p)$: By (R_BLAME_C), $\Sigma \rhd (\mathsf{blame}\, p)\langle c \rangle \longrightarrow \Sigma \rhd \mathsf{blame}\, p$.

Case $\Sigma \rhd M' \longrightarrow \Sigma' \rhd M''(\exists \Sigma', M'')$: By (R_CTX_C), $\Sigma \rhd M'\langle c \rangle \longrightarrow \Sigma' \rhd M''\langle c \rangle$.

□

The proof of preservation starts with various weakening lemmas.

Lemma E.5. If $\Sigma \mid \Gamma_1, \Gamma_2 \vdash c : A \rightsquigarrow B$ and $\Sigma \vdash \Gamma_1, \Gamma$ and $\Gamma \# \Gamma_2$, then $\Sigma \mid \Gamma_1, \Gamma, \Gamma_2 \vdash c : A \rightsquigarrow B$.

Proof. By straightforward induction on $\Sigma \mid \Gamma_1, \Gamma_2 \vdash c : A \rightsquigarrow B$ with Lemma D.1. □

Lemma E.6. If $\Sigma \mid \Gamma \vdash c : A \rightsquigarrow B$ and $\Sigma' \supseteq \Sigma$ and $\vdash \Sigma'$, then $\Sigma' \mid \Gamma \vdash c : A \rightsquigarrow B$.

Proof. By straightforward induction on $\Sigma \mid \Gamma \vdash c : A \rightsquigarrow B$ with Lemma D.2. □

Lemma E.7. If $\Sigma \mid \Gamma_1, \Gamma_2 \vdash M : A$ and $\Sigma \vdash \Gamma_1, \Gamma$ and $\Gamma \# \Gamma_2$, then $\Sigma \mid \Gamma_1, \Gamma, \Gamma_2 \vdash M : A$.

Proof. By straightforward induction on $\Sigma \mid \Gamma_1, \Gamma_2 \vdash M : A$ with Lemmas D.1 and E.5. □

Lemma E.8. If $\Sigma \mid \Gamma \vdash M : A$ and $\Sigma' \supseteq \Sigma$ and $\vdash \Sigma'$, then $\Sigma' \mid \Gamma \vdash M : A$.

Proof. By straightforward induction on $\Sigma \mid \Gamma \vdash M : A$ with Lemmas D.2 and E.6. □

Lemma E.9 (Agreement (1))**.** If $\Sigma \mid \Gamma \vdash c : A \rightsquigarrow B$, then $\Sigma \vdash \Gamma$ and $\Sigma \mid \Gamma \vdash A$ and $\Sigma \mid \Gamma \vdash B$ and $\mathrm{ftv}(c) \subseteq \mathrm{dom}(\Gamma)$.

Proof. By straightforward induction on $\Sigma \mid \Gamma \vdash c : A \rightsquigarrow B$ with Lemmas D.4 (1) and D.1 (1). □

Lemma E.10 (Agreement (2))**.** If $\Sigma \mid \Gamma \vdash M : A$, then $\vdash \Sigma$ and $\Sigma \vdash \Gamma$ and $\Sigma \mid \Gamma \vdash A$.

Proof. By straightforward induction on $\Sigma \mid \Gamma \vdash M : A$ with Lemmas D.4 (2), D.3 (1), D.5 (1), and E.9. □

Lemma E.11 (Strengthening Type Environment in Coercion Typing)**.** $\Sigma \mid \Gamma_1, x : C, \Gamma_2 \vdash c : A \rightsquigarrow B$, then $\Sigma \mid \Gamma_1, \Gamma_2 \vdash c : A \rightsquigarrow B$.

Proof. By straightforward induction on $\Sigma \mid \Gamma_1, x : C, \Gamma_2 \vdash c : A \rightsquigarrow B$ with Lemma D.3. □

Lemma E.12 (Value Substitution)**.** If $\Sigma \mid \Gamma_1, x : A, \Gamma_2 \vdash M : B$ and $\Sigma \mid \Gamma_1 \vdash V : A$, then $\Sigma \mid \Gamma_1, \Gamma_2 \vdash M[x := V] : B$.

Proof. By straightforward induction on $\Sigma \mid \Gamma_1, x : A, \Gamma_2 \vdash M : B$ with Lemmas D.3, E.7, and E.11. □

Lemma E.13 (Type Name Substitution (Coercion))**.** If $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash c : A \rightsquigarrow B$ and $\alpha \in \mathrm{dom}(\Sigma)$, then $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \vdash c[X := \alpha] : A[X := \alpha] \rightsquigarrow B[X := \alpha]$.

Proof. By induction on $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash c : A \rightsquigarrow B$ with case analysis on the last rule applied in the typing derivation.

Case (Ct_Id_C): We have

$$c = \mathsf{id}_{A'}, \quad A = A', \quad B = A', \quad \vdash \Sigma, \quad \Sigma \vdash \Gamma_1, X, \Gamma_2, \quad \Sigma \mid \Gamma_1, X, \Gamma_2 \vdash A' \quad (\exists A') \; .$$

Since $\mathsf{id}_{A'}[X := \alpha] = \mathsf{id}_{A'[X:=\alpha]}$, it suffices to show $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \vdash \mathsf{id}_{A'[X:=\alpha]} : A'[X := \alpha] \rightsquigarrow A'[X := \alpha]$. Because $\Sigma \mid \Gamma_1 \vdash \alpha$ by (Tw_Name), Lemma D.5 implies $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \vdash A'[X := \alpha]$ and $\Sigma \vdash \Gamma_1, \Gamma_2[X := \alpha]$. Thus, by (Ct_Id_C), $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \vdash \mathsf{id}_{A'[X:=\alpha]} : A'[X := \alpha] \rightsquigarrow A'[X := \alpha]$.

Case (Ct_Inj_C), (Ct_Proj_C), (Ct_Fail_C): Similarly to the case for (Ct_Id_C).

Case (Ct_Conceal_C): We have

$$c = \beta^-, \quad A = \mathbb{C}, \quad B = \beta, \quad \vdash \Sigma, \quad \Sigma \vdash \Gamma_1, X, \Gamma_2, \quad \beta := \mathbb{C} \in \Sigma \quad (\exists \beta, \mathbb{C}) \; .$$

We have $\beta^-[X := \alpha] = \beta^-$ and $\beta[X := \alpha] = \beta$. By Lemma D.4 (1), $\Sigma \mid \emptyset \vdash \mathbb{C}$. By Lemma D.8, $\mathrm{ftv}(\mathbb{C}) \subseteq \emptyset$. Thus, $\mathbb{C}$ is closed and $\mathbb{C}[X := \alpha] = \mathbb{C}$. Thus, it suffices to show $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \vdash \beta^- : \mathbb{C} \rightsquigarrow \beta$. By Lemma D.5 (2), $\Sigma \vdash \Gamma_1, \Gamma_2[X := \alpha]$. By (Ct_Conceal_C), $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \vdash \beta^- : \mathbb{C} \rightsquigarrow \beta$.

Case (Ct_Reveal_C): Similarly to the case for (Ct_Conceal_C).

Case (Ct_Arrow_C), (Ct_Seq_C), (Ct_All_C): By the IH(s).

□

Lemma E.14 (Type Name Substitution)**.** If $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash M : A$ and $\alpha \in \mathrm{dom}(\Sigma)$, then $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \vdash M[X := \alpha] : A[X := \alpha]$.

Proof. By straightforward induction on $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash M : A$ with Lemmas D.5 and E.13. □

Lemma E.15 (Dynamic Type Substitution (Coercion))**.** If $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash c : A \rightsquigarrow B$, then $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash c[X := \star] : A[X := \star] \rightsquigarrow B[X := \star]$.

Proof. By induction on $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash c : A \rightsquigarrow B$ with case analysis on the last rule applied in the typing derivation.

Case (Ct_Id_C): We have

$$c = \mathsf{id}_{A'}, \quad A = A', \quad B = A', \quad \vdash \Sigma, \quad \Sigma \vdash \Gamma_1, X, \Gamma_2, \quad \Sigma \mid \Gamma_1, X, \Gamma_2 \vdash A' \quad (\exists A') \; .$$

Since $\mathsf{id}_{A'}[X := \star] = \mathsf{id}_{A'[X:=\star]}$, it suffices to show $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash \mathsf{id}_{A'[X:=\star]} : A'[X := \star] \rightsquigarrow A'[X := \star]$. Because $\Sigma \mid \Gamma_1 \vdash \star$ by (Tw_Star), Lemma D.5 implies $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash A'[X := \star]$ and $\Sigma \vdash \Gamma_1, \Gamma_2[X := \star]$. Thus, by (Ct_Id_C), $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash \mathsf{id}_{A'[X:=\star]} : A'[X := \star] \rightsquigarrow A'[X := \star]$.

Case (Ct_Fail_C): Similarly to the case for (Ct_Id_C).

Case (Ct_Inj_C): We have

$$c = G!, \quad A = G, \quad B = \star, \quad \vdash \Sigma, \quad \Sigma \vdash \Gamma_1, X, \Gamma_2, \quad \Sigma \mid \Gamma_1, X, \Gamma_2 \vdash G \quad (\exists G) \; .$$

By case analysis on G.

Case $G = X$: Since $G![X := \star] = \mathsf{id}_\star$ and $G[X := \star] = \star$, it suffices to show $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash \mathsf{id}_\star : \star \rightsquigarrow \star$. By (Tw_Star), $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash \star$. Because $\Sigma \mid \Gamma_1 \vdash \star$ again by (Tw_Star), Lemma D.5 (2) implies $\Sigma \vdash \Gamma_1, \Gamma_2[X := \star]$. Thus, by (Ct_Id_C), $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash \mathsf{id}_\star : \star \rightsquigarrow \star$.

Case $G \neq X$: Since $G![X := \star] = G!$ and $G[X := \star] = G$, it suffices to show that $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash G! : G \rightsquigarrow \star$. By Lemma D.5, $\Sigma \vdash \Gamma_1, \Gamma_2[X := \star]$ and $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash G[X := \star] (= G)$. Thus, by (Ct_Inj_C), $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash G! : G \rightsquigarrow \star$.

Case (Ct_Proj_C): Similarly to the case for (Ct_Inj_C).

Case (Ct_Conceal_C): We have

$$c = \alpha^-, \quad A = \mathbb{C}, \quad B = \alpha, \quad \vdash \Sigma, \quad \Sigma \vdash \Gamma_1, X, \Gamma_2, \quad \alpha := \mathbb{C} \in \Sigma \quad (\exists \alpha, \mathbb{C}) .$$

We have $\alpha^-[X := \star] = \alpha^-$ and $\alpha[X := \star] = \alpha$. By Lemma D.4 (1), $\Sigma \mid \emptyset \vdash \mathbb{C}$. By Lemma D.8, $\mathrm{ftv}(\mathbb{C}) \subseteq \emptyset$. Thus, $\mathbb{C}$ is closed and $\mathbb{C}[X := \star] = \mathbb{C}$. Thus, it suffices to show $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash \alpha^- : \mathbb{C} \rightsquigarrow \alpha$. By Lemma D.5 (2), $\Sigma \vdash \Gamma_1, \Gamma_2[X := \star]$. By (Ct_Conceal_C), $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash \alpha^- : \mathbb{C} \rightsquigarrow \alpha$.

Case (Ct_Reveal_C): Similarly to the case for (Ct_Conceal_C).

Case (Ct_Arrow_C), (Ct_Seq_C), (Ct_All_C): By the IH(s).

□

Lemma E.16 (Dynamic Type Substitution)**.** If $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash M : A$, then $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash M[X := \star] : A[X := \star]$.

Proof. By straightforward induction on $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash M : A$ with Lemmas D.5 and E.15. □

Lemma E.17 (Coercion Generation is Well Typed)**.** Assume that $\vdash \Sigma$ and $\alpha := \mathbb{B} \in \Sigma$ and $\Sigma \vdash \Gamma_1, X, \Gamma_2$ and $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash A$ and α does not occur in type A. Then, the following holds:

- $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \vdash \mathit{coerce}^+_\alpha(A[X := \alpha]) : A[X := \alpha] \rightsquigarrow A[X := \mathbb{B}]$; and
- $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \vdash \mathit{coerce}^-_\alpha(A[X := \alpha]) : A[X := \mathbb{B}] \rightsquigarrow A[X := \alpha]$.

Proof. By induction on A. Note that, because $\Sigma \mid \Gamma_1 \vdash \alpha$ by (Tw_Name), Lemma D.5 implies $\Sigma \vdash \Gamma_1, \Gamma_2[X := \alpha]$ and $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \vdash A[X := \alpha]$. We proceed by case analysis on A.

Case $A = \iota (\exists \iota)$: We have

$$\mathit{coerce}^+_\alpha(\iota[X := \alpha]) = \mathit{coerce}^+_\alpha(\iota) = \mathsf{id}_\iota, \quad \mathit{coerce}^-_\alpha(\iota[X := \alpha]) = \mathit{coerce}^-_\alpha(\iota) = \mathsf{id}_\iota, \quad \iota[X := \alpha] = \iota[X := \mathbb{B}] = \iota .$$

It suffices to show $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \vdash \mathsf{id}_\iota : \iota \rightsquigarrow \iota$, which is implied by (Ct_Id_C).

Case $A = \star$: Similar to the case where $A = \iota$.

Case $A = A' \to B' (\exists A', B')$: We have

$$\begin{aligned}
\mathit{coerce}^+_\alpha((A' \to B')[X := \alpha]) &= \mathit{coerce}^+_\alpha(A'[X := \alpha] \to B'[X := \alpha]) = \mathit{coerce}^-_\alpha(A'[X := \alpha]) \to \mathit{coerce}^+_\alpha(B'[X := \alpha]),\\
\mathit{coerce}^-_\alpha((A' \to B')[X := \alpha]) &= \mathit{coerce}^-_\alpha(A'[X := \alpha] \to B'[X := \alpha]) = \mathit{coerce}^+_\alpha(A'[X := \alpha]) \to \mathit{coerce}^-_\alpha(B'[X := \alpha]),\\
(A' \to B')[X := \alpha] &= A'[X := \alpha] \to B'[X := \alpha], \quad (A' \to B')[X := \mathbb{B}] = A'[X := \mathbb{B}] \to B'[X := \mathbb{B}] .
\end{aligned}$$

Thus, it suffices to show

- $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \vdash \mathit{coerce}^-_\alpha(A'[X := \alpha]) \to \mathit{coerce}^+_\alpha(B'[X := \alpha]) : (A'[X := \alpha] \to B'[X := \alpha]) \rightsquigarrow (A'[X := \mathbb{B}] \to B'[X := \mathbb{B}])$ and
- $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \vdash \mathit{coerce}^+_\alpha(A'[X := \alpha]) \to \mathit{coerce}^-_\alpha(B'[X := \alpha]) : (A'[X := \mathbb{B}] \to B'[X := \mathbb{B}]) \rightsquigarrow (A'[X := \alpha] \to B'[X := \alpha])$.

From $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash A' \to B'$, we have $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash A'$ and $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash B'$. By the IHs,

$$\begin{aligned}
\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] &\vdash \mathit{coerce}^+_\alpha(A'[X := \alpha]) : A'[X := \alpha] \rightsquigarrow A'[X := \mathbb{B}],\\
\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] &\vdash \mathit{coerce}^-_\alpha(A'[X := \alpha]) : A'[X := \mathbb{B}] \rightsquigarrow A'[X := \alpha],\\
\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] &\vdash \mathit{coerce}^+_\alpha(B'[X := \alpha]) : B'[X := \alpha] \rightsquigarrow B'[X := \mathbb{B}],\\
\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] &\vdash \mathit{coerce}^-_\alpha(B'[X := \alpha]) : B'[X := \mathbb{B}] \rightsquigarrow B'[X := \alpha] .
\end{aligned}$$

(Ct_Arrow_C) finishes the case.

Case $A = \forall Y.A'(\exists Y, A')$: We can assume $Y \neq X$ without loss of generality. We have

$$coerce^+_\alpha((\forall Y.A')[X := \alpha]) = coerce^+_\alpha(\forall Y.A'[X := \alpha]) = \forall Y.coerce^+_\alpha(A'[X := \alpha]),$$
$$coerce^-_\alpha((\forall Y.A')[X := \alpha]) = coerce^-_\alpha(\forall Y.A'[X := \alpha]) = \forall Y.coerce^-_\alpha(A'[X := \alpha]),$$
$$(\forall Y.A')[X := \alpha] = \forall Y.A'[X := \alpha], \quad (\forall Y.A')[X := \mathbb{B}] = \forall Y.A'[X := \mathbb{B}] \ .$$

Thus, it suffices to show

- $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \vdash \forall Y.coerce^+_\alpha(A'[X := \alpha]) : \forall Y.A'[X := \alpha] \rightsquigarrow \forall Y.A'[X := \mathbb{B}]$ and
- $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \vdash \forall Y.coerce^-_\alpha(A'[X := \alpha]) : \forall Y.A'[X := \mathbb{B}] \rightsquigarrow \forall Y.A'[X := \alpha]$.

Form $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash \forall Y.A'$, we have $\Sigma \mid \Gamma_1, X, \Gamma_2, Y \vdash A'$. By (Tew_Tyvar), $\Sigma \vdash \Gamma_1, X, \Gamma_2, Y$. By the IH, we have

$$\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha], Y \vdash coerce^+_\alpha(A'[X := \alpha]) : A'[X := \alpha] \rightsquigarrow A'[X := \mathbb{B}],$$
$$\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha], Y \vdash coerce^-_\alpha(A'[X := \alpha]) : A'[X := \mathbb{B}] \rightsquigarrow A'[X := \alpha] \ .$$

(Ct_All_C) finishes the case.

Case $A = Y(\exists Y)$: We have the following two subcases.

Case $Y = X$: We have

$$coerce^+_\alpha(X[X := \alpha]) = coerce^+_\alpha(\alpha) = \alpha^+, \quad coerce^-_\alpha(X[X := \alpha]) = coerce^-_\alpha(\alpha) = \alpha^-,$$
$$X[X := \alpha] = \alpha, \quad X[X := \mathbb{B}] = \mathbb{B} \ .$$

It suffices to show

- $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \vdash \alpha^+ : \alpha \rightsquigarrow \mathbb{B}$ and
- $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \vdash \alpha^- : \mathbb{B} \rightsquigarrow \alpha$,

which follows from (Ct_Conceal_C) and (Ct_Reveal_C).

Case $Y \neq X$: We have

$$coerce^+_\alpha(Y[X := \alpha]) = coerce^+_\alpha(Y) = \mathsf{id}_Y, \quad coerce^-_\alpha(Y[X := \alpha]) = coerce^-_\alpha(Y) = \mathsf{id}_Y,$$
$$Y[X := \alpha] = Y[X := \mathbb{B}] = Y \ .$$

It suffices to show $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \vdash \mathsf{id}_Y : Y \rightsquigarrow Y$, which follows from (Ct_Id_C).

Case $A = \beta(\exists\beta)$: Since α does not occur in type A, $\beta \neq \alpha$. We have

$$coerce^+_\alpha(\beta[X := \alpha]) = coerce^+_\alpha(\beta) = \mathsf{id}_\beta, \quad coerce^-_\alpha(\beta[X := \alpha]) = coerce^-_\alpha(\beta) = \mathsf{id}_\beta,$$
$$\beta[X := \alpha] = \beta[X := \mathbb{B}] = \beta \ .$$

It suffices to show $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \vdash \mathsf{id}_\beta : \beta \rightsquigarrow \beta$, which follows from (Ct_Id_C).

□

Lemma E.18. If $\Sigma \mid \emptyset \vdash M : A$ and $\Sigma \triangleright M \longrightarrow \Sigma' \triangleright M'$, then either

- $\Sigma' = \Sigma$ or
- $\Sigma' = \Sigma, \alpha := \mathbb{B}$ and $\Sigma \mid \emptyset \vdash \mathbb{B}$ for some $\mathbb{B}$ and $\alpha \notin \mathrm{dom}(\Sigma)$.

Proof. By easy induction on $\Sigma \triangleright M \longrightarrow \Sigma' \triangleright M'$. □

Theorem E.19 (Preservation (Theorem **??** of the paper))**.** If $\Sigma \mid \emptyset \vdash M : A$ and $\Sigma \triangleright M \longrightarrow \Sigma' \triangleright M'$, then $\Sigma' \mid \emptyset \vdash M' : A$.

Proof. By induction on the derivation of $\Sigma \triangleright M \longrightarrow \Sigma' \triangleright M'$ with case analysis on the last rule used.

Case (R_DELTA_C): We have

$$M = k_1\, k_2, \quad M' = \delta(k_1, k_2), \quad \Sigma' = \Sigma \quad (\exists k_1, k_2)\ .$$

From $\Sigma \mid \emptyset \vdash k_1\, k_2 : A$, we have $\Sigma \mid \emptyset \vdash k_1 : \iota \to A$ and $\Sigma \mid \emptyset \vdash k_2 : \iota$. Then, by the assumption on δ, $\Sigma \mid \emptyset \vdash \delta(k_1, k_2) : A$.

Case (R_BETA_C): We have

$$M = (\lambda x\!:\!A'.M'')\ V, \quad M' = M''[x := V], \quad \Sigma = \Sigma' \quad (\exists A', x, M'', V)\ .$$

From $\Sigma \mid \emptyset \vdash (\lambda x\!:\!A'.M'')\ V : A$, we have

$$\Sigma \mid \emptyset, x : A' \vdash M'' : A, \quad \Sigma \mid \emptyset \vdash V : A'\ .$$

Thus, by Lemma E.12, $\Sigma \mid \emptyset \vdash M''[x := V] : A$.

Case (R_ID_C): We have

$$M = V\langle \mathsf{id}_{A'} \rangle, \quad M' = V, \quad \Sigma' = \Sigma \quad (\exists A', V)\ .$$

From $\Sigma \mid \emptyset \vdash V\langle \mathsf{id}_{A'} \rangle : A$, we have

$$A = A', \quad \Sigma \mid \emptyset \vdash V : A\ .$$

Case (R_WRAP_C): We have

$$M = (V\langle c \to d \rangle)\ V', \quad M' = (V\ (V'\langle c \rangle))\langle d \rangle, \quad \Sigma' = \Sigma \quad (\exists c, d, V, V')\ .$$

From $\Sigma \mid \emptyset \vdash (V\langle c \to d \rangle)\ V' : A$, we have

$$\Sigma \mid \emptyset \vdash V : B \to C, \quad \Sigma \mid \emptyset \vdash V' : D, \quad \Sigma \mid \emptyset \vdash c : D \rightsquigarrow B,$$
$$\Sigma \mid \emptyset \vdash d : C \rightsquigarrow A \quad (\exists B, C, D)\ .$$

Thus, $\Sigma \mid \emptyset \vdash (V\ (V'\langle c \rangle))\langle d \rangle : A$ by (T_CRC_C), (T_APP_C), and (T_CRC_C).

Case (R_FAIL_C): We have

$$M = V\langle \bot^p_{A' \rightsquigarrow B'} \rangle, \quad M' = \mathsf{blame}\, p, \quad \Sigma' = \Sigma \quad (\exists p, A', B', V)\ .$$

By Lemma E.10, we have $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. By (T_BLAME_C), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p : A$.

Case (R_COLLAPSE_C): We have

$$M = V\langle G! \rangle\langle G?^p \rangle, \quad M' = V, \quad \Sigma' = \Sigma \quad (\exists p, G, V)\ .$$

From $\Sigma \mid \emptyset \vdash V\langle G! \rangle\langle G?^p \rangle : A$, we have

$$\Sigma \mid \emptyset \vdash V : G, \quad \Sigma \mid \emptyset \vdash G! : G \rightsquigarrow \star, \quad \Sigma \mid \emptyset \vdash G?^p : \star \rightsquigarrow G, \quad A = G\ .$$

Thus, $\Sigma \mid \emptyset \vdash V : A$.

Case (R_CONFLICT_C): We have

$$M = V\langle G! \rangle\langle H?^p \rangle, \quad M' = \mathsf{blame}\, p, \quad \Sigma = \Sigma', \quad G \neq H \quad (\exists p, G, V)\ .$$

By Lemma E.10, we have $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Thus, by (T_BLAME_C), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p : A$.

Case (R_REMOVE_C): We have

$$M = V\langle \alpha^- \rangle\langle \alpha^+ \rangle, \quad M' = V, \quad \Sigma' = \Sigma \quad (\exists \alpha, V) .$$

From $\Sigma \mid \emptyset \vdash V\langle \alpha^- \rangle\langle \alpha^+ \rangle : A$, we have

$$\Sigma \mid \emptyset \vdash V : \mathbb{B}, \quad \Sigma \mid \emptyset \vdash \alpha^- : \mathbb{B} \rightsquigarrow \alpha, \quad \Sigma \mid \emptyset \vdash \alpha^+ : \alpha \rightsquigarrow \mathbb{B}, \quad A = \mathbb{B} \quad (\exists \mathbb{B}) ,$$

finishing the case.

Case (R_SPLIT_C): We have

$$M = V\langle c\,;d \rangle, \quad M' = V\langle c \rangle\langle d \rangle, \quad \Sigma' = \Sigma \quad (\exists c, d, V) .$$

From $\Sigma \mid \emptyset \vdash V\langle c\,;d \rangle : A$, we have

$$\Sigma \mid \emptyset \vdash V : C, \quad \Sigma \mid \emptyset \vdash c : C \rightsquigarrow B, \quad \Sigma \mid \emptyset \vdash d : B \rightsquigarrow A \quad (\exists B, C) .$$

Thus, $\Sigma \mid \emptyset \vdash V\langle c \rangle\langle d \rangle : A$ by using (T_CRC_C) twice.

Case (R_TYBETADYN_C): We have

$$M = ((\Lambda X.(M'' : A_0))\overline{\langle \forall X.c \rangle})\,\star, \quad M' = (M''\overline{\langle c \rangle})[X := \star], \quad \Sigma' = \Sigma, \quad (\exists X, A_0, \overline{\langle c \rangle}, M'') .$$

From $\Sigma \mid \emptyset \vdash ((\Lambda X.(M'' : A_0))\overline{\langle \forall X.c \rangle})\,\star : A$, we have

$$\Sigma \mid \emptyset \vdash (\Lambda X.(M'' : A_0))\overline{\langle \forall X.c \rangle} : \forall X.D, \quad A = D[X := \star] \quad (\exists D) .$$

By Lemma E.3, $\Sigma \mid \emptyset \vdash \Lambda X.(M'' : A_0) : \forall X.A_0$ and $\Sigma \vdash \overline{\langle \forall X.c \rangle} : \forall X.A_0 \rightsquigarrow \forall X.D$. From $\Sigma \mid \emptyset \vdash \Lambda X.(M'' : A_0) : \forall X.A_0$, we have $\Sigma \mid \emptyset, X \vdash M'' : A_0$. By Lemma E.16, $\Sigma \mid \emptyset \vdash M''[X := \star] : A_0[X := \star]$. By Lemma E.15, $\Sigma \vdash \overline{\langle c[X := \star] \rangle} : A_0[X := \star] \rightsquigarrow D[X := \star]$. Thus, $\Sigma \mid \emptyset \vdash M''[X := \star]\overline{\langle c[X := \star] \rangle} : D[X := \star]$, which is what we need to show.

Case (R_TYBETA_C): We have

$$M = ((\Lambda X.(M'' : A_0))\overline{\langle \forall X.c \rangle})\,\mathbb{B}, \quad M' = (M''\overline{\langle c \rangle})[X := \alpha]\langle coerce^+_\alpha(A_n[X := \alpha]) \rangle,$$
$$\Sigma' = \Sigma, \alpha := \mathbb{B}, \quad \Sigma \vdash \overline{\langle \forall X.c \rangle} : \forall X.A_0 \rightsquigarrow \forall X.A_n, \quad \alpha \notin \mathrm{dom}(\Sigma) \quad (\exists X, \alpha, A_0, \mathbb{B}, A_n, \overline{\langle c \rangle}, M'') .$$

From $\Sigma \mid \emptyset \vdash ((\Lambda X.(M'' : A_0))\overline{\langle \forall X.c \rangle})\,\mathbb{B} : A$, we have

$$\Sigma \mid \emptyset \vdash (\Lambda X.(M'' : A_0))\overline{\langle \forall X.c \rangle} : \forall X.D, \quad \Sigma \mid \emptyset \vdash \mathbb{B}, \quad A = D[X := \mathbb{B}] \quad (\exists D) .$$

By Lemmas E.3 and E.1, $\Sigma \mid \emptyset \vdash \Lambda X.(M'' : A_0) : \forall X.A_0$ and $A_n = D$. Thus, it suffices to show

$$\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash (M''\overline{\langle c \rangle})[X := \alpha]\langle coerce^+_\alpha(A_n[X := \alpha]) \rangle : A_n[X := \mathbb{B}] .$$

From $\Sigma \mid \emptyset \vdash \Lambda X.(M'' : A_0) : \forall X.A_0$, we have $\Sigma \mid \emptyset, X \vdash M'' : A_0$. By Lemma E.10, we have $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash \forall X.A_n$. Then, because $\vdash \Sigma, \alpha := \mathbb{B}$ by (SW_BINDING), Lemma E.8 implies $\Sigma, \alpha := \mathbb{B} \mid \emptyset, X \vdash M'' : A_0$. By Lemma E.14, $\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash M''[X := \alpha] : A_0[X := \alpha]$. By Lemmas E.6 and E.13, $\Sigma, \alpha := \mathbb{B} \vdash \overline{\langle c[X := \alpha] \rangle} : A_0[X := \alpha] \rightsquigarrow A_n[X := \alpha]$. Thus,

$$\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash M''[X := \alpha]\overline{\langle c[X := \alpha] \rangle} : A_n[X := \alpha] .$$

From $\Sigma \mid \emptyset \vdash \forall X.A_n$, we have $\Sigma \mid \emptyset, X \vdash A_n$. Because $\alpha \notin \mathrm{dom}(\Sigma)$, α does not occur in A_n. By Lemma D.2 (1), $\Sigma, \alpha := \mathbb{B} \mid \emptyset, X \vdash A_n$. By (TEW_EMPTY) and (TEW_TYVAR), $\Sigma, \alpha := \mathbb{B} \vdash \emptyset, X$. Therefore, by Lemma E.17, $\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash coerce^+_\alpha(A_n[X := \alpha]) : A_n[X := \alpha] \rightsquigarrow A_n[X := \mathbb{B}]$. By (T_CRC_C), $\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash M''[X := \alpha]\overline{\langle c[X := \alpha] \rangle}\langle coerce^+_\alpha(A_n[X := \alpha]) \rangle : A_n[X := \mathbb{B}]$.

Case (R_BLAME_C): Similar to the case for (R_CONFLICT_C).

Case (R_CtxE_C): We have

$$M = E[M_1], \quad M' = E[M_1'], \quad \Sigma \triangleright M_1 \longrightarrow \Sigma' \triangleright M_1' \quad (\exists E, M_1, M_1') .$$

Case analysis on E.

Case $E = \square\, M''(\exists M'')$: We have $E[M_1] = M_1\, M''$. From $\Sigma \mid \emptyset \vdash M_1\, M'' : A$,

$$\Sigma \mid \emptyset \vdash M_1 : B \to A, \quad \Sigma \mid \emptyset \vdash M'' : B \quad (\exists B) .$$

By Lemma E.18, we consider the two cases below.

Case $\Sigma' = \Sigma$: It suffices to show $\Sigma \mid \emptyset \vdash M_1'\, M'' : A$. By the IH, $\Sigma \mid \emptyset \vdash M_1' : B \to A$. By (T_App_C), $\Sigma \mid \emptyset \vdash M_1'\, M'' : A$.

Case $\Sigma' = \Sigma, \alpha := \mathbb{C}(\exists \alpha, \mathbb{C})$: We have $\alpha \notin \mathrm{dom}(\Sigma)$ and $\Sigma \mid \emptyset \vdash \mathbb{C}$. It suffices to show $\Sigma, \alpha := \mathbb{C} \mid \emptyset \vdash M_1'\, M'' : A$. By Lemma E.10 and (Sw_Binding), $\vdash \Sigma, \alpha := \mathbb{C}$. By Lemma E.8, $\Sigma, \alpha := \mathbb{C} \mid \emptyset \vdash M'' : B$. By the IH, $\Sigma, \alpha := \mathbb{C} \mid \emptyset \vdash M_1' : B \to A$. By (T_App_C), $\Sigma, \alpha := \mathbb{C} \mid \emptyset \vdash M_1'\, M'' : A$.

Otherwise: The other cases follow similarly (with Lemmas D.2 (1) and E.6).

□

Corollary E.20 (Preservation (multi step))**.** If $\Sigma \mid \emptyset \vdash M : A$ and $\Sigma \triangleright M \longrightarrow^* \Sigma' \triangleright M'$, then $\Sigma' \mid \emptyset \vdash M' : A$.

Theorem E.21 (Type Safety (Theorem 3.2 of the paper))**.** If $\Sigma \mid \emptyset \vdash M : A$, then one of the followings holds:

- $\Sigma \triangleright M \longrightarrow^* \Sigma' \triangleright V$ for some store Σ' and value V such that $\Sigma' \mid \emptyset \vdash V : A$;
- $\Sigma \triangleright M \longrightarrow^* \Sigma' \triangleright \mathsf{blame}\, p$ for some store Σ' and blame label p; or
- $\Sigma \triangleright M \Uparrow$.

Proof. By Theorem E.4 and Corollary E.20. □

E.2 $\lambda \mathbf{S}^{\forall}_{mp}$

The proof of type safety of $\lambda \mathrm{S}^{\forall}_{mp}$ is similar to that of $\lambda \mathrm{C}^{\forall}_{mp}$. A main difference is the coercion composition $s \fatsemi t$. We first prove Lemma E.24 that states that, if s and t are well typed and the target type of s and the source type of t agree, then $s \fatsemi t$ is well defined and well typed.

Lemma E.22 (Agreement (1))**.** If $\Sigma \mid \Gamma \vdash s : A \rightsquigarrow B$, then $\vdash \Sigma$ and $\emptyset \vdash \Gamma$ and $\emptyset \mid \Gamma \vdash A$ and $\emptyset \mid \Gamma \vdash B$ and $\mathrm{ftv}(s) \subseteq \mathrm{dom}(\Gamma)$.

Proof. By straightforward induction on $\Sigma \mid \Gamma \vdash s : A \rightsquigarrow B$ with Lemma D.10(1). □

Lemma E.23. For any $\mathbb{A}$ and Σ, $\Sigma(\mathbb{A}) \neq \star$.

Proof. By induction on the size of Σ. Obvious in the cases where $\mathbb{A}$ is not a type name. Assume that $\mathbb{A} = \alpha$ for some α. If $\alpha \in \mathrm{dom}(\Sigma)$, then $\Sigma = \Sigma_1, \alpha := \mathbb{B}, \Sigma_2$ for some $\mathbb{B}, \Sigma_1, \Sigma_2$. By definition, $\Sigma(\alpha) = \Sigma_1(\mathbb{B})$. By the IH, $\Sigma_1(\mathbb{B}) \neq \star$. Therefore, $\Sigma(\alpha) \neq \star$. Otherwise, if $\alpha \notin \mathrm{dom}(\Sigma)$, then $\Sigma(\alpha) = \alpha \neq \star$. □

Lemma E.24 (Coercion Composition is Well Typed)**.** If $\Sigma \mid \Gamma \vdash s : A \rightsquigarrow B$ and $\Sigma \mid \Gamma \vdash t : B \rightsquigarrow C$, then $s' = s \fatsemi t$ and $\Sigma \mid \Gamma \vdash s' : A \rightsquigarrow C$ for some s'. Moreover:

- if $s = b$, then either $b' = s \fatsemi t$ for some b', or $A = \star$;
- if $s = i$ and $t = h$, then either $g' = s \fatsemi t$ for some g', or $C = \star$; and
- if $s = g$ and $t = h$, then $g' = s \fatsemi t$ for some g'.

Proof. By induction on the sum of the sizes of s and t with case analysis on the shape of s.

Case $s = G?^p\ ;\ b(\exists p, G, b)$: From $\Sigma \mid \Gamma \vdash\ G?^p\ ;\ b : A \rightsquigarrow B$,

$$A = \star, \quad \Sigma \mid \Gamma \vdash\ b : \Sigma(G) \rightsquigarrow B\ .$$

By Lemma E.23, $\Sigma(G) \neq \star$. Therefore, by the IH, there exists some b' such that $b' = b \fatsemi t$ and $\Sigma \mid \Gamma \vdash\ b' : \Sigma(G) \rightsquigarrow C$. Since $s \fatsemi t = (G?^p\ ;\ b) \fatsemi t = G?^p\ ;\ (b \fatsemi t) = G?^p\ ;\ b'$, it suffices to show $\Sigma \mid \Gamma \vdash\ G?^p\ ;\ b' : \star \rightsquigarrow C$, which follows from (Ct_Proj_S).

Case $s = \perp^p(\exists p)$: By definition, $\perp^p \fatsemi t = \perp^p$. It suffices to show $\Sigma \mid \Gamma \vdash\ \perp^p : A \rightsquigarrow C$. By Lemma E.22, $\vdash \Sigma$ and $\emptyset \vdash \Gamma$ and $\emptyset \mid \Gamma \vdash A$ and $\emptyset \mid \Gamma \vdash C$. By (Ct_Fail_S), $\Sigma \mid \Gamma \vdash\ \perp^p : A \rightsquigarrow C$.

Case $s = g\ ;\ G!(\exists G, g)$: From $\Sigma \mid \Gamma \vdash\ g\ ;\ G! : A \rightsquigarrow B$, we have

$$B = \star, \quad \Sigma \mid \Gamma \vdash\ g : A \rightsquigarrow \Sigma(G)\ .$$

Case analysis on the shape of t.

Case $t = G?^p\ ;\ b(\exists p, b)$: From $\Sigma \mid \Gamma \vdash\ G?^p\ ;\ b : \star \rightsquigarrow C$,

$$\Sigma \mid \Gamma \vdash\ b : \Sigma(G) \rightsquigarrow C\ .$$

By the IH, there exists some s' such that $s' = g \fatsemi b$ and $\Sigma \mid \Gamma \vdash\ s' : A \rightsquigarrow C$. By definition, $s \fatsemi t = (g\ ;\ G!) \fatsemi (G?^p\ ;\ b) = g \fatsemi b = s'$. The IH also implies $s' = b'$ for some b', or $A = \star$. For the former case, $s \fatsemi t = b'$, so we finish the case.

Case $t = H?^p\ ;\ b(H \neq G)(\exists p, H, b)$: We have $s \fatsemi t = (g\ ;\ G!) \fatsemi (H?^p\ ;\ b) = \perp^p$ and it suffices to show $\Sigma \mid \Gamma \vdash\ \perp^p : A \rightsquigarrow C$. By Lemma E.22, $\vdash \Sigma$ and $\emptyset \vdash \Gamma$ and $\emptyset \mid \Gamma \vdash A$ and $\emptyset \mid \Gamma \vdash C$. By (Ct_Fail_S), $\Sigma \mid \Gamma \vdash\ \perp^p : A \rightsquigarrow C$.

Case $t = \perp^p(\exists p)$: By definition, $s \fatsemi t = (g\ ;\ G!) \fatsemi \perp^p = \perp^p$. It suffices to show $\Sigma \mid \Gamma \vdash\ \perp^p : A \rightsquigarrow C$. By Lemma E.22, $\vdash \Sigma$ and $\emptyset \vdash \Gamma$ and $\emptyset \mid \Gamma \vdash A$ and $\emptyset \mid \Gamma \vdash C$. Thus, by (Ct_Fail_S), $\Sigma \mid \Gamma \vdash\ \perp^p : A \rightsquigarrow C$.

Case $t = h\ ;\ H!(\exists H, h)$: From $\Sigma \mid \Gamma \vdash\ h\ ;\ H! : \star \rightsquigarrow C$, we have

$$C = \star, \quad \Sigma \mid \Gamma \vdash\ h : \star \rightsquigarrow \Sigma(H)\ .$$

By Lemma E.23, $\Sigma(H) \neq \star$. Therefore, by the IH, there exists some g' such that $g' = (g\ ;\ G!) \fatsemi h$ and $\Sigma \mid \Gamma \vdash\ g' : A \rightsquigarrow \Sigma(H)$. Since $s \fatsemi t = (g\ ;\ G!) \fatsemi (h\ ;\ H!) = ((g\ ;\ G!) \fatsemi h)\ ;\ H! = g'\ ;\ H!$, it suffices to show $\Sigma \mid \Gamma \vdash\ g'\ ;\ H! : A \rightsquigarrow \star$, which follows from (Ct_Inj_S).

Case $t = \mathsf{id}$: We have $s \fatsemi t = (g\ ;\ G!) \fatsemi \mathsf{id} = g\ ;\ G!$. From $\Sigma \mid \Gamma \vdash\ \mathsf{id} : B \rightsquigarrow C$ and $B = \star$, we have $C = \star$. Thus, it suffices to show $\Sigma \mid \Gamma \vdash\ g\ ;\ G! : A \rightsquigarrow B$, which we already have.

Case $t = s' \to t'(\exists s', t')$ or $t = \forall X.t'(\exists X, t')$: These cases cannot happen because they contradict $B = \star$.

Case $s = \mathsf{id}$: From $\Sigma \mid \Gamma \vdash\ \mathsf{id} : A \rightsquigarrow B$, we have

$$A = B = \Sigma(A'), \quad \vdash \Sigma, \quad \emptyset \vdash \Gamma, \quad \Sigma \mid \Gamma \vdash A'\ (\exists A').$$

Case analysis on the shape of t.

Case $t = h\ ;\ H!(\exists H, h)$: From $\Sigma \mid \Gamma \vdash\ h\ ;\ H! : \Sigma(A') \rightsquigarrow C$, we have

$$C = \star, \quad \Sigma \mid \Gamma \vdash\ h : \Sigma(A') \rightsquigarrow \Sigma(H)\ .$$

By the IH, there exists some g' such that $g' = \mathsf{id} \fatsemi h$ and $\Sigma \mid \Gamma \vdash\ g' : \Sigma(A') \rightsquigarrow \Sigma(H)$. We have $s \fatsemi t = \mathsf{id} \fatsemi (h\ ;\ H!) = (\mathsf{id} \fatsemi h)\ ;\ H! = g'\ ;\ H!$. By (Ct_Inj_S), we have $\Sigma \mid \Gamma \vdash\ g'\ ;\ H! : \Sigma(A') \rightsquigarrow \star$.

Case $t = \mathsf{id}$: From $\Sigma \mid \Gamma \vdash\ \mathsf{id} : B \rightsquigarrow C$, we have

$$C = B = \Sigma(B'), \quad \Sigma \mid \Gamma \vdash B'\ (\exists B').$$

Because $B = \Sigma(A')$, we have $\Sigma(A') = \Sigma(B')$. We have $s \fatsemi t = \mathsf{id} \fatsemi \mathsf{id} = \mathsf{id}$. Therefore, it suffices to show that $\Sigma \mid \Gamma \vdash\ \mathsf{id} : \Sigma(A') \rightsquigarrow \Sigma(A')$, which we already have.

Otherwise: We have $s \fatsemi t = \mathsf{id} \fatsemi t = t$ and it suffices to show that $\Sigma \mid \Gamma \vdash t : \Sigma(A') \rightsquigarrow C$, which we already have. Furthermore, we show the following additional properties.

- We show that either $b' = t$ for some b', or $A = \star$. Assume that there exists no b' such that $b' = t$. Then, $t = H?^p ; b$ for some H, p, and b. From $\Sigma \mid \Gamma \vdash H?^p ; b : \Sigma(A') \rightsquigarrow C$, $\Sigma(A') = A = \star$.
- if $t = h$, then $s \fatsemi t = h$.

Case $s = s' \to t' (\exists s', t')$: From $\Sigma \mid \Gamma \vdash s' \to t' : A \rightsquigarrow B$, we have

$$A = A' \to B', \quad B = A'' \to B'', \quad \Sigma \mid \Gamma \vdash s' : A'' \rightsquigarrow A', \quad \Sigma \mid \Gamma \vdash t' : B' \rightsquigarrow B'' \quad (\exists A', A'', B', B'') .$$

Case analysis on the shape of t.

Case $t = G?^p ; b(\exists p, G, b)$ or $t = \forall X.t''(\exists X, t'')$: These cases cannot happen since they contradict $B = A'' \to B''$.

Case $t = \bot^p(\exists p)$: By definition, $s \fatsemi t = (s' \to t') \fatsemi \bot^p = \bot^p$. It suffices to show $\Sigma \mid \Gamma \vdash \bot^p : (A' \to B') \rightsquigarrow C$. By Lemma E.22, $\vdash \Sigma$ and $\emptyset \vdash \Gamma$ and $\emptyset \mid \Gamma \vdash A' \to B'$ and $\emptyset \mid \Gamma \vdash C$. Thus, by (Ct_Fail_S), $\Sigma \mid \Gamma \vdash \bot^p : (A' \to B') \rightsquigarrow C$.

Case $t = h ; H!(\exists H, h)$: From $\Sigma \mid \Gamma \vdash h ; H! : (A'' \to B'') \rightsquigarrow C$, we have

$$C = \star, \quad \Sigma \mid \Gamma \vdash h : (A'' \to B'') \rightsquigarrow \Sigma(H) .$$

By the IH, there exists some g' such that $g' = (s' \to t') \fatsemi h$ and $\Sigma \mid \Gamma \vdash g' : A' \to B' \rightsquigarrow \Sigma(H)$. We have $s \fatsemi t = (s' \to t') \fatsemi (h ; H!) = (s' \to t' \fatsemi h) ; H! = g' ; H!$. By (Ct_Inj_S), we have $\Sigma \mid \Gamma \vdash g' ; H! : A' \to B' \rightsquigarrow \star$.

Case $t = \mathsf{id}$: From $\Sigma \mid \Gamma \vdash \mathsf{id} : A'' \to B'' \rightsquigarrow C$, we have

$$A'' \to B'' = C .$$

We have $s \fatsemi t = (s' \to t') \fatsemi \mathsf{id} = s' \to t'$. Therefore, it suffices to show that $\Sigma \mid \Gamma \vdash s' \to t' : A' \to B' \rightsquigarrow A'' \to B''$, which we already have.

Case $t = s'' \to t''(\exists s'', t'')$: From $\Sigma \mid \Gamma \vdash s'' \to t'' : (A'' \to B'') \rightsquigarrow C$, we have

$$C = A''' \to B''', \quad \Sigma \mid \Gamma \vdash s'' : A''' \rightsquigarrow A'', \quad \Sigma \mid \Gamma \vdash t'' : B'' \rightsquigarrow B''' \quad (\exists A''', B''') .$$

By the IHs, there exist some s''' and t''' such that $s''' = s'' \fatsemi s'$ and $\Sigma \mid \Gamma \vdash s''' : A''' \rightsquigarrow A'$ and $t''' = t' \fatsemi t''$ and $\Sigma \mid \Gamma \vdash t''' : B' \rightsquigarrow B'''$. Since $s \fatsemi t = (s' \to t') \fatsemi (s'' \to t'') = (s'' \fatsemi s') \to (t' \fatsemi t'') = s''' \to t'''$, it suffices to show $\Sigma \mid \Gamma \vdash s''' \to t''' : (A' \to B') \rightsquigarrow (A''' \to B''')$, which follows from (Ct_Arrow).

Case $s = \forall X.s' \,,, s''(\exists X, s', s'')$: From $\Sigma \mid \Gamma \vdash \forall X.s' : A \rightsquigarrow B$, we have

$$A = \forall X.A', \quad B = \forall X.B', \quad \Sigma \mid \Gamma, X \vdash s' : A' \rightsquigarrow B', \quad \Sigma \mid \Gamma \vdash s'' : A'[X := \star] \rightsquigarrow B'[X := \star] \quad (\exists A', B') .$$

Case analysis on the shape of t.

Case $t = H?^p ; b(\exists p, H, b)$ or $t = s'' \to t''(\exists s'', t'')$: These cases cannot happen since they contradict $B = \forall X.B'$.

Case $t = \bot^p(\exists p)$: By definition, $s \fatsemi t = (\forall X.s') \fatsemi \bot^p = \bot^p$. It suffices to show $\Sigma \mid \Gamma \vdash \bot^p : \forall X.A' \rightsquigarrow C$. By Lemma E.22, $\vdash \Sigma$ and $\emptyset \vdash \Gamma$ and $\emptyset \mid \Gamma \vdash \forall X.A'$ and $\emptyset \mid \Gamma \vdash C$. Thus, by (Ct_Fail_S), $\Sigma \mid \Gamma \vdash \bot^p : \forall X.A' \rightsquigarrow C$.

Case $t = h ; H!(\exists h, H)$: From $\Sigma \mid \Gamma \vdash h ; H! : \forall X.B' \rightsquigarrow C$, we have

$$C = \star, \quad \Sigma \mid \Gamma \vdash h : \forall X.B' \rightsquigarrow \Sigma(H) .$$

By the IH, there exists some g' such that $g' = (\forall X.s' \,,, s'') \fatsemi h$ and $\Sigma \mid \Gamma \vdash g' : \forall X.A' \rightsquigarrow \Sigma(H)$. We have $s \fatsemi t = (\forall X.s' \,,, s'') \fatsemi (h ; H!) = ((\forall X.s' \,,, s'') \fatsemi h) ; H! = g' ; H!$. By (Ct_Inj_S), we have $\Sigma \mid \Gamma \vdash g' ; H! : \forall X.A' \rightsquigarrow \star$.

Case $t = \mathsf{id}$: From $\Sigma \mid \Gamma \vdash \mathsf{id} : \forall X.B' \rightsquigarrow C$, we have

$$\forall X.B' = C\ .$$

We have $s \mathbin{⨾} t = (\forall X.s' \mathbin{,,} s'') \mathbin{⨾} \mathsf{id} = \forall X.s' \mathbin{,,} s''$. Therefore, it suffices to show that $\Sigma \mid \Gamma \vdash \forall X.s' \mathbin{,,} s'' : \forall X.A' \rightsquigarrow \forall X.B'$, which we already have.

Case $t = \forall Y.t' \mathbin{,,} t''(\exists Y, t', t'')$: From $\Sigma \mid \Gamma \vdash \forall Y.t' : \forall X.B' \rightsquigarrow C$, we have

$$Y = X, \quad C = \forall X.C', \quad \Sigma \mid \Gamma, X \vdash t' : B' \rightsquigarrow C', \Sigma \mid \Gamma \vdash t'' : B'[X := \star] \rightsquigarrow C'[X := \star] \quad (\exists C')\ .$$

By the IH, there exists some s''' and t''' such that $s''' = s' \mathbin{⨾} t'$ and $\Sigma \mid \Gamma, X \vdash s''' : A' \rightsquigarrow C'$ and $t''' = s'' \mathbin{⨾} t''$ and $\Sigma \mid \Gamma \vdash t''' : A'[X := \star] \rightsquigarrow C'[X := \star]$ Since $s \mathbin{⨾} t = (\forall X.s' \mathbin{,,} s'') \mathbin{⨾} (\forall X.t' \mathbin{,,} t'') = \forall X.(s' \mathbin{⨾} t') \mathbin{,,} (s'' \mathbin{,,} t'') = \forall X.s''' \mathbin{,,} t'''$, it suffices to show $\Sigma \mid \Gamma \vdash \forall X.s''' \mathbin{,,} t''' : \forall X.A' \rightsquigarrow \forall X.C'$, which follows from (Ct_All_S).

□

Next we prove the canonical forms lemma and Progress.

Lemma E.25 (Canonical forms)**.** If $\Sigma \mid \emptyset \vdash V : A$, then one of the followings holds:

- $V = k$ and $A = ty(k)$ for some k;
- $V = \lambda x\!:\!A'.M$ and $A = \Sigma(A') \to B$ for some A', B, x, M;
- $V = \Lambda X.M$ and $A = \forall X.A'$ for some X, A', M;
- $V = U\langle g\ ;\ G!\rangle$ and $A = \star$ for some G, g, U;
- $V = U\langle s \to t\rangle$ and $A = A' \to B'$ for some A', B', s, t, U; or
- $V = U\langle \forall X.s \mathbin{,,} t\rangle$ and $A = \forall X.A'$ for some X, A', s, t, U.

Proof. Straightforward by case analysis on V. □

Theorem E.26 (Progress)**.** If $\Sigma \mid \emptyset \vdash M : A$, then one of the followings holds:

- $M = V$ for some V;
- $M = \mathsf{blame}\, p$ for some p; or
- $\Sigma \rhd M \longrightarrow \Sigma' \rhd M'$ for some Σ', M'.

Proof. By induction on $\Sigma \mid \emptyset \vdash M : A$ with case analysis on the last rule used. Most cases are similar to the proof of Theorem E.4, using Lemma E.25.

Case (T_Const_S), (T_Abs_S), (T_Tyabs_S), (T_Blame_S): Immediate.

Case (T_Var_S): Cannot happen.

Case (T_App_S): Similar to the case of (T_App_C) in Theorem E.4.

Case (T_Tyapp_S): We have

$$M = M'\, A', \quad A = B[X := \Sigma(A')], \quad \Sigma \mid \emptyset \vdash M' : \forall X.B, \quad \Sigma \mid \emptyset \vdash A' \quad (\exists X, A', B, M')\ .$$

By the IH, we have three subcases.

Case $M' = V(\exists V)$: By Lemma E.25, there are four cases for V and A'.

Case $V = \Lambda X.M'', A' = \mathbb{B}(\exists M'', \mathbb{B})$: By (R_Tybeta_S), $\Sigma \rhd (\Lambda X.M'')\, \mathbb{B} \longrightarrow \Sigma, \alpha := \mathbb{B} \rhd M''[X := \alpha]$.

Case $V = U\langle \forall X.s \mathbin{,,} t\rangle, A' = \mathbb{B}(\exists s, t, U, \mathbb{B})$: From $\Sigma \mid \emptyset \vdash U\langle \forall X.s\rangle : \forall X.B$, we have

$$\Sigma \mid \emptyset \vdash U : \forall X.A'', \quad \Sigma \mid \emptyset \vdash \forall X.s \mathbin{,,} t : \forall X.A'' \rightsquigarrow \forall X.B \quad (\exists A'')\ .$$

By Lemma E.25, $U = \Lambda X.M''$ for some M''. By (R_TybetaC_S), $\Sigma \rhd (\Lambda X.M'')\langle \forall X.s \mathbin{,,} t\rangle\, \mathbb{B} \longrightarrow \Sigma, \alpha := \mathbb{B} \rhd (M''\langle s\rangle)[X := \alpha]$.

Case $V = \Lambda X.M''$, $A' = \star(\exists M'')$: By (R_TYBETADYN_S), $\Sigma \triangleright (\Lambda X.M'')\star \longrightarrow \Sigma \triangleright M''[X := \star]$.

Case $V = U\langle\forall X.s \,,, t\rangle$, $A' = \star(\exists s, t, U)$: By $\Sigma \mid \emptyset \vdash U\langle\forall X.s \,,, t\rangle : \forall X.B$ and Lemma E.25, $U = \Lambda X.M''$ for some M''. By (R_TYBETADYNC_S), $\Sigma \triangleright (\Lambda X.M'')\langle\forall X.s \,,, t\rangle\star \longrightarrow \Sigma \triangleright (M''\langle t\rangle)[X := \star]$.

Case $M' = \mathsf{blame}\, p(\exists p)$: By (R_BLAMEE_S), $\Sigma \triangleright (\mathsf{blame}\, p)\, A' \longrightarrow \Sigma \triangleright \mathsf{blame}\, p$.

Case $\Sigma \triangleright M' \longrightarrow \Sigma' \triangleright M''(\exists \Sigma', M'')$: By (R_CTXE_S), $\Sigma \triangleright M'\, A' \longrightarrow \Sigma' \triangleright M''\, A'$.

Case (T_CRC_S): We have

$$M = M'\langle s\rangle, \quad \Sigma \mid \emptyset \vdash M' : B, \quad \Sigma \mid \emptyset \vdash s : B \rightsquigarrow A \quad (\exists B, s, M')\ .$$

By the IH, we have three subcases.

Case $M' = V(\exists V)$: Case analysis on V.

Case $V = U\langle t\rangle(\exists U, t)$: From $\Sigma \mid \emptyset \vdash U\langle t\rangle : B$, we have $\Sigma \mid \emptyset \vdash U : C$ and $\Sigma \mid \emptyset \vdash t : C \rightsquigarrow B$ for some C. By Lemma E.24, $t \,⨟\, s$ is well defined. By (R_MERGE_S), $\Sigma \triangleright U\langle t\rangle\langle s\rangle \longrightarrow \Sigma \triangleright U\langle t \,⨟\, s\rangle$.

Case $V = U(\exists U)$: Case analysis on s.

Case $s = G?^p\, ;\, b(\exists p, G, b)$: From $\Sigma \mid \emptyset \vdash G?^p\, ;\, b : B \rightsquigarrow A$, we have $B = \star$. Thus, $\Sigma \mid \emptyset \vdash U : \star$ and, by Lemma E.25, there exist some H, h, U' such that $U = U'\langle h\, ;\, H!\rangle$. Contradiction.

Case $s = \bot^p(\exists p)$: By (R_FAIL_S), $\Sigma \triangleright U\langle\bot^p\rangle \longrightarrow \Sigma \triangleright \mathsf{blame}\, p$.

Case $s = g\, ;\, G!(\exists G, g)$: $U\langle g\, ;\, G!\rangle$ is a value.

Case $s = \mathsf{id}$: By (R_ID_S), $\Sigma \triangleright U\langle\mathsf{id}\rangle \longrightarrow \Sigma \triangleright U$.

Case $s = s' \to t(\exists s', t)$: $U\langle s' \to t\rangle$ is a value.

Case $s = \forall X.s' \,,, s''(\exists X, s', s'')$: $U\langle\forall X.s' \,,, s''\rangle$ is a value.

Case $M' = \mathsf{blame}\, p(\exists p)$: By (R_BLAMEC_S), $\Sigma \triangleright (\mathsf{blame}\, p)\langle s\rangle \longrightarrow \Sigma \triangleright \mathsf{blame}\, p$.

Case $\Sigma \triangleright M' \longrightarrow \Sigma' \triangleright M''(\exists \Sigma', M'')$: We have two cases depending on $\longrightarrow$ is $\xrightarrow{e}$ or $\xrightarrow{c}$.

Case $\xrightarrow{e}$: By (R_CTXC_S), $\Sigma \triangleright M'\langle s\rangle \longrightarrow \Sigma' \triangleright M''\langle s\rangle$.

Case $\xrightarrow{c}$: By the definition of $\xrightarrow{c}$, it must be the case that M' is a coercion application of the form $M'''\langle t\rangle$ for some t, M'''. From $\Sigma \mid \emptyset \vdash M'''\langle t\rangle : B$, we have $\Sigma \mid \emptyset \vdash M''' : C$ and $\Sigma \mid \emptyset \vdash t : C \rightsquigarrow B$ for some C. By Lemma E.24, $t \,⨟\, s$ is well defined. By (R_MERGE_S), $\Sigma \triangleright M'''\langle t\rangle\langle s\rangle \longrightarrow \Sigma \triangleright M'''\langle t \,⨟\, s\rangle$.

□

The proof of preservation starts with various weakeninng lemmas.

Lemma E.27. If $\Sigma \mid \Delta_2 \vdash s : A \rightsquigarrow B$ and $\Delta_1 \# \Delta_2$, then $\Sigma \mid \Delta_1, \Delta_2 \vdash s : A \rightsquigarrow B$.

Proof. By straightforward induction on $\Sigma \mid \Delta_2 \vdash s : A \rightsquigarrow B$ with Lemma D.1. □

Lemma E.28. If $\vdash \Sigma, \Sigma'$ and $\Sigma \mid \Gamma \vdash A$, then $\Sigma(A) = (\Sigma, \Sigma')(A)$.

Proof. By induction on Σ'.

Case $\Sigma' = \emptyset$: Obvious.

Case $\Sigma' = \Sigma'', \alpha := \mathbb{B}(\exists \Sigma'', \alpha, \mathbb{B})$: From $\vdash \Sigma, \Sigma'', \alpha := \mathbb{B}$, we have $\vdash \Sigma, \Sigma''$ and $\alpha \notin \mathrm{dom}(\Sigma)$. Because $\Sigma \mid \Gamma \vdash A$, type name α does not occur in type A. Therefore, $(\Sigma, \Sigma')(A) = (\Sigma, \Sigma'')(A[\alpha := \mathbb{B}]) = (\Sigma, \Sigma'')(A)$. By the IH, $(\Sigma, \Sigma'')(A) = \Sigma(A)$.

□

Lemma E.29. If $\vdash \Sigma$ and $\emptyset \mid \Gamma \vdash \Sigma(A)$, then $\Sigma \mid \Gamma \vdash A$.

Proof. By straightforward induction on A. □

Lemma E.30. If $\Sigma \mid \Gamma \vdash s : A \rightsquigarrow B$ and $\vdash \Sigma, \Sigma'$, then $\Sigma, \Sigma' \mid \Gamma \vdash s : A \rightsquigarrow B$.

Proof. By induction on $\Sigma \mid \Gamma \vdash s : A \rightsquigarrow B$.

Case (CT_ID_S): By Lemmas D.2 (1) and E.28 and (CT_ID_S).

Case (CT_FAIL_S): By (CT_FAIL_S).

Case (CT_INJ_S): We have

$$s = g\,;G!, \quad B = \star, \quad \Sigma \mid \Gamma \vdash g : A \rightsquigarrow \Sigma(G)\ (\exists g, G)\ .$$

By the IH, $\Sigma, \Sigma' \mid \Gamma \vdash g : A \rightsquigarrow \Sigma(G)$. By (CT_INJ_S), it suffices to show that $\Sigma(G) = (\Sigma, \Sigma')(G)$. By Lemma E.22, $\vdash \Sigma$ and $\emptyset \mid \Gamma \vdash \Sigma(G)$. By Lemma E.29, $\Sigma \mid \Gamma \vdash G$. By Lemma E.28, $\Sigma(G) = (\Sigma, \Sigma')(G)$.

Case (CT_PROJ_S): Similar to the case of (CT_INJ_S).

Case (CT_ARROW_S) and (CT_ALL_S): By the IH(s) and the corresponding coercion typing rule.

□

Lemma E.31. If $\Sigma \mid \Gamma_2 \vdash M : A$ and $\emptyset \vdash \Gamma_1$ and $\Gamma_1 \# \Gamma_2$, then $\Sigma \mid \Gamma_1, \Gamma_2 \vdash M : A$.

Proof. By straightforward induction on $\Sigma \mid \Gamma_2 \vdash M : A$ with Lemma D.1. □

Lemma E.32 (Agreement (2))**.** If $\Sigma \mid \Gamma \vdash M : A$, then $\vdash \Sigma$ and $\emptyset \vdash \Gamma$ and $\emptyset \mid \Gamma \vdash A$.

Proof. By straightforward induction on $\Sigma \mid \Gamma \vdash M : A$ with Lemmas D.4 (2), D.3 (1), D.10 (4), D.5 (1), E.22, and D.1. □

Lemma E.33. If $\Sigma \mid \Gamma \vdash M : A$ and $\vdash \Sigma, \Sigma'$, then $\Sigma, \Sigma' \mid \Gamma \vdash M : A$.

Proof. By induction on $\Sigma \mid \Gamma \vdash M : A$. Most cases are proven easily. We mention only the interesting cases.

Case (T_ABS_S): We have

$$M = \lambda x\!:\!A'.M', \quad A = \Sigma(A') \to B, \quad \Sigma \mid \Gamma, x : \Sigma(A') \vdash M' : B\ (\exists x, A', B, M')\ .$$

By the IH, $\Sigma, \Sigma' \mid \Gamma, x : \Sigma(A') \vdash M' : B$. By Lemma E.32, $\vdash \Sigma$ and $\emptyset \vdash \Gamma, x : \Sigma(A')$. Therefore, $\emptyset \mid \Gamma \vdash \Sigma(A')$. By Lemma E.29, $\Sigma \mid \Gamma \vdash A'$. By Lemma E.28, $\Sigma(A') = (\Sigma, \Sigma')(A')$. Therefore, we have $\Sigma, \Sigma' \mid \Gamma, x : (\Sigma, \Sigma')(A') \vdash M' : B$. Then, by (T_ABS_S), $\Sigma, \Sigma' \mid \Gamma \vdash \lambda x\!:\!A'.M' : (\Sigma, \Sigma')(A') \to B$. Because $(\Sigma, \Sigma')(A') = \Sigma(A')$, we finish the case.

Case (T_TYAPP_S): We have

$$M = M'\,A', \quad A = B[X := \Sigma(A')], \quad \Sigma \mid \Gamma \vdash M' : \forall X.B, \quad \Sigma \mid \Gamma \vdash A'\ (\exists X, A', B, M')\ .$$

By the IH, $\Sigma, \Sigma' \mid \Gamma \vdash M' : \forall X.B$. By Lemma D.2 (1), $\Sigma, \Sigma' \mid \Gamma \vdash A'$. By (T_TYAPP_S), $\Sigma, \Sigma' \mid \Gamma \vdash M'\,A' : B[X := (\Sigma, \Sigma')(A')]$. Because $(\Sigma, \Sigma')(A') = \Sigma(A')$ by Lemma E.28, we finish the case.

Case (T_CRC_S): By the IH, Lemma E.30, and (T_CRC_S).

□

Lemma E.34 (Value Substitution)**.** If $\Sigma \mid x : A, \Gamma \vdash M : B$ and $\Sigma \mid \emptyset \vdash V : A$, then $\Sigma \mid \Gamma \vdash M[x := V] : B$.

Proof. By straightforward induction on $\Sigma \mid x : A, \Gamma \vdash M : B$ with Lemmas D.3, E.31. □

Lemma E.35. If $\Sigma \mid \Gamma \vdash A$ and $\mathrm{dom}(\Gamma) = \mathrm{dom}(\Gamma')$, then $\Sigma \mid \Gamma' \vdash A$.

Proof. By straightforward induction on $\Sigma \mid \Gamma \vdash A$. □

Lemma E.36. If $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash A$ and $\alpha \in \mathrm{dom}(\Sigma)$, then $\Sigma \mid \Gamma_1, \Gamma_2[X := \Sigma(\alpha)] \vdash A[X := \alpha]$.

Proof. By (TW_NAME), $\Sigma \mid \Gamma_1 \vdash \alpha$. By Lemma D.5 (1), $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \vdash A[X := \alpha]$. By Lemma E.35, $\Sigma \mid \Gamma_1, \Gamma_2[X := \Sigma(\alpha)] \vdash A[X := \alpha]$. □

Lemma E.37. Assume that $\vdash \Sigma$ and $\alpha \in \mathrm{dom}(\Sigma)$.

1. If $\emptyset \mid \Gamma_1, X, \Gamma_2 \vdash A$, then $\emptyset \mid \Gamma_1, \Gamma_2[X := \Sigma(\alpha)] \vdash A[X := \Sigma(\alpha)]$.

2. If $\emptyset \vdash \Gamma_1, X, \Gamma_2$, then $\emptyset \vdash \Gamma_1, \Gamma_2[X := \Sigma(\alpha)]$.

Proof.

1. By (Tw_Name), $\Sigma \mid \Gamma_1 \vdash \alpha$. By Lemma D.10 (1), $\emptyset \mid \Gamma_1 \vdash \Sigma(\alpha)$. Then, by Lemma D.5 (1), we have the conclusion.

2. By (Tw_Name) and Lemmas Lemma D.10 (1) and D.5 (2).

□

Lemma E.38. If $\vdash \Sigma$, then $\Sigma(A[X := B]) = \Sigma(A)[X := \Sigma(B)]$.

Proof. By induction on Σ.

Case $\Sigma = \emptyset$: Obvious.

Case $\Sigma = \Sigma', \alpha := \mathbb{C} (\exists \Sigma', \alpha, \mathbb{C})$: From $\vdash \Sigma$, we have $\vdash \Sigma'$ and $\Sigma' \mid \emptyset \vdash \mathbb{C}$. Therefore, $X \notin \mathrm{ftv}(\mathbb{C})$. Then, $\Sigma(A[X := B]) = (\Sigma', \alpha := \mathbb{C})(A[X := B]) = \Sigma'(A[X := B][\alpha := \mathbb{C}]) = \Sigma'(A[\alpha := \mathbb{C}][X := B[\alpha := \mathbb{C}]])$. By the IH, $\Sigma'(A[\alpha := \mathbb{C}][X := B[\alpha := \mathbb{C}]]) = \Sigma'(A[\alpha := \mathbb{C}])[X := \Sigma'(B[\alpha := \mathbb{C}])] = \Sigma(A)[X := \Sigma(B)]$.

□

Lemma E.39 (Type Name Substitution (Coercion))**.** If $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash s : A \rightsquigarrow B$ and $\alpha \in \mathrm{dom}(\Sigma)$, then $\Sigma \mid \Gamma_1, \Gamma_2[X := \Sigma(\alpha)] \vdash s[X := \alpha] : A[X := \Sigma(\alpha)] \rightsquigarrow B[X := \Sigma(\alpha)]$.

Proof. By induction on $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash s : A \rightsquigarrow B$.

Case (Ct_Id_S): We have

$$s = \mathsf{id}, \quad A = B = \Sigma(A'), \quad \vdash \Sigma, \quad \emptyset \vdash \Gamma_1, X, \Gamma_2, \quad \Sigma \mid \Gamma_1, X, \Gamma_2 \vdash A'$$

for some A' s.t. A' is neither a function nor universal type. By Lemma E.36, $\Sigma \mid \Gamma_1, \Gamma_2[X := \Sigma(\alpha)] \vdash A'[X := \alpha]$. Moreover, $A'[X := \alpha]$ is neither a function nor universal type. By Lemma E.37 (2), $\emptyset \vdash \Gamma_1, \Gamma_2[X := \Sigma(\alpha)]$. Therefore, by (Ct_Id_S), $\Sigma \mid \Gamma_1, \Gamma_2[X := \Sigma(\alpha)] \vdash \mathsf{id} : \Sigma(A'[X := \alpha]) \rightsquigarrow \Sigma(A'[X := \alpha])$. By Lemma E.38, $\Sigma(A'[X := \alpha]) = \Sigma(A')[X := \Sigma(\alpha)]$. Therefore, we have $\Sigma \mid \Gamma_1, \Gamma_2[X := \Sigma(\alpha)] \vdash \mathsf{id} : \Sigma(A')[X := \Sigma(\alpha)] \rightsquigarrow \Sigma(A')[X := \Sigma(\alpha)]$.

Case (Ct_Fail_S): By Lemma E.37 and (Ct_Fail_S).

Case (Ct_Inj_S): We have

$$s = g \,;\, G!, \quad B = \star, \quad \Sigma \mid \Gamma_1, X, \Gamma_2 \vdash g : A \rightsquigarrow \Sigma(G) \ (\exists g, G) \ .$$

By the IH, $\Sigma \mid \Gamma_1, \Gamma_2[X := \Sigma(\alpha)] \vdash g[X := \alpha] : A[X := \Sigma(\alpha)] \rightsquigarrow \Sigma(G)[X := \Sigma(\alpha)]$. By (Ct_Inj_S), it suffices to show that $\Sigma(G)[X := \Sigma(\alpha)] = \Sigma(G[X := \alpha])$, which follows from Lemmas E.22 and E.38.

Case (Ct_Proj_S): Similar to the case of (Ct_Inj_S).

Case (Ct_Arrow_S) and (Ct_All_S): By the IH(s) and the corresponding coercion typing rule.

□

Lemma E.40 (Type Name Substitution)**.** If $\Sigma \mid X, \Gamma \vdash M : A$ and $\alpha \in \mathrm{dom}(\Sigma)$, then $\Sigma \mid \Gamma[X := \Sigma(\alpha)] \vdash M[X := \alpha] : A[X := \Sigma(\alpha)]$.

Proof. By induction on $\Sigma \mid X, \Gamma \vdash M : A$. Most cases are proven easily using Lemma E.37. We show only the interesting cases.

Case (T_ABS_S): We have

$$M = \lambda x : A'.M', \quad A = \Sigma(A') \to B, \quad \Sigma \mid X, \Gamma, x : \Sigma(A') \vdash M' : B \ (\exists x, A', B, M') .$$

By the IH, $\Sigma \mid \Gamma[X := \Sigma(\alpha)], x : \Sigma(A')[X := \Sigma(\alpha)] \vdash M'[X := \alpha] : B[X := \Sigma(\alpha)]$. By (T_ABS_S), it suffices to show that $\Sigma(A')[X := \Sigma(\alpha)] = \Sigma(A'[X := \alpha])$, which follows from Lemmas E.32 and E.38.

Case (T_TYAPP_S):

$$M = M'\, A', \quad A = B[Y := \Sigma(A')], \quad \Sigma \mid X, \Gamma \vdash M' : \forall Y.B, \quad \Sigma \mid X, \Gamma \vdash A' \ (\exists Y, A', B, M') .$$

Without loss of generality, we can suppose that Y does not occur in X, Γ and Σ. By the IH, $\Sigma \mid \Gamma[X := \Sigma(\alpha)] \vdash M'[X := \alpha] : (\forall Y.B)[X := \Sigma(\alpha)]$. By Lemma E.36, $\Sigma \mid \Gamma[X := \Sigma(\alpha)] \vdash A'[X := \alpha]$. By (T_TYAPP_S), $\Sigma \mid \Gamma[X := \Sigma(\alpha)] \vdash (M'[X := \alpha])\,(A'[X := \alpha]) : B[X := \Sigma(\alpha)][Y := \Sigma(A'[X := \alpha])]$. Because $\vdash \Sigma$ by Lemma E.32, we have $B[X := \Sigma(\alpha)][Y := \Sigma(A'[X := \alpha])] = B[X := \Sigma(\alpha)][Y := \Sigma(A')[X := \Sigma(\alpha)]] = B[Y := \Sigma(A')][X := \Sigma(\alpha)]$, using Lemma E.38.

Case (T_CRC_S): We are given

$$M = M'\langle s \rangle, \quad \Sigma \mid X, \Gamma \vdash M' : B, \quad \Sigma \mid X, \Gamma \vdash s : B \rightsquigarrow A \ (\exists M', s, B) .$$

By the IH, $\Sigma \mid \Gamma[X := \Sigma(\alpha)] \vdash M'[X := \alpha] : B[X := \Sigma(\alpha)]$. Lemma E.39 and rule (T_CRC_S) finish the case.

□

Lemma E.41. If $\vdash \Sigma$, then $\mathrm{ftv}(A) = \mathrm{ftv}(\Sigma(A))$.

Proof. By induction on $\vdash \Sigma$.

Case $\Sigma = \emptyset$: Obvious.

Case $\Sigma = \Sigma', \alpha := \mathbb{B}$ $(\exists \Sigma', \alpha, \mathbb{B})$: Because $\vdash \Sigma$, we have $\vdash \Sigma'$ and $\Sigma' \mid \emptyset \vdash \mathbb{B}$. Therefore, by the IH, $\mathrm{ftv}(A) = \mathrm{ftv}(A[\alpha := \mathbb{B}]) = \mathrm{ftv}(\Sigma'(A[\alpha := \mathbb{B}])) = \mathrm{ftv}(\Sigma(A))$.

□

Lemma E.42 (Dynamic-Type Substitution (Coercion))**.** If $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash s : A \rightsquigarrow B$, then $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash s[X := \star] : A[X := \star] \rightsquigarrow B[X := \star]$.

Proof. By induction on $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash s : A \rightsquigarrow B$ with case analysis on the last rule applied in the typing derivation.

Case (CT_ID_S): By Lemmas D.5 and E.38 and (CT_ID_S).

Case (CT_FAIL_S): By Lemma D.5 and (CT_FAIL_S).

Case (CT_INJ_S): We have

$$s = g\,; G!, \quad B = \star, \quad \Sigma \mid \Gamma_1, X, \Gamma_2 \vdash g : A \rightsquigarrow \Sigma(G) \quad (\exists G, g) .$$

Case analysis on G.

Case $G = X$: Since $(g\,; G!)[X := \star] = g[X := \star]$, it suffices to show $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash g[X := \star] : A[X := \star] \rightsquigarrow \star$, which follows from the IH.

Case $G \neq X$: We have $G[X := \star] = G$. Since $(g\,; G!)[X := \star] = g[X := \star]\,; G!$, it suffices to show that $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash g[X := \star]\,; G! : A[X := \star] \rightsquigarrow \star$. By Lemmas E.22 and E.41, $X \notin \mathrm{ftv}(\Sigma(G))$. Therefore, by the IH, $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash g[X := \star] : A[X := \star] \rightsquigarrow \Sigma(G)$. Thus, by (CT_INJ_S), $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash g[X := \star]\,; G! : A[X := \star] \rightsquigarrow \star$. Note that $g[X := \star] = g'$ for some g'.

Case (Ct_Proj_S): We have

$$s = G?^p \,;\, b, \quad A = \star, \quad \Sigma \mid \Gamma_1, X, \Gamma_2 \vdash b : \Sigma(G) \rightsquigarrow B \quad (\exists p, G, b) \;.$$

Case analysis on G.

Case $G = X$: Since $(G?^p \,;\, b)[X := \star] = b[X := \star]$, it suffices to show $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash b[X := \star] : \star \rightsquigarrow B[X := \star]$, which follows from the IH.

Case $G \neq X$: We have $G[X := \star] = G$. Since $(G?^p \,;\, b)[X := \star] = G?^p \,;\, b[X := \star]$, it suffices to show that $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash G?^p \,;\, b[X := \star] : \star \rightsquigarrow B[X := \star]$. By Lemmas E.22 and E.41, $X \notin \mathrm{ftv}(\Sigma(G))$. Therefore, by the IH, $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash b[X := \star] : \Sigma(G) \rightsquigarrow B[X := \star]$. Thus, by (Ct_Proj_S), $\Sigma \mid \Gamma_1, \Gamma_2[X := \star] \vdash G?^p \,;\, b[X := \star] : \star \rightsquigarrow B[X := \star]$. Note that $b[X := \star] = b'$ for some b'.

Case (Ct_Arrow_S) and (Ct_All_S): By the IH(s).

□

Lemma E.43 (Dynamic-Type Substitution)**.** If $\Sigma \mid X, \Gamma \vdash M : A$, then $\Sigma \mid \Gamma[X := \star] \vdash M[X := \star] : A[X := \star]$.

Proof. By straightforward induction on $\Sigma \mid X, \Gamma \vdash M : A$ with Lemmas D.5, E.32, E.38, and E.42.

□

Lemma E.44. If $\Sigma \mid \emptyset \vdash M : A$ and $\Sigma \triangleright M \longrightarrow \Sigma' \triangleright M'$, then either

- $\Sigma' = \Sigma$ or
- $\Sigma' = \Sigma, \alpha := \mathbb{B}$ and $\Sigma \mid \emptyset \vdash \mathbb{B}$ for some $\mathbb{B}$ and $\alpha \notin \mathrm{dom}(\Sigma)$.

Proof. By easy induction on $\Sigma \triangleright M \longrightarrow \Sigma' \triangleright M'$. □

Theorem E.45 (Preservation)**.** If $\Sigma \mid \emptyset \vdash M : A$ and $\Sigma \triangleright M \longrightarrow \Sigma' \triangleright M'$, then $\Sigma' \mid \emptyset \vdash M' : A$.

Proof. By induction on the derivation of $\Sigma \triangleright M \longrightarrow \Sigma' \triangleright M'$ with case analysis on the last rule used.

Case (R_Delta_S): We have

$$M = k_1\, k_2, \quad M' = \delta(k_1, k_2), \quad \Sigma' = \Sigma \quad (\exists k_1, k_2) \;.$$

From $\Sigma \mid \emptyset \vdash k_1\, k_2 : A$, we have $\Sigma \mid \emptyset \vdash k_1 : \iota \to A$ and $\Sigma \mid \emptyset \vdash k_2 : \iota$. Then, by the assumption on δ, $\Sigma \mid \emptyset \vdash \delta(k_1, k_2) : A$.

Case (R_Beta_S): We have

$$M = (\lambda x\!:\!A'.M'')\; V, \quad M' = M''[x := V], \quad \Sigma = \Sigma' \quad (\exists A', x, M'', V) \;.$$

From $\Sigma \mid \emptyset \vdash (\lambda x\!:\!A'.M'')\; V : A$, we have

$$\Sigma \mid \emptyset, x : \Sigma(A') \vdash M'' : A, \quad \Sigma \mid \emptyset \vdash V : \Sigma(A') \;.$$

Thus, by Lemma E.34, $\Sigma \mid \emptyset \vdash M''[x := V] : A$.

Case (R_Id_S): We have

$$M = U\langle \mathsf{id} \rangle, \quad M' = U, \quad \Sigma' = \Sigma \quad (\exists U) \;.$$

From $\Sigma \mid \emptyset \vdash U\langle \mathsf{id} \rangle : A$, we have $\Sigma \mid \emptyset \vdash U : A$.

Case (R_WRAP_S): We have

$$M = (U\langle s \to t\rangle)\, V', \quad M' = (U\,(V'\langle s\rangle))\langle t\rangle, \quad \Sigma' = \Sigma \quad (\exists s, t, U, V') .$$

From $\Sigma \mid \emptyset \vdash (U\langle s \to t\rangle)\, V' : A$, we have

$$\Sigma \mid \emptyset \vdash U : B \to C, \quad \Sigma \mid \emptyset \vdash V' : D, \quad \Sigma \mid \emptyset \vdash s : D \rightsquigarrow B,$$
$$\Sigma \mid \emptyset \vdash t : C \rightsquigarrow A \quad (\exists B, C, D) .$$

Thus, $\Sigma \mid \emptyset \vdash (U\,(V'\langle s\rangle))\langle t\rangle : A$ by (CCT_ATOM_S), (T_CRC_S), (T_APP_S), and (T_CRC_S).

Case (R_FAIL_S): We have

$$M = U\langle \bot^p\rangle, \quad M' = \mathsf{blame}\, p, \quad \Sigma' = \Sigma \quad (\exists p, U) .$$

By Lemma E.32, we have $\vdash \Sigma$ and $\emptyset \vdash \emptyset$ and $\emptyset \mid \emptyset \vdash A$. By (T_BLAME_S), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p : A$.

Case (R_MERGE_S): We have

$$M = M''\langle s\rangle\langle t\rangle, \quad M' = M''\langle s \mathbin{;} t\rangle \quad (\exists s, t, M'') .$$

From $\Sigma \mid \emptyset \vdash M''\langle s\rangle\langle t\rangle : A$, we have

$$\Sigma \mid \emptyset \vdash M'' : C, \quad \Sigma \mid \emptyset \vdash s : C \rightsquigarrow B, \quad \Sigma \mid \emptyset \vdash t : B \rightsquigarrow A \quad (\exists B, C) .$$

By Lemma E.24, $\Sigma \mid \emptyset \vdash s \mathbin{;} t : C \rightsquigarrow A$. By (CCT_ATOM_S) and (T_CRC_S), $\Sigma \mid \emptyset \vdash M''\langle s \mathbin{;} t\rangle : A$.

Case (R_TYBETADYN_S): We have

$$M = (\Lambda X.M'')\,\star, \quad M' = M''[X := \star], \quad \Sigma' = \Sigma, \quad (\exists X, M'') .$$

From $\Sigma \mid \emptyset \vdash (\Lambda X.M'')\,\star : A$, we have

$$\Sigma \mid X \vdash M'' : B, \quad A = B[X := \star] \quad (\exists B) .$$

By Lemma E.43, $\Sigma \mid \emptyset \vdash M''[X := \star] : B[X := \star]$.

Case (R_TYBETADYNC_S): We have

$$M = (\Lambda X.M'')\langle \forall X.s \,,, t\rangle\,\star, \quad M' = (M''[X := \star])\langle t\rangle, \quad \Sigma' = \Sigma, \quad (\exists X, M'', s, t) .$$

From $\Sigma \mid \emptyset \vdash (\Lambda X.M'')\langle \forall X.s \,,, t\rangle\,\star : A$, we have

$$\Sigma \mid X \vdash M'' : C, \quad \Sigma \mid \emptyset \vdash t : C[X := \star] \rightsquigarrow B[X := \star], \quad A = B[X := \star] \quad (\exists B, C) .$$

By Lemma E.43, $\Sigma \mid \emptyset \vdash M''[X := \star] : C[X := \star]$. By (CCT_ATOM_S) and (T_CRC_S), we have $\Sigma \mid \emptyset \vdash (M''[X := \star])\langle t\rangle : B[X := \star]$.

Case (R_TYBETA_S): We have

$$M = (\Lambda X.M'')\,\mathbb{B}, \quad M' = M''[X := \alpha],$$
$$\Sigma' = \Sigma, \alpha := \mathbb{B}, \quad \alpha \notin \mathrm{dom}(\Sigma) \quad (\exists X, \alpha, \mathbb{B}, M'') .$$

From $\Sigma \mid \emptyset \vdash (\Lambda X.M'')\,\mathbb{B} : A$, we have

$$\Sigma \mid X \vdash M'' : C, \quad \Sigma \mid \emptyset \vdash \mathbb{B}, \quad A = C[X := \Sigma(\mathbb{B})] \quad (\exists C) .$$

By Lemma E.32, $\vdash \Sigma$. By (SW_BINDING), we have $\vdash \Sigma, \alpha := \mathbb{B}$. By Lemma E.33, $\Sigma, \alpha := \mathbb{B} \mid X \vdash M'' : C$. By Lemma E.40, $\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash M''[X := \alpha] : C[X := (\Sigma, \alpha := \mathbb{B})(\alpha)]$. Because $(\Sigma, \alpha := \mathbb{B})(\alpha) = \Sigma(\mathbb{B})$, we have $\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash M''[X := \alpha] : C[X := \Sigma(\mathbb{B})]$.

Case (R_TybetaC_S): We have

$$M = (\Lambda X.M'')\langle \forall X.s \,,, t\rangle \mathbb{B}, \quad M' = (M''\langle s\rangle)[X := \alpha],$$
$$\Sigma' = \Sigma, \alpha := \mathbb{B}, \quad \alpha \notin \mathrm{dom}(\Sigma) \quad (\exists X, \alpha, \mathbb{B}, M'', s) .$$

From $\Sigma \mid \emptyset \vdash (\Lambda X.M'')\langle \forall X.s \,,, t\rangle \mathbb{B} : A$, we have

$$\Sigma \mid X \vdash M'' : D, \quad \Sigma \mid X \vdash s : D \rightsquigarrow C, \quad \Sigma \mid \emptyset \vdash \mathbb{B}, \quad A = C[X := \Sigma(\mathbb{B})] \quad (\exists C, D) .$$

By Lemma E.32, $\vdash \Sigma$. By (Sw_Binding), we have $\vdash \Sigma, \alpha := \mathbb{B}$. By Lemma E.33, $\Sigma, \alpha := \mathbb{B} \mid X \vdash M'' : D$. By Lemma E.40, $\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash M''[X := \alpha] : D[X := (\Sigma, \alpha := \mathbb{B})(\alpha)]$. By Lemma E.30, $\Sigma, \alpha := \mathbb{B} \mid X \vdash s : D \rightsquigarrow C$. By Lemma E.39, $\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash s[X := \alpha] : D[X := (\Sigma, \alpha := \mathbb{B})(\alpha)] \rightsquigarrow C[X := (\Sigma, \alpha := \mathbb{B})(\alpha)]$. By (Cct_Atom_S) and (T_Crc_S), $\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash (M''\langle s\rangle)[X := \alpha] : C[X := (\Sigma, \alpha := \mathbb{B})(\alpha)]$. Because $(\Sigma, \alpha := \mathbb{B})(\alpha) = \Sigma(\mathbb{B})$, we have $\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash (M''\langle s\rangle)[X := \alpha] : C[X := \Sigma(\mathbb{B})]$.

Case (R_BlameE_S) and (R_BlameC_S): By Lemma E.32 and (T_Blame_S).

Case (R_CtxE_S): We have

$$M = E[M_1], \quad M' = E[M_1'], \quad \Sigma \triangleright M_1 \longrightarrow \Sigma' \triangleright M_1' \quad (\exists E, M_1, M_1') .$$

Case analysis on E.

Case $E = \Box\, M''(\exists M'')$: We have $E[M_1] = M_1\, M''$. From $\Sigma \mid \emptyset \vdash M_1\, M'' : A$,

$$\Sigma \mid \emptyset \vdash M_1 : B \to A, \quad \Sigma \mid \emptyset \vdash M'' : B \quad (\exists B) .$$

By Lemma E.44, we consider the two cases below.

Case $\Sigma' = \Sigma$: It suffices to show $\Sigma \mid \emptyset \vdash M_1'\, M'' : A$. By the IH, $\Sigma \mid \emptyset \vdash M_1' : B \to A$. By (T_App_S), $\Sigma \mid \emptyset \vdash M_1'\, M'' : A$.

Case $\Sigma' = \Sigma, \alpha := \mathbb{C}(\exists \alpha, \mathbb{C})$: We have $\alpha \notin \mathrm{dom}(\Sigma)$ and $\Sigma \mid \emptyset \vdash \mathbb{C}$. It suffices to show $\Sigma, \alpha := \mathbb{C} \mid \emptyset \vdash M_1'\, M'' : A$. By Lemma E.32 and (Sw_Binding), $\vdash \Sigma, \alpha := \mathbb{C}$. By Lemma E.33, $\Sigma, \alpha := \mathbb{C} \mid \emptyset \vdash M'' : B$. By the IH, $\Sigma, \alpha := \mathbb{C} \mid \emptyset \vdash M_1' : B \to A$. By (T_App_S), $\Sigma, \alpha := \mathbb{C} \mid \emptyset \vdash M_1'\, M'' : A$.

Case $E = V\,\Box(\exists V)$: Similar to the case where $E = \Box\, M''$.

Case $E = \Box\, B(\exists B)$: We have $E[M_1] = M_1\, B$. From $\Sigma \mid \emptyset \vdash M_1\, B : A$,

$$\Sigma \mid \emptyset \vdash M_1 : \forall X.C, \quad \Sigma \mid \emptyset \vdash B, \quad A = C[X := \Sigma(B)] \quad (\exists X, C) .$$

By Lemma E.44, we consider the two cases below.

Case $\Sigma' = \Sigma$: By the IH and (T_Tyapp_S).

Case $\Sigma' = \Sigma, \alpha := \mathbb{A}_0(\exists \alpha, \mathbb{A}_0)$: We have $\alpha \notin \mathrm{dom}(\Sigma)$ and $\Sigma \mid \emptyset \vdash \mathbb{A}_0$. It suffices to show $\Sigma, \alpha := \mathbb{A}_0 \mid \emptyset \vdash M_1'\, B : C[X := \Sigma(B)]$. By Lemma D.2 (1), $\Sigma, \alpha := \mathbb{A}_0 \mid \emptyset \vdash B$. By the IH, $\Sigma, \alpha := \mathbb{A}_0 \mid \emptyset \vdash M_1' : \forall X.C$. By (T_Tyapp_S), $\Sigma, \alpha := \mathbb{A}_0 \mid \emptyset \vdash M_1'\, B : C[X := (\Sigma, \alpha := \mathbb{A}_0)(B)]$. Because $\Sigma \mid \emptyset \vdash B$ implies that α does not occur in B, we have $(\Sigma, \alpha := \mathbb{A}_0)(B) = \Sigma(B)$. Therefore, we have $\Sigma, \alpha := \mathbb{A}_0 \mid \emptyset \vdash M_1'\, B : C[X := \Sigma(B)]$.

Case (R_CtxC_S): Similar to the case of (R_CtxE_S), that is, by the IH and Lemmas E.44 and E.30.

□

Corollary E.46 (Preservation (multi step))**.** If $\Sigma \mid \emptyset \vdash M : A$ and $\Sigma \triangleright M \longrightarrow^* \Sigma' \triangleright M'$, then $\Sigma' \mid \emptyset \vdash M' : A$.

Theorem E.47 (Type Safety)**.** If $\Sigma \mid \emptyset \vdash M : A$, then one of the followings holds:

- $\Sigma \triangleright M \longrightarrow^* \Sigma' \triangleright V$ for some store Σ' and value V such that $\Sigma' \mid \emptyset \vdash V : A$;
- $\Sigma \triangleright M \longrightarrow^* \Sigma' \triangleright \mathsf{blame}\, p$ for some store Σ' and blame label p; or
- $\Sigma \triangleright M \Uparrow$.

Proof. By Theorem E.26 and Corollary E.46. □

F Determinacy of Reduction

F.1 $\lambda \mathbf{C}^{\forall}_{mp}$

We extend α-equivalence to $\Sigma \triangleright M$ in a straightforward manner—by considering type variables defined in Σ bound in M. For example, $\alpha := \mathsf{Int} \triangleright 42\langle \alpha^- \rangle$ and $\beta := \mathsf{Int} \triangleright 42\langle \beta^- \rangle$ are α-equivalent and, thus, identified.

Theorem F.1 (Determinacy of Reduction (Theorem 3.1 of the paper))**.** Assume that $\Sigma \mid \emptyset \vdash M : A$. If $\Sigma \triangleright M \longrightarrow \Sigma_1 \triangleright M_1$ and $\Sigma \triangleright M \longrightarrow \Sigma_2 \triangleright M_2$, then $\Sigma_1 = \Sigma_2$ and $M_1 = M_2$.

Proof. By straightforward induction on $\Sigma \triangleright M \longrightarrow \Sigma_1 \triangleright M_1$. We use Lemma E.1 in the case for (R_TYBETA_C) to show that the input to the coercion generation function is unique. Note that values and blame cannot be reduced. □

Corollary F.2. If $\Sigma \mid \emptyset \vdash M : A$, then the reduction sequence starting from given $\Sigma \triangleright M$ is unique.

F.2 $\lambda \mathbf{S}^{\forall}_{mp}$

Theorem F.3 (Determinacy of Reduction)**.** If $\Sigma \triangleright M \longrightarrow \Sigma_1 \triangleright M_1$ and $\Sigma \triangleright M \longrightarrow \Sigma_2 \triangleright M_2$, then $\Sigma_1 = \Sigma_2$ and $M_1 = M_2$.

Proof. By induction on $\Sigma \triangleright M \longrightarrow \Sigma_1 \triangleright M_1$. The cases except for (R_MERGE_S) and (R_CTXC_S) are easy to show, using the fact that values and blame cannot be reduced.

Consider the case for (R_MERGE_S). We have $M = M'\langle s \rangle\langle t \rangle$ for some M', s, and t. It is obvious that the reduction rules except for (R_MERGE_S) and (R_CTXC_S) cannot be applied to $M'\langle s \rangle\,\langle t \rangle$. If (R_MERGE_S) is applied to obtain both $\Sigma_1 \triangleright M_1$ and $\Sigma_2 \triangleright M_2$, then we finish the case. Otherwise, assume that (R_CTX_S) is applied to obtain $\Sigma_2 \triangleright M_2$. Then, $\Sigma \triangleright M'\langle s \rangle \xrightarrow{e} \Sigma' \triangleright M''$ for some Σ' and M''. Then, by the definition of $\xrightarrow{e}$, $M'\langle s \rangle$ is not a coercion application, which is a contradiction.

The case for (R_CTXC_S) is proven similarly.

□

Corollary F.4. The reduction sequence starting from given $\Sigma \triangleright M$ is unique.

Note that only Theorem F.1 assumes the typability of reduced term M. The semantics of $\lambda \mathrm{C}^{\forall}_{mp}$ rests on the coercion generation function, and to ensure its determinacy, type names chosen at run time for type application should not occur in the reduced term M (more precisely, should not occur in the input type to the coercion generation function). The typability of M under store Σ guarantees it.

G Properties of the Logical Relation

The statements described in this section are on $\lambda \mathrm{C}^{\forall}_{mp}$.

Lemma G.1 (Evaluation under Contexts)**.**

1. If $\Sigma_1 \triangleright M_1 \longrightarrow \Sigma_2 \triangleright M_2$, then $\Sigma_1 \triangleright F[M_1] \longrightarrow \Sigma_2 \triangleright F[M_2]$ for any F.
2. If $\Sigma_1 \triangleright M_1 \longrightarrow^n \Sigma_2 \triangleright M_2$, then $\Sigma_1 \triangleright F[M_1] \longrightarrow^n \Sigma_2 \triangleright F[M_2]$.
3. If $\Sigma_1 \triangleright M \longrightarrow^n \Sigma_2 \triangleright \mathsf{blame}\, p$, then $\Sigma_1 \triangleright F[M] \longrightarrow^m \Sigma_2 \triangleright \mathsf{blame}\, p$ for some $m \geq n$.

Proof.

1. Straightforward by induction on F with (R_CTX_C).
2. Straightforward by induction on n, using the case (1).
3. The case (2) implies $\Sigma_1 \triangleright F[M] \longrightarrow^n \Sigma_2 \triangleright F[\mathsf{blame}\, p]$. It is easy to show that $\Sigma_2 \triangleright F[\mathsf{blame}\, p] \longrightarrow^* \Sigma_2 \triangleright \mathsf{blame}\, p$ by induction on F with (R_CTX_C) and (R_BLAME_C).

□

Lemma G.2 (Successive Approximation). Suppose $n \leq m$.

- $\lfloor \lfloor R \rfloor_m \rfloor_n = \lfloor \lfloor R \rfloor_n \rfloor_m = \lfloor R \rfloor_n$.
- $\lfloor \lfloor \kappa \rfloor_m \rfloor_n = \lfloor \lfloor \kappa \rfloor_n \rfloor_m = \lfloor \kappa \rfloor_n$.

Proof. The second case is implied by the first case. Therefore, we consider only the first case in what follows. We have the conclusion by the following:

$$\begin{array}{rcl} (W, M_1, M_2) \in \lfloor \lfloor R \rfloor_m \rfloor_n & \Leftrightarrow & (W, M_1, M_2) \in \lfloor R \rfloor_m \wedge W.n < n \\ & \Leftrightarrow & (W, M_1, M_2) \in R \wedge W.n < m \wedge W.n < n \\ & \Leftrightarrow & (W, M_1, M_2) \in R \wedge W.n < n \\ & \Leftrightarrow & (W, M_1, M_2) \in \lfloor R \rfloor_n , \end{array}$$

and

$$\begin{array}{rcl} (W, M_1, M_2) \in \lfloor \lfloor R \rfloor_n \rfloor_m & \Leftrightarrow & (W, M_1, M_2) \in \lfloor R \rfloor_n \wedge W.n < m \\ & \Leftrightarrow & (W, M_1, M_2) \in R \wedge W.n < n \wedge W.n < m \\ & \Leftrightarrow & (W, M_1, M_2) \in R \wedge W.n < n \\ & \Leftrightarrow & (W, M_1, M_2) \in \lfloor R \rfloor_n . \end{array}$$

□

Lemma G.3 (Worlds are Stratified). If $W \in \text{World}$ and $(W', M_1, M_2) \in W.\kappa(\alpha)$, then $W'.n < W.n$.

Proof. Because $W \in \text{World}$, we have $W.\kappa(\alpha) \in \text{Rel}_{W.n} [\![W.\Sigma_1(\alpha), W.\Sigma_2(\alpha)]\!]$, which implies $W.\kappa(\alpha) \subseteq \text{Atom}^{\text{val}}_{W.n} [\![W.\Sigma_1(\alpha), W.\Sigma_2(\alpha)]\!]$. Therefore, by definition, for any $(W', M_1, M_2) \in W.\kappa(\alpha)$, we have $W' \in \text{World}_{W.n}$, which implies $W'.n < W.n$. □

Lemma G.4 (Idempotent Approximation). $W.\kappa = \lfloor W.\kappa \rfloor_{W.n}$ for any $W \in \text{World}$.

Proof. By definition, $\text{dom}(W.\kappa) = \text{dom}(\lfloor W.\kappa \rfloor_{W.n})$. Let $\alpha \in \text{dom}(W.\kappa)$. By definition, it suffices to show that $W.\kappa(\alpha) = \lfloor W.\kappa(\alpha) \rfloor_{W.n}$. We have $\lfloor W.\kappa(\alpha) \rfloor_{W.n} \subseteq W.\kappa(\alpha)$ trivially. For showing the converse, let $(W', M_1, M_2) \in W.\kappa(\alpha)$. By Lemma G.3, $W'.n < W.n$. Thus, $(W', M_1, M_2) \in \lfloor W.\kappa(\alpha) \rfloor_{W.n}$. □

Lemma G.5 (World Extension is Reflexive and Transitive).

- For any $W \in \text{World}$, $W \sqsupseteq W$.
- For any W_1, W_2, $W_3 \in \text{World}$, if $W_1 \sqsupseteq W_2$ and $W_2 \sqsupseteq W_3$, then $W_1 \sqsupseteq W_3$.

Proof.

- Let $W \in \text{World}$. It suffices to show that $W.\kappa \sqsupseteq \lfloor W.\kappa \rfloor_{W.n}$. First, we have $\lfloor W.\kappa \rfloor_{W.n} = W.\kappa$ by Lemma G.4. Furthermore, $W.\kappa \sqsupseteq W.\kappa$ holds trivially. Therefore, we have the conclusion.
- Let W_1, W_2, $W_3 \in \text{World}$ such that $W_1 \sqsupseteq W_2$ and $W_2 \sqsupseteq W_3$. It suffices to show that $W_1.\kappa \sqsupseteq \lfloor W_3.\kappa \rfloor_{W_1.n}$, that is,

  $$\forall \alpha \in \text{dom}(\lfloor W_3.\kappa \rfloor_{W_1.n}).\ W_1.\kappa(\alpha) = \lfloor W_3.\kappa \rfloor_{W_1.n}(\alpha) .$$

  Let $\alpha \in \text{dom}(\lfloor W_3.\kappa \rfloor_{W_1.n})$. By definition, $\alpha \in \text{dom}(W_3.\kappa)$. Since $\text{dom}(W_3.\kappa) = \text{dom}(\lfloor W_3.\kappa \rfloor_{W_2.n})$, we have $\alpha \in \text{dom}(\lfloor W_3.\kappa \rfloor_{W_2.n})$. Since $W_2 \sqsupseteq W_3$, we have $W_2.\kappa \sqsupseteq \lfloor W_3.\kappa \rfloor_{W_2.n}$. Thus, $W_2.\kappa(\alpha) = \lfloor W_3.\kappa \rfloor_{W_2.n}(\alpha)$. By definition,

  $$\lfloor W_2.\kappa \rfloor_{W_1.n}(\alpha) = \lfloor W_2.\kappa(\alpha) \rfloor_{W_1.n} = \lfloor \lfloor W_3.\kappa \rfloor_{W_2.n}(\alpha) \rfloor_{W_1.n} = \lfloor \lfloor W_3.\kappa \rfloor_{W_2.n} \rfloor_{W_1.n}(\alpha) .$$

  Since $W_1 \sqsupseteq W_2$, we have $W_1.\kappa(\alpha) = \lfloor W_2.\kappa \rfloor_{W_1.n}(\alpha)$, that is, $W_1.\kappa(\alpha) = \lfloor \lfloor W_3.\kappa \rfloor_{W_2.n} \rfloor_{W_1.n}(\alpha)$.

  Now, it suffices to show that $\lfloor \lfloor W_3.\kappa \rfloor_{W_2.n} \rfloor_{W_1.n}(\alpha) = \lfloor W_3.\kappa \rfloor_{W_1.n}(\alpha)$, which is implied by Lemma G.2 with $W_1.n \leq W_2.n$.

□

Lemma G.6 (Addition Extends Worlds)**.** If $W \in \text{World}$ and $W.\Sigma_1 \mid \emptyset \vdash \mathbb{A}_1$ and $W.\Sigma_2 \mid \emptyset \vdash \mathbb{A}_2$ and $R \in \text{Rel}_{W.n}[\![\mathbb{A}_1, \mathbb{A}_2]\!]$ and $\alpha \notin \text{dom}(W.\Sigma_1) \cup \text{dom}(W.\Sigma_2)$, then $W \boxplus (\alpha, \mathbb{A}_1, \mathbb{A}_2, R) \sqsupseteq W$.

Proof. By definition, it suffices to show the following.

- $W.n \leq W.n$: Obvious.
- $W.\Sigma_1, \alpha := \mathbb{A}_1 \supseteq W.\Sigma_1$ and $W.\Sigma_2, \alpha := \mathbb{A}_2 \supseteq W.\Sigma_2$: Obvious.
- $W.\kappa\{\alpha \mapsto R\} \sqsupseteq \lfloor W.\kappa \rfloor_{W.n}$: Let $\beta \in \text{dom}(\lfloor W.\kappa \rfloor_{W.n})$. We show that $W.\kappa\{\alpha \mapsto R\}(\beta) = \lfloor W.\kappa \rfloor_{W.n}(\beta)$. By definition, $\beta \in \text{dom}(W.\kappa)$. Then, $W \in \text{World}$ implies $\beta \in \text{dom}(W.\Sigma_1) \cap \text{dom}(W.\Sigma_2)$. Since $\alpha \notin \text{dom}(W.\Sigma_1) \cup \text{dom}(W.\Sigma_2)$, we have $\alpha \neq \beta$. Thus, $W.\kappa\{\alpha \mapsto R\}(\beta) = W.\kappa(\beta)$. By Lemma G.4, we have the conclusion.
- $W \boxplus (\alpha, \mathbb{A}_1, \mathbb{A}_2, R) \in \text{World}$: $W \in \text{World}$ implies that there exists some n such that $W \in \text{World}_n$. We show that $W \boxplus (\alpha, \mathbb{A}_1, \mathbb{A}_2, R) \in \text{World}_n$. Let $W' = W \boxplus (\alpha, \mathbb{A}_1, \mathbb{A}_2, R)$. We have the conclusion by the following.
  - $W'.n < n$: This is implied by $W \in \text{World}_n$ and $W.n = W'.n$.
  - $\vdash W'.\Sigma_1$ and $\vdash W'.\Sigma_2$: These are implied by $W \in \text{World}_n$ and $W.\Sigma_1 \mid \emptyset \vdash \mathbb{A}_1$ and $W.\Sigma_2 \mid \emptyset \vdash \mathbb{A}_2$ and $\alpha \notin \text{dom}(W.\Sigma_1) \cup \text{dom}(W.\Sigma_2)$ and (SW_BINDING).
  - $\forall \beta \in \text{dom}(W'.\kappa).\ W'.\kappa(\beta) \in \text{Rel}_{W'.n}[\![W'.\Sigma_1(\beta), W'.\Sigma_2(\beta)]\!]$: Let $\beta \in \text{dom}(W'.\kappa)$. By definition, $\beta = \alpha$ or $\beta \in \text{dom}(W.\kappa)$. If $\beta = \alpha$, then $W'.\kappa(\beta) = R$. Because $R \in \text{Rel}_{W.n}[\![\mathbb{A}_1, \mathbb{A}_2]\!] = \text{Rel}_{W'.n}[\![W'.\Sigma_1(\beta), W'.\Sigma_2(\beta)]\!]$, we have the conclusion. Otherwise, if $\beta \neq \alpha$ and $\beta \in \text{dom}(W.\kappa)$, then $W \in \text{World}_n$ implies the conclusion

$$W'.\kappa(\beta) = W.\kappa(\beta) \in \text{Rel}_{W.n}[\![W.\Sigma_1(\beta), W.\Sigma_2(\beta)]\!] = \text{Rel}_{W'.n}[\![W'.\Sigma_1(\beta), W'.\Sigma_2(\beta)]\!]\ .$$

□

Lemma G.7 (Properties of $\blacktriangleright$)**.** Let $W \in \text{World}$ such that $\blacktriangleright W$ is well defined.

1. $\blacktriangleright W \in \text{World}$.
2. $\blacktriangleright W \sqsupseteq_1 W$.
3. For any W' such that $W' \sqsupseteq W$ and $\blacktriangleright W'$ is well defined, $\blacktriangleright W' \sqsupseteq \blacktriangleright W$.

Proof.

1. Since $\blacktriangleright W$ is well defined, there exists some n such that $W.n = n+1$. $W \in \text{World}$ implies $W \in \text{World}_m$ for some m. Since $W.n = n+1 < m$, there exists some m_0 such that $m = m_0 + 1$. The conclusion is implied by $\blacktriangleright W \in \text{World}_{m_0}$, which is shown by the following.
   - $(\blacktriangleright W).n < m_0$ because $(\blacktriangleright W).n = n < m_0$; note that $n+1 < m = m_0 + 1$.
   - $\vdash (\blacktriangleright W).\Sigma_1$ because $(\blacktriangleright W).\Sigma_1 = W.\Sigma_1$ and $\vdash W.\Sigma_1$ by $W \in \text{World}_m$.
   - $\vdash (\blacktriangleright W).\Sigma_2$ because $(\blacktriangleright W).\Sigma_2 = W.\Sigma_2$ and $\vdash W.\Sigma_2$ by $W \in \text{World}_m$.
   - $\forall \alpha \in \text{dom}((\blacktriangleright W).\kappa).\ (\blacktriangleright W).\kappa(\alpha) \in \text{Rel}_{(\blacktriangleright W).n}[\![(\blacktriangleright W).\Sigma_1(\alpha), (\blacktriangleright W).\Sigma_2(\alpha)]\!]$: Let $\alpha \in \text{dom}((\blacktriangleright W).\kappa)$. By definition, it suffices to show that $\lfloor W.\kappa \rfloor_{W.n-1}(\alpha) \in \text{Rel}_{W.n-1}[\![W.\Sigma_1(\alpha), W.\Sigma_2(\alpha)]\!]$. The conclusion is implied by the following.
     - $\lfloor W.\kappa \rfloor_{W.n-1}(\alpha) \subseteq \text{Atom}^{\text{val}}_{W.n-1}[\![W.\Sigma_1(\alpha), W.\Sigma_2(\alpha)]\!]$: Let $(W', M_1, M_2) \in \lfloor W.\kappa \rfloor_{W.n-1}(\alpha)$. By definition, $(W', M_1, M_2) \in W.\kappa(\alpha)$ and $W'.n < W.n - 1$. Since $W \in \text{World}_m$, we have $(W', M_1, M_2) \in W.\kappa(\alpha) \subseteq \text{Atom}^{\text{val}}_{W.n}[\![W.\Sigma_1(\alpha), W.\Sigma_2(\alpha)]\!]$. Since $W'.n < W.n - 1$, we have the conclusion.
     - Let $(W_1, V_1, V_2) \in \lfloor W.\kappa \rfloor_{W.n-1}(\alpha)$ and $W_2 \sqsupseteq W_1$. Then, we show that $(W_2, V_1, V_2) \in \lfloor W.\kappa \rfloor_{W.n-1}(\alpha)$. By definition, $(W_1, V_1, V_2) \in W.\kappa(\alpha)$ and $W_1.n < W.n - 1$. Since $W \in \text{World}_m$, $W.\kappa(\alpha)$ satisfies monotonicity, so $(W_2, V_1, V_2) \in W.\kappa(\alpha)$. Since $W_2.n \leq W_1.n < W.n - 1$, we have the conclusion $(W_2, V_1, V_2) \in \lfloor W.\kappa \rfloor_{W.n-1}(\alpha)$.

2. By the following.

   - $(\blacktriangleright W).n = W.n - 1$, so $(\blacktriangleright W).n < W.n$.
   - $(\blacktriangleright W).\Sigma_1 = W.\Sigma_1$ and $(\blacktriangleright W).\Sigma_2 = W.\Sigma_2$.
   - $\blacktriangleright W \in \mathrm{World}$ by the case (1).
   - $(\blacktriangleright W).\kappa \sqsupseteq \lfloor W.\kappa \rfloor_{(\blacktriangleright W).n}$ by Lemma G.5 and $(\blacktriangleright W).\kappa = \lfloor W.\kappa \rfloor_{(\blacktriangleright W).n}$.

3. The conclusion is implied by the following.

   - $(\blacktriangleright W').n \leq (\blacktriangleright W).n$: Because $W' \sqsupseteq W$ implies $W'.n \leq W.n$, we have $(\blacktriangleright W').n = W'.n - 1 \leq W.n - 1 = (\blacktriangleright W).n$.
   - $(\blacktriangleright W').\Sigma_1 \supseteq (\blacktriangleright W).\Sigma_1$ and $(\blacktriangleright W').\Sigma_2 \supseteq (\blacktriangleright W).\Sigma_2$: By definition with $W' \sqsupseteq W$.
   - $(\blacktriangleright W').\kappa \sqsupseteq \lfloor (\blacktriangleright W).\kappa \rfloor_{(\blacktriangleright W').n}$: Let $\alpha \in \mathrm{dom}(\lfloor (\blacktriangleright W).\kappa \rfloor_{(\blacktriangleright W').n})$. Then, the conclusion is implied by:

$$\begin{array}{rcll} \lfloor (\blacktriangleright W).\kappa \rfloor_{(\blacktriangleright W').n}(\alpha) & = & \lfloor (\blacktriangleright W).\kappa(\alpha) \rfloor_{(\blacktriangleright W').n} & \text{(by definition)} \\ & = & \lfloor \lfloor W.\kappa(\alpha) \rfloor_{(\blacktriangleright W).n} \rfloor_{(\blacktriangleright W').n} & \text{(by definition)} \\ & = & \lfloor W.\kappa(\alpha) \rfloor_{(\blacktriangleright W').n} & \text{(by Lemma G.2 with } (\blacktriangleright W').n \leq (\blacktriangleright W).n) \\ & = & \lfloor \lfloor W.\kappa(\alpha) \rfloor_{W'.n} \rfloor_{(\blacktriangleright W').n} & \text{(by Lemma G.2 with } (\blacktriangleright W').n < W'.n) \\ & = & \lfloor W'.\kappa(\alpha) \rfloor_{(\blacktriangleright W').n} & \text{(by } W' \sqsupseteq W) \\ & = & \lfloor W'.\kappa \rfloor_{(\blacktriangleright W').n}(\alpha) & \text{(by definition)} \\ & = & (\blacktriangleright W').\kappa(\alpha) & \text{(by definition)} . \end{array}$$

   - $\blacktriangleright W' \in \mathrm{World}$ and $\blacktriangleright W \in \mathrm{World}$: By the case (1); note that $W' \in \mathrm{World}$ is implied by $W' \sqsupseteq W$.

□

Lemma G.8 (Monotonicity of Later Relations)**.** If $(W, V_1, V_2) \in \blacktriangleright(W.\kappa(\alpha))$ and $W' \sqsupseteq W$, then $(W', V_1, V_2) \in \blacktriangleright(W'.\kappa(\alpha))$.

Proof. If $W'.n = 0$, then we have the conclusion by the definition of $\blacktriangleright$.

Suppose that $W'.n > 0$. Then, it suffices to show that $(\blacktriangleright W', V_1, V_2) \in W'.\kappa(\alpha)$. $W' \sqsupseteq W$ implies $0 < W'.n \leq W.n$. Therefore, $(W, V_1, V_2) \in \blacktriangleright(W.\kappa(\alpha))$ implies $(\blacktriangleright W, V_1, V_2) \in W.\kappa(\alpha)$. $W' \sqsupseteq W$ implies $\blacktriangleright W' \sqsupseteq \blacktriangleright W$ by Lemma G.7 (3). $W' \sqsupseteq W$ implies $W \in \mathrm{World}$, which further implies $W.\kappa(\alpha)$ satisfies monotonicity. Thus, since $(\blacktriangleright W, V_1, V_2) \in W.\kappa(\alpha)$ and $\blacktriangleright W' \sqsupseteq \blacktriangleright W$, we have $(\blacktriangleright W', V_1, V_2) \in W.\kappa(\alpha)$. Because $(\blacktriangleright W').n < W'.n$, we have $(\blacktriangleright W', V_1, V_2) \in \lfloor W.\kappa(\alpha) \rfloor_{W'.n}$. Because $W' \sqsupseteq W$ implies $W'.\kappa \sqsupseteq \lfloor W.\kappa \rfloor_{W'.n}$, we have the conclusion $(\blacktriangleright W', V_1, V_2) \in W'.\kappa(\alpha)$. □

Lemma G.9 (Monotonicity of Atom)**.** If $W' \sqsupseteq W$ and $(W, V_1, V_2) \in \mathrm{Atom}^{\mathrm{val}}\,[\![A]\!]\,\rho$, then $(W', V_1, V_2) \in \mathrm{Atom}^{\mathrm{val}}\,[\![A]\!]\,\rho$.

Proof. Because $(W, V_1, V_2) \in \mathrm{Atom}^{\mathrm{val}}\,[\![A]\!]\,\rho$, we have $W \in \mathrm{World}_n$ for some n, and $W.\Sigma_1 \mid \emptyset \vdash V_1 : \rho(A)$ and $W.\Sigma_2 \mid \emptyset \vdash V_2 : \rho(A)$. Because $W' \sqsupseteq W$, we have $W'.\Sigma_1 \supseteq W.\Sigma_1$ and $\vdash W'.\Sigma_1$ and $W'.\Sigma_2 \supseteq W.\Sigma_2$ and $\vdash W'.\Sigma_2$. Therefore, by Lemma E.8, $W'.\Sigma_1 \mid \emptyset \vdash V_1 : \rho(A)$ and $W'.\Sigma_2 \mid \emptyset \vdash V_2 : \rho(A)$. Furthermore, $W' \sqsupseteq W$ and $W \in \mathrm{World}_n$ implies $W' \in \mathrm{World}_n$. Therefore, we have the conclusion. □

Lemma G.10 (Monotonicity)**.** If $W' \sqsupseteq W$ and $(W, V_1, V_2) \in \mathcal{V}\,[\![A]\!]\,\rho$, then $(W', V_1, V_2) \in \mathcal{V}\,[\![A]\!]\,\rho$.

Proof. By induction on $W.n$. Note that $(W', V_1, V_2) \in \mathrm{Atom}^{\mathrm{val}}\,[\![A]\!]\,\rho$ by Lemma G.9. We proceed by case analysis on A.

Case $A = \iota$: Obvious.

Case $A = \alpha$: By Lemma G.8.

Case $A = B \to C$: Let $W'' \sqsupseteq W'$ and V_1'' and V_2'' be values such that $(W'', V_1'', V_2'') \in \mathcal{V}[\![B]\!]\rho$. Then, it suffices to show that

$$(W'', V_1\, V_1'', V_2\, V_2'') \in \mathcal{E}[\![C]\!]\rho .$$

$W'' \sqsupseteq W'$ and $W' \sqsupseteq W$ imply $W'' \sqsupseteq W$ by Lemma G.5. Because $(W, V_1, V_2) \in \mathcal{V}[\![A]\!]\rho = \mathcal{V}[\![B \to C]\!]\rho$, we have the conclusion $(W'', V_1\, V_1', V_2\, V_2'') \in \mathcal{E}[\![C]\!]\rho$.

Case $A = \forall X.B$: By the following two cases.

- Let $W'' \sqsupseteq W'$ and $\mathbb{C}_1, \mathbb{C}_2, R, M_1, M_2, \alpha$ such that
  - $W''.\Sigma_1 \mid \emptyset \vdash \mathbb{C}_1$,
  - $W''.\Sigma_2 \mid \emptyset \vdash \mathbb{C}_2$,
  - $R \in \mathrm{Rel}_{W''.n}[\![\mathbb{C}_1, \mathbb{C}_2]\!]$,
  - $W''.\Sigma_1 \triangleright V_1\, \mathbb{C}_1 \longrightarrow W''.\Sigma_1, \alpha := \mathbb{C}_1 \triangleright M_1\langle coerce^{+}_{\alpha}(\rho(B)[X := \alpha])\rangle$, and
  - $W''.\Sigma_2 \triangleright V_2\, \mathbb{C}_2 \longrightarrow W''.\Sigma_2, \alpha := \mathbb{C}_2 \triangleright M_2\langle coerce^{+}_{\alpha}(\rho(B)[X := \alpha])\rangle$.

  Then, we show that

  $$(W'' \boxplus (\alpha, \mathbb{C}_1, \mathbb{C}_2, R), M_1, M_2) \in \blacktriangleright\mathcal{E}[\![B]\!]\rho\{X \mapsto \alpha\} .$$

  $W'' \sqsupseteq W'$ and $W' \sqsupseteq W$ imply $W'' \sqsupseteq W$ by Lemma G.5. Because $(W, V_1, V_2) \in \mathcal{V}[\![A]\!]\rho = \mathcal{V}[\![\forall X.B]\!]\rho$, we have the conclusion.
- Let $W'' \sqsupseteq W'$. We show that $(W'', V_1\, \star, V_2\, \star) \in \mathcal{E}[\![B]\!]\rho\{X \mapsto \star\}$, which is implied by Lemma G.5 and $(W, V_1, V_2) \in \mathcal{V}[\![\forall X.B]\!]\rho$.

Case $A = \star$: Since $(W, V_1, V_2) \in \mathcal{V}[\![A]\!]\rho = \mathcal{V}[\![\star]\!]\rho$, there exist some G, V_1', and V_2' such that

- $V_1 = V_1'\langle G!\rangle$,
- $V_2 = V_2'\langle G!\rangle$, and
- $(W, V_1', V_2') \in \blacktriangleright\mathcal{V}[\![G]\!]\emptyset$.

To prove the conclusion, it suffices to show that

$$(W', V_1', V_2') \in \blacktriangleright\mathcal{V}[\![G]\!]\emptyset .$$

If $W'.n = 0$, then the conclusion holds trivially. Otherwise, suppose that $W'.n > 0$. Then, it suffices to show that $(\blacktriangleright W', V_1', V_2') \in \mathcal{V}[\![G]\!]\emptyset$. Because $(W, V_1', V_2') \in \blacktriangleright\mathcal{V}[\![G]\!]\emptyset$ and $0 < W'.n \leq W.n$ by $W' \sqsupseteq W$, we have $(\blacktriangleright W, V_1', V_2') \in \mathcal{V}[\![G]\!]\emptyset$. $W' \sqsupseteq W$ implies $\blacktriangleright W' \sqsupseteq \blacktriangleright W$ by Lemma G.7 (3). Since $(\blacktriangleright W).n < W.n$, we have the conclusion $(\blacktriangleright W', V_1', V_2') \in \mathcal{V}[\![G]\!]\emptyset$ by the IH.

Case $A = X$: By the case for $A = \alpha$ or that for $A = \star$.

□

Lemma G.11 (World Extension is Closed Under Domains of Interpretation Mappings)**.** If $W' \sqsupseteq W$, then $\mathrm{dom}(W'.\kappa) \supseteq \mathrm{dom}(W.\kappa)$.

Proof. Let $\alpha \in \mathrm{dom}(W.\kappa)$. By definition, $\alpha \in \mathrm{dom}(\lfloor W.\kappa \rfloor_{W'.n})$. Because $W' \sqsupseteq W$ implies $W'.\kappa \sqsupseteq \lfloor W.\kappa \rfloor_{W'.n}$, we find $W'.\kappa(\alpha)$ well defined, that is, $\alpha \in \mathrm{dom}(W'.\kappa)$. □

Lemma G.12 (Substitution Monotonicity)**.** If $W' \sqsupseteq W$ and $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$, then $(W', \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$.

Proof. By induction on Γ.

Case $\Gamma = \emptyset$: Obvious.

Case $\Gamma = \Gamma', x : A$: By the IH and Lemma G.10.

Case $\Gamma = \Gamma', X$: By the IH Lemma G.11.

□

Lemma G.13 (Name Store Monotonicity). If $W' \sqsupseteq W$ and $W \in \mathcal{S}[\![\Sigma]\!]$, then $W' \in \mathcal{S}[\![\Sigma]\!]$.

Proof. By induction on Σ.

Case $\Sigma = \emptyset$: Obvious. Note that $W' \sqsupseteq W$ implies $W' \in$ World.

Case $\Sigma = \Sigma', \alpha := \mathbb{A}$: Because $W \in \mathcal{S}[\![\Sigma]\!] = \mathcal{S}[\![\Sigma', \alpha := \mathbb{A}]\!]$, we have the following:

- $W \in \mathcal{S}[\![\Sigma']\!]$,
- $W.\Sigma_1(\alpha) = \mathbb{A}$,
- $W.\Sigma_2(\alpha) = \mathbb{A}$, and
- $W.\kappa(\alpha) = \lfloor \mathcal{V}[\![\mathbb{A}]\!]\,\emptyset \rfloor_{W.n}$.

The conclusion is implied by the following.

- $W' \in \mathcal{S}[\![\Sigma']\!]$: By the IH.
- $W'.\Sigma_1(\alpha) = \mathbb{A}$: Because $W' \sqsupseteq W$ implies $W'.\Sigma_1 \supseteq W.\Sigma_1$, and $W.\Sigma_1(\alpha) = \mathbb{A}$, we have $W'.\Sigma_1(\alpha) = \mathbb{A}$.
- $W'.\Sigma_2(\alpha) = \mathbb{A}$: Proven similarly to the case for $W'.\Sigma_1$.
- $W'.\kappa(\alpha) = \lfloor \mathcal{V}[\![\mathbb{A}]\!]\,\emptyset \rfloor_{W'.n}$: Noting $W'.\kappa \sqsupseteq \lfloor W.\kappa \rfloor_{W'.n}$ (by $W' \sqsupseteq W$), $\alpha \in \mathrm{dom}(W.\kappa) = \mathrm{dom}(\lfloor W.\kappa \rfloor_{W'.n})$, and $W'.n \le W.n$ (by $W' \sqsupseteq W$), we can prove the conclusion as follows:

$$\begin{array}{rcll} W'.\kappa(\alpha) & = & \lfloor W.\kappa \rfloor_{W'.n}(\alpha) & (\text{by } W'.\kappa \sqsupseteq \lfloor W.\kappa \rfloor_{W'.n}) \\ & = & \lfloor W.\kappa(\alpha) \rfloor_{W'.n} & (\text{by definition}) \\ & = & \lfloor \lfloor \mathcal{V}[\![\mathbb{A}]\!]\,\emptyset \rfloor_{W.n} \rfloor_{W'.n} & (\text{by } W.\kappa(\alpha) = \lfloor \mathcal{V}[\![\mathbb{A}]\!]\,\emptyset \rfloor_{W.n}) \\ & = & \lfloor \mathcal{V}[\![\mathbb{A}]\!]\,\emptyset \rfloor_{W'.n} & (\text{by Lemma G.2 and } W'.n \le W.n)\ . \end{array}$$

□

Lemma G.14 (Related Values are Related Terms). If $(W, V_1, V_2) \in \mathcal{V}[\![A]\!]\,\rho$, then $(W, V_1, V_2) \in \mathcal{E}[\![A]\!]\,\rho$.

Proof. If $W.n = 0$, then the conclusion holds obviously. Otherwise, if $W.n > 0$, then it suffices to show that $W \sqsupseteq_0 W$, which is implied by Lemma G.5. □

Lemma G.15 ($\mathcal{E}$ is Closed Under Anti-Reduction). Suppose that

- $W.\Sigma_1 \triangleright M_1 \longrightarrow^n \Sigma_1' \triangleright M_1'$ and
- If $n < W.n$, then there exist some W' and M_2' such that
  - $W.\Sigma_2 \triangleright M_2 \longrightarrow^* W'.\Sigma_2 \triangleright M_2'$,
  - $W' \sqsupseteq_n W$,
  - $W'.\Sigma_1 = \Sigma_1'$, and
  - $(W', M_1', M_2') \in \mathcal{E}[\![A]\!]\,\rho$.

Then, $(W, M_1, M_2) \in \mathrm{Atom}[\![A]\!]\,\rho$ implies $(W, M_1, M_2) \in \mathcal{E}[\![A]\!]\,\rho$.

Proof. By case analysis on the termination of $W.\Sigma_1 \triangleright M_1$.

Case $\exists\, m, \Sigma_1, V_1.\ m < W.n \wedge W.\Sigma_1 \triangleright M_1 \longrightarrow^m \Sigma_1 \triangleright V_1$: In this case, we must show that there exist some W'' and V_2 such that

- $W.\Sigma_2 \triangleright M_2 \longrightarrow^* W''.\Sigma_2 \triangleright V_2$,
- $W'' \sqsupseteq_m W$,
- $W''.\Sigma_1 = \Sigma_1$, and
- $(W'', V_1, V_2) \in \mathcal{V}[\![A]\!]\,\rho$.

[TS: The following "WLOG" is okay?] Because $W.\Sigma_1 \triangleright M_1 \longrightarrow^n \Sigma'_1 \triangleright M'_1$ and $W.\Sigma_1 \triangleright M_1 \longrightarrow^m \Sigma_1 \triangleright V_1$, Theorem F.1 implies that, without loss of generality, we can assume that there exists some n_0 such that $m = n + n_0$ and $\Sigma'_1 \triangleright M'_1 \longrightarrow^{n_0} \Sigma_1 \triangleright V_1$. $m < W.n$ and $m = n + n_0$ imply $n < W.n$. Thus, by the assumption, there exists some W' and M'_2 such that ⟵

- $W.\Sigma_2 \triangleright M_2 \longrightarrow^* W'.\Sigma_2 \triangleright M'_2$,
- $W' \sqsupseteq_n W$,
- $W'.\Sigma_1 = \Sigma'_1$, and
- $(W', M'_1, M'_2) \in \mathcal{E}[\![A]\!]\rho$.

Because $W' \sqsupseteq_n W$ implies $W'.n = W.n - n = W.n - (m - n_0) = (W.n - m) + n_0$, and $m < W.n$, we have $n_0 < W'.n$. Thus, because $(W', M'_1, M'_2) \in \mathcal{E}[\![A]\!]\rho$ and $W'.\Sigma_1 \triangleright M'_1 = \Sigma'_1 \triangleright M'_1 \longrightarrow^{n_0} \Sigma_1 \triangleright V_1$, there exist some W'' and V_2 such that

- $W'.\Sigma_2 \triangleright M'_2 \longrightarrow^* W''.\Sigma_2 \triangleright V_2$,
- $W'' \sqsupseteq_{n_0} W'$,
- $W''.\Sigma_1 = \Sigma_1$, and
- $(W'', V_1, V_2) \in \mathcal{V}[\![A]\!]\rho$.

Now, we have the conclusion by the following, in addition to $W''.\Sigma_1 = \Sigma_1$ and $(W'', V_1, V_2) \in \mathcal{V}[\![A]\!]\rho$, which have been proven.

- $W.\Sigma_2 \triangleright M_2 \longrightarrow^* W'.\Sigma_2 \triangleright M'_2 \longrightarrow^* W''.\Sigma_2 \triangleright V_2$.
- $W'' \sqsupseteq_m W$ by Lemma G.5 with $W'' \sqsupseteq_{n_0} W'$ and $W' \sqsupseteq_n W$, and $m = n + n_0$.

Case $\exists\, m, \Sigma_1, p.\ m < W.n \wedge W.\Sigma_1 \triangleright M_1 \longrightarrow^m \Sigma_1 \triangleright \mathsf{blame}\, p$: In this case, we must show that there exists some Σ_2 such that $W.\Sigma_2 \triangleright M_2 \longrightarrow^* \Sigma_2 \triangleright \mathsf{blame}\, p$. [TS: The following "WLOG" is okay?] Because $W.\Sigma_1 \triangleright M_1 \longrightarrow^n \Sigma'_1 \triangleright M'_1$ and $W.\Sigma_1 \triangleright M_1 \longrightarrow^m \Sigma_1 \triangleright \mathsf{blame}\, p$, Theorem F.1 implies that, without loss of generality, we can assume that there exists some n_0 such that $m = n + n_0$ and $\Sigma'_1 \triangleright M'_1 \longrightarrow^{n_0} \Sigma_1 \triangleright \mathsf{blame}\, p$. $m < W.n$ and $m = n + n_0$ imply $n < W.n$. Thus, by the assumption, there exists some W' and M'_2 such that ⟵

- $W.\Sigma_2 \triangleright M_2 \longrightarrow^* W'.\Sigma_2 \triangleright M'_2$,
- $W' \sqsupseteq_n W$,
- $W'.\Sigma_1 = \Sigma'_1$, and
- $(W', M'_1, M'_2) \in \mathcal{E}[\![A]\!]\rho$.

Because $W' \sqsupseteq_n W$ implies $W'.n = (W.n - m) + n_0$, and $m < W.n$, we have $n_0 < W'.n$. Thus, because $(W', M'_1, M'_2) \in \mathcal{E}[\![A]\!]\rho$ and $W'.\Sigma_1 \triangleright M'_1 = \Sigma'_1 \triangleright M'_1 \longrightarrow^{n_0} \Sigma_1 \triangleright \mathsf{blame}\, p$, there exists some Σ_2 such that $W'.\Sigma_2 \triangleright M'_2 \longrightarrow^* \Sigma_2 \triangleright \mathsf{blame}\, p$. Since $W.\Sigma_2 \triangleright M_2 \longrightarrow^* W'.\Sigma_2 \triangleright M'_2$, we have the conclusion.

Otherwise: In this case, we have no proof obligation.

□

Lemma G.16 (Monadic Bind)**.** Suppose that

- $(W, M_1, M_2) \in \mathcal{E}[\![A]\!]\rho$ and
- $\forall\, W', V_1, V_2.\ W' \sqsupseteq W \wedge (W', V_1, V_2) \in \mathcal{V}[\![A]\!]\rho \Longrightarrow (W', F_1[V_1], F_2[V_2]) \in \mathcal{E}[\![B]\!]\rho$.

Then, $(W, F_1[M_1], F_2[M_2]) \in \mathrm{Atom}[\![B]\!]\rho$ implies $(W, F_1[M_1], F_2[M_2]) \in \mathcal{E}[\![B]\!]\rho$.

Proof. By case analysis on the termination of $W.\Sigma_1 \triangleright M_1$.

Case $\exists\, n, \Sigma_1, V_1.\ n < W.n \wedge W.\Sigma_1 \triangleright M_1 \longrightarrow^n \Sigma_1 \triangleright V_1$: Since $(W, M_1, M_2) \in \mathcal{E}[\![A]\!]\rho$, there exist some W' and V_2 such that

- $W.\Sigma_2 \triangleright M_2 \longrightarrow^* W'.\Sigma_2 \triangleright V_2$,

- $W' \sqsupseteq_n W$,
- $W'.\Sigma_1 = \Sigma_1$, and
- $(W', V_1, V_2) \in \mathcal{V}[\![A]\!]\rho$.

By the assumption, we have

$$(W', F_1[V_1], F_2[V_2]) \in \mathcal{E}[\![B]\!]\rho \ .$$

Since $W.\Sigma_1 \triangleright M_1 \longrightarrow^n \Sigma_1 \triangleright V_1 = W'.\Sigma_1 \triangleright V_1$, we have

$$W.\Sigma_1 \triangleright F_1[M_1] \longrightarrow^n W'.\Sigma_1 \triangleright F_1[V_1]$$

by Lemma G.1 (2). Since $W.\Sigma_2 \triangleright M_2 \longrightarrow^* W'.\Sigma_2 \triangleright V_2$, we have

$$W.\Sigma_2 \triangleright F_2[M_2] \longrightarrow^* W'.\Sigma_2 \triangleright F_2[V_2]$$

by Lemma G.1 (2). Then, by Lemma G.15, we have the conclusion.

Case $\exists\, n, \Sigma_1, p.\ n < W.n \wedge W.\Sigma_1 \triangleright M_1 \longrightarrow^n \Sigma_1 \triangleright \mathsf{blame}\, p$: Since $W.\Sigma_1 \triangleright M_1 \longrightarrow^n \Sigma_1 \triangleright \mathsf{blame}\, p$, Lemma G.1 (3) implies $W.\Sigma_1 \triangleright F_1[M_1] \longrightarrow^m \Sigma_1 \triangleright \mathsf{blame}\, p$ for some $m \geq n$. If $m \geq W.n$, then the conclusion holds trivially by Theorem F.1. Otherwise, if $m < W.n$, then we must show that there exists some Σ_2 such that $W.\Sigma_2 \triangleright F_2[M_2] \longrightarrow^* \Sigma_2 \triangleright \mathsf{blame}\, p$. Since $(W, M_1, M_2) \in \mathcal{E}[\![A]\!]\rho$, there exists some Σ_2 such that $W.\Sigma_2 \triangleright M_2 \longrightarrow^* \Sigma_2 \triangleright \mathsf{blame}\, p$. By Lemma G.1 (3), we have $W.\Sigma_2 \triangleright F_2[M_2] \longrightarrow^* \Sigma_2 \triangleright \mathsf{blame}\, p$. Thus, we have the conclusion.

Otherwise: There is no n, Σ_1, and M_1 such that

- $n < W.n$,
- $W.\Sigma_1 \triangleright M_1 \longrightarrow^n \Sigma_1 \triangleright M_1$, and
- M_1 is a value or blame.

Let $n < W.n$, Σ_1 be a name store, and M_1 be a term such that $W.\Sigma_1 \triangleright M_1 \longrightarrow^n \Sigma_1 \triangleright M_1$. Then, M_1 is neither a value nor blame. By Lemma G.1 (2), $W.\Sigma_1 \triangleright F_1[M_1] \longrightarrow^n \Sigma_1 \triangleright F_1[M_1]$. Since M_1 is neither a value nor blame, $F_1[M_1]$ is not either. Therefore, by Theorem F.1, for any Σ' and M'_1 such $W.\Sigma_1 \triangleright F_1[M_1] \longrightarrow^n \Sigma'_1 \triangleright M'_1$, we find that M'_1 is neither a value nor blame. Then, we have no proof obligation for proving the conclusion.

□

Lemma G.17 (Compositionality)**.** Suppose that $X \notin \mathrm{dom}(\rho)$.

1. $\mathcal{V}[\![A]\!]\rho\{X \mapsto \alpha^\star\} = \mathcal{V}[\![A[X := \alpha^\star]]\!]\rho$.
2. $\mathcal{E}[\![A]\!]\rho\{X \mapsto \alpha^\star\} = \mathcal{E}[\![A[X := \alpha^\star]]\!]\rho$.

Proof. We show that

1. $\forall\, W \in \mathrm{World}.\ \forall\, V_1, V_2.\ (W, V_1, V_2) \in \mathcal{V}[\![A]\!]\rho\{X \mapsto \alpha^\star\} \iff (W, V_1, V_2) \in \mathcal{V}[\![A[X := \alpha^\star]]\!]\rho$ and
2. $\forall\, W \in \mathrm{World}.\ \forall\, M_1, M_2.\ (W, M_1, M_2) \in \mathcal{E}[\![A]\!]\rho\{X \mapsto \alpha^\star\} \iff (W, M_1, M_2) \in \mathcal{E}[\![A[X := \alpha^\star]]\!]\rho$.

We prove both direction of $\iff$ by lexicographic induction on the pair of $W.n$ and A. To avoid repetition, we first show the case (1) using IHs for the case (2), and then the case (2) without using IHs directly but by assuming the case (1). Note that the proof of the case (2) can be unfolded in the proof of the case (1). Note that $\mathrm{Atom}[\![A]\!]\rho\{X \mapsto \alpha^\star\} = \mathrm{Atom}[\![A[X := \alpha^\star]]\!]\rho$.

1. Let $W \in \mathrm{World}$ and V_1 and V_2 be values. We show that

$$(W, V_1, V_2) \in \mathcal{V}[\![A]\!]\rho\{X \mapsto \alpha^\star\} \iff (W, V_1, V_2) \in \mathcal{V}[\![A[X := \alpha^\star]]\!]\rho \ .$$

We proceed by case analysis on A.

Case $A = \iota$: Obvious.

Case $A = \beta$: Because $\beta[X := \alpha^\star] = \beta$, it suffices to show that

$$(W, V_1, V_2) \in \mathcal{V}[\![\beta]\!]\rho\{X \mapsto \alpha^\star\} \iff (W, V_1, V_2) \in \mathcal{V}[\![\beta]\!]\rho\ ,$$

which holds obviously as $\mathcal{V}[\![\beta]\!]\rho$ is determined independently of given ρ.

Case $A = \star$: Obvious because $\star[X := \alpha^\star] = \star$ and $\mathcal{V}[\![\star]\!]\rho$ is determined independently of given ρ.

Case $A = Y$: If $Y = X$, then

$$\mathcal{V}[\![A]\!]\rho\{X \mapsto \alpha^\star\} = \mathcal{V}[\![X]\!]\rho\{X \mapsto \alpha^\star\} = \mathcal{V}[\![\alpha^\star]\!]\rho\{X \mapsto \alpha^\star\} = \mathcal{V}[\![\alpha^\star]\!]\rho = \mathcal{V}[\![X[X := \alpha^\star]]\!]\rho = \mathcal{V}[\![A[X := \alpha^\star]]\!]\rho\ .$$

Note that $\mathcal{V}[\![\alpha^\star]\!]\rho\{X \mapsto \alpha^\star\} = \mathcal{V}[\![\alpha^\star]\!]\rho$.
If $Y \neq X$, then

$$\mathcal{V}[\![A]\!]\rho\{X \mapsto \alpha^\star\} = \mathcal{V}[\![\rho(Y)]\!]\rho\{X \mapsto \alpha^\star\} = \mathcal{V}[\![\rho(Y)]\!]\rho = \mathcal{V}[\![Y]\!]\rho = \mathcal{V}[\![A[X := \alpha^\star]]\!]\rho\ .$$

Case $A = B \to C$: We show every case of $\iff$.

Case $\Longrightarrow$: Suppose that $(W, V_1, V_2) \in \mathcal{V}[\![B \to C]\!]\rho\{X \mapsto \alpha^\star\}$. We must show that

$$(W, V_1, V_2) \in \mathcal{V}[\![(B \to C)[X := \alpha^\star]]\!]\rho\ .$$

Let $W' \sqsupseteq W$, and V_1' and V_2' be values such that $(W', V_1', V_2') \in \mathcal{V}[\![B[X := \alpha^\star]]\!]\rho$. Then, it suffices to show that

$$(W', V_1\ V_1', V_2\ V_2') \in \mathcal{E}[\![C[X := \alpha^\star]]\!]\rho\ .$$

Because $W'.n \leq W.n$ by $W' \sqsupseteq W$, the IH implies that it suffices to show that

$$(W', V_1\ V_1', V_2\ V_2') \in \mathcal{E}[\![C]\!]\rho\{X \mapsto \alpha^\star\}\ .$$

Because $(W', V_1', V_2') \in \mathcal{V}[\![B[X := \alpha^\star]]\!]\rho$, we have $(W', V_1', V_2') \in \mathcal{V}[\![B]\!]\rho\{X \mapsto \alpha^\star\}$ by the IH. Since $(W, V_1, V_2) \in \mathcal{V}[\![B \to C]\!]\rho\{X \mapsto \alpha^\star\}$, we have the conclusion.

Case $\Longleftarrow$: Suppose that $(W, V_1, V_2) \in \mathcal{V}[\![(B \to C)[X := \alpha^\star]]\!]\rho$. We must show that

$$(W, V_1, V_2) \in \mathcal{V}[\![B \to C]\!]\rho\{X \mapsto \alpha^\star\}\ .$$

Let $W' \sqsupseteq W$, and V_1' and V_2' be values such that $(W', V_1', V_2') \in \mathcal{V}[\![B]\!]\rho\{X \mapsto \alpha^\star\}$. Then, it suffices to show that

$$(W', V_1\ V_1', V_2\ V_2') \in \mathcal{E}[\![C]\!]\rho\{X \mapsto \alpha^\star\}\ .$$

Because $W'.n \leq W.n$ by $W' \sqsupseteq W$, the IH implies that it suffices to show that

$$(W', V_1\ V_1', V_2\ V_2') \in \mathcal{E}[\![C[X := \alpha^\star]]\!]\rho\ .$$

Because $(W', V_1', V_2') \in \mathcal{V}[\![B]\!]\rho\{X \mapsto \alpha^\star\}$, we have $(W', V_1', V_2') \in \mathcal{V}[\![B[X := \alpha^\star]]\!]\rho$ by the IH. Since $(W, V_1, V_2) \in \mathcal{V}[\![(B \to C)[X := \alpha^\star]]\!]\rho$, we have the conclusion.

Case $A = \forall Y.B$: Without loss of generality, we can suppose that $Y \notin \mathrm{dom}(\rho\{X \mapsto \alpha^\star\})$. We show every case of $\iff$.

Case $\Longrightarrow$: Suppose that $(W, V_1, V_2) \in \mathcal{V}[\![\forall Y.B]\!]\rho\{X \mapsto \alpha^\star\}$. We must show that

$$(W, V_1, V_2) \in \mathcal{V}[\![(\forall Y.B)[X := \alpha^\star]]\!]\rho\ ,$$

which is implied by the following two cases.

- Let $W', \mathbb{C}_1, \mathbb{C}_2, R, M_1', M_2', \beta$ such that
  - $W' \sqsupseteq W$,
  - $W'.\Sigma_1 \mid \emptyset \vdash \mathbb{C}_1$,
  - $W'.\Sigma_2 \mid \emptyset \vdash \mathbb{C}_2$,

– $R \in \mathrm{Rel}_{W'.n}[\![\mathbb{C}_1, \mathbb{C}_2]\!]$,
– $W'.\Sigma_1 \rhd V_1\, \mathbb{C}_1 \longrightarrow W'.\Sigma_1, \beta := \mathbb{C}_1 \rhd M_1' \langle \mathit{coerce}^+_\beta(\rho(B[X := \alpha^\star])[Y := \beta]) \rangle$, and
– $W'.\Sigma_2 \rhd V_2\, \mathbb{C}_2 \longrightarrow W'.\Sigma_2, \beta := \mathbb{C}_2 \rhd M_2' \langle \mathit{coerce}^+_\beta(\rho(B[X := \alpha^\star])[Y := \beta]) \rangle$.

We show that

$$(W' \boxplus (\beta, \mathbb{C}_1, \mathbb{C}_2, R), M_1', M_2') \in \blacktriangleright\mathcal{E}[\![B[X := \alpha^\star]]\!]\, \rho\{Y \mapsto \beta\}\ .$$

Suppose that $(W' \boxplus (\beta, \mathbb{C}_1, \mathbb{C}_2, R)).n > 0$ (i.e., $W'.n > 0$). Then, it suffices to show that

$$(\blacktriangleright(W' \boxplus (\beta, \mathbb{C}_1, \mathbb{C}_2, R)), M_1', M_2') \in \mathcal{E}[\![B[X := \alpha^\star]]\!]\, \rho\{Y \mapsto \beta\}\ .$$

Because $W' \sqsupseteq W$ implies $W'.n \le W.n$, we have $\blacktriangleright(W' \boxplus (\beta, \mathbb{C}_1, \mathbb{C}_2, R)).n = (\blacktriangleright W').n = W'.n - 1 \le W.n - 1 < W.n$. Thus, by the IH, it suffices to show that

$$(\blacktriangleright(W' \boxplus (\beta, \mathbb{C}_1, \mathbb{C}_2, R)), M_1', M_2') \in \mathcal{E}[\![B]\!]\, \rho\{X \mapsto \alpha^\star\}\{Y \mapsto \beta\}\ .$$

Because of $(W, V_1, V_2) \in \mathcal{V}[\![\forall Y.B]\!]\, \rho\{X \mapsto \alpha^\star\}$, noting that $\rho(B[X := \alpha^\star]) = \rho\{X \mapsto \alpha^\star\}(B)$, we have the conclusion.

- Let $W' \sqsupseteq W$. We show that $(W', V_1 \star, V_2 \star) \in \mathcal{E}[\![B[X := \alpha^\star]]\!]\, \rho\{Y \mapsto \star\}$. Because $W'.n \le W.n$ by $W' \sqsupseteq W$, the IH implies that it suffices to show that $(W', V_1 \star, V_2 \star) \in \mathcal{E}[\![B]\!]\, \rho\{X \mapsto \alpha^\star\}\{Y \mapsto \star\}$, which follows from $(W, V_1, V_2) \in \mathcal{E}[\![\forall Y.B]\!]\, \rho\{X \mapsto \alpha^\star\}$.

Case $\Longleftarrow$: Suppose that $(W, V_1, V_2) \in \mathcal{V}[\![(\forall Y.B)[X := \alpha^\star]]\!]\, \rho$. We must show that

$$(W, V_1, V_2) \in \mathcal{V}[\![\forall Y.B]\!]\, \rho\{X \mapsto \alpha^\star\}\ ,$$

which is implied by the following two cases.

- Let $W', \mathbb{C}_1, \mathbb{C}_2, R, M_1', M_2', \beta$ such that
  – $W' \sqsupseteq W$,
  – $W'.\Sigma_1 \mid \emptyset \vdash \mathbb{C}_1$,
  – $W'.\Sigma_2 \mid \emptyset \vdash \mathbb{C}_2$,
  – $R \in \mathrm{Rel}_{W'.n}[\![\mathbb{C}_1, \mathbb{C}_2]\!]$,
  – $W'.\Sigma_1 \rhd V_1\, \mathbb{C}_1 \longrightarrow W'.\Sigma_1, \beta := \mathbb{C}_1 \rhd M_1' \langle \mathit{coerce}^+_\beta((\rho\{X \mapsto \alpha^\star\}(B))[Y := \beta]) \rangle$, and
  – $W'.\Sigma_2 \rhd V_2\, \mathbb{C}_2 \longrightarrow W'.\Sigma_2, \beta := \mathbb{C}_2 \rhd M_2' \langle \mathit{coerce}^+_\beta((\rho\{X \mapsto \alpha^\star\}(B))[Y := \beta]) \rangle$.

  We show that

$$(W' \boxplus (\beta, \mathbb{C}_1, \mathbb{C}_2, R), M_1', M_2') \in \blacktriangleright\mathcal{E}[\![B]\!]\, \rho\{X \mapsto \alpha^\star\}\{Y \mapsto \beta\}\ .$$

  Suppose that $(W' \boxplus (\beta, \mathbb{C}_1, \mathbb{C}_2, R)).n > 0$ (i.e., $W'.n > 0$). Then, it suffices to show that

$$(\blacktriangleright(W' \boxplus (\beta, \mathbb{C}_1, \mathbb{C}_2, R)), M_1', M_2') \in \mathcal{E}[\![B]\!]\, \rho\{X \mapsto \alpha^\star\}\{Y \mapsto \beta\}\ .$$

  Because $W' \sqsupseteq W$ implies $W'.n \le W.n$, we have $\blacktriangleright(W' \boxplus (\beta, \mathbb{C}_1, \mathbb{C}_2, R)).n = (\blacktriangleright W').n = W'.n - 1 \le W.n - 1 < W.n$. Thus, by the IH, it suffices to show that

$$(\blacktriangleright(W' \boxplus (\beta, \mathbb{C}_1, \mathbb{C}_2, R)), M_1', M_2') \in \mathcal{E}[\![B[X := \alpha^\star]]\!]\, \rho\{Y \mapsto \beta\}\ .$$

  Because of $(W, V_1, V_2) \in \mathcal{V}[\![(\forall Y.B)[X := \alpha^\star]]\!]\, \rho$, noting that $\rho\{X \mapsto \alpha^\star\}(B) = \rho(B[X := \alpha^\star])$, we have the conclusion.

- Let $W' \sqsupseteq W$. We show that $(W', V_1 \star, V_2 \star) \in \mathcal{E}[\![B]\!]\, \rho\{X \mapsto \alpha^\star\}\{Y \mapsto \star\}$. Because $W'.n \le W.n$ by $W' \sqsupseteq W$, the IH implies that it suffices to show that $(W', V_1 \star, V_2 \star) \in \mathcal{E}[\![B[X := \alpha^\star]]\!]\, \rho\{Y \mapsto \star\}$, which follows from $(W, V_1, V_2) \in \mathcal{E}[\![(\forall Y.B)[X := \alpha^\star]]\!]\, \rho$.

2. Let $W \in \mathrm{World}$ and M_1 and M_2 be terms. We show that

$$(W, M_1, M_2) \in \mathcal{E}[\![A]\!]\, \rho\{X \mapsto \alpha^\star\} \iff (W, M_1, M_2) \in \mathcal{E}[\![A[X := \alpha^\star]]\!]\, \rho\ .$$

By case analysis on the termination of $W.\Sigma_1 \rhd M_1$.

Case $\exists\, n, \Sigma_1, V_1.\ n < W.n \ \wedge\ W.\Sigma_1 \triangleright M_1 \ \longrightarrow^n\ \Sigma_1 \triangleright V_1$: In both directions of $\Longleftrightarrow$, there exist some W' and V_2 such that

- $W.\Sigma_2 \triangleright M_2 \ \longrightarrow^*\ W'.\Sigma_2 \triangleright V_2$,
- $W' \sqsupseteq_n W$, and
- $W'.\Sigma_1 = \Sigma_1$.

Thus, it suffices to show that

$$(W', V_1, V_2) \in \mathcal{V}[\![A]\!]\, \rho\{X \mapsto \alpha^\star\} \iff (W', V_1, V_2) \in \mathcal{V}[\![A[X := \alpha^\star]]\!]\, \rho\ ,$$

which is proven by the case (1).

Case $\exists\, n, \Sigma_1, p.\ n < W.n \ \wedge\ W.\Sigma_1 \triangleright M_1 \ \longrightarrow^n\ \Sigma_1 \triangleright \mathsf{blame}\, p$: In both directions of $\Longleftrightarrow$, there exists some Σ_2 such that $W.\Sigma_2 \triangleright M_2 \ \longrightarrow^*\ \Sigma_2 \triangleright \mathsf{blame}\, p$. Thus, we have the conclusion.

Otherwise: No proof obligation in this case.

□

Definition G.1 (Loose Coercion Typing)**.** A judgment $\Sigma \mid \Gamma \Vdash c : A \rightsquigarrow B$ is derived by the same inference rules as those for coercion typing judgment $\Sigma \mid \Gamma \vdash c : A \rightsquigarrow B$ except that the rules for $\Sigma \mid \Gamma \Vdash c : A \rightsquigarrow B$ do not assume $\vdash \Sigma$. We call the rule corresponding to (Ct_*_C) (Ct0_*_C).

[TS: Loose coercion typing is used in Lemma G.29 to type the coercion generated by type application, that is, $\Sigma, \alpha := \mathbb{A}_0 \mid \Gamma \Vdash \mathit{coerce}^{+}_{\alpha}(B[X := \alpha]) : B[X := \alpha] \rightsquigarrow B[X := \mathbb{A}_0]$, where $\Sigma \mid \emptyset \vdash \mathbb{A}_0$ may not hold because type names in $\mathbb{A}_0$ occur only in stores $W.\Sigma_1$ and $W.\Sigma_2$ in world W (and $W.\Sigma_1 \neq W.\Sigma_2$ in general).] ⟵

Lemma G.18 (Loosely Typing Coercions)**.** If $\Sigma \mid \Gamma \Vdash c : A \rightsquigarrow B$ and $\Sigma \subseteq \Sigma'$ and $\vdash \Sigma'$, then $\Sigma' \mid \Gamma \vdash c : A \rightsquigarrow B$.

Proof. Straightforward by induction on the derivation of $\Sigma \mid \Gamma \Vdash c : A \rightsquigarrow B$ with Lemmas D.2. □

Lemma G.19 (Coercion Typed Under Type Environments in Worlds)**.** If $\Sigma \mid \Gamma \Vdash c : A \rightsquigarrow B$ and $W \in \mathcal{S}[\![\Sigma]\!]$ and $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$, then $W.\Sigma_1 \mid \emptyset \vdash \rho(c) : \rho(A) \rightsquigarrow \rho(B)$ and $W.\Sigma_2 \mid \emptyset \vdash \rho(c) : \rho(A) \rightsquigarrow \rho(B)$.

Proof. $W \in \mathcal{S}[\![\Sigma]\!]$ implies $W \in \mathrm{World}$, $W.\Sigma_1 \supseteq \Sigma$, and $W.\Sigma_2 \supseteq \Sigma$. $W \in \mathrm{World}$ implies $\vdash W.\Sigma_1$ and $\vdash W.\Sigma_2$. Hence, Lemma G.18 with $\Sigma \mid \Gamma \vdash c : A \rightsquigarrow B$ implies

- $W.\Sigma_1 \mid \Gamma \vdash c : A \rightsquigarrow B$ and
- $W.\Sigma_2 \mid \Gamma \vdash c : A \rightsquigarrow B$.

Furthermore, $\forall X \in \mathrm{dom}(\Gamma).\ \rho(X) = \star \vee (\exists \alpha.\ \rho(X) = \alpha \wedge \alpha \in \mathrm{dom}(W.\Sigma_1) \cap \mathrm{dom}(W.\Sigma_2))$ because:

- $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$ implies $\forall X \in \mathrm{dom}(\Gamma).\ \rho(X) = \star \vee (\exists \alpha.\ \alpha = \rho(X) \wedge \alpha \in \mathrm{dom}(W.\kappa))$; and
- $W \in \mathrm{World}$ implies $\mathrm{dom}(W.\kappa) \subseteq \mathrm{dom}(W.\Sigma_1) \cap \mathrm{dom}(W.\Sigma_2)$.

Hence, by Lemmas E.11, E.13, and E.15, we have the conclusion. □

Lemma G.20 ($\mathcal{E}$ is Closed Under Reduction)**.** Assume that $(W, M_1, M_2) \in \mathcal{E}[\![A]\!]\, \rho$ and $W.\Sigma_1 \triangleright M_1 \ \longrightarrow^n\ \Sigma'_1 \triangleright M'_1$ and $W.\Sigma_2 \triangleright M_2 \ \longrightarrow^*\ \Sigma'_2 \triangleright M'_2$. Let $W' = (W.n - n, \Sigma'_1, \Sigma'_2, \lfloor W.\kappa \rfloor_{W.n-n})$. Then, $(W', M'_1, M'_2) \in \mathcal{E}[\![A]\!]\, \rho$.

Proof. We have $(W', M'_1, M'_2) \in \mathrm{Atom}[\![A]\!]\, \rho$ by Corollary E.20. By case analysis on the termination of $W'.\Sigma_1 \triangleright M'_1$.

Case $\exists\, m, \Sigma_1, V_1.\ m < W'.n \ \wedge\ W'.\Sigma_1 \triangleright M'_1 \ \longrightarrow^m\ \Sigma_1 \triangleright V_1$: We must show that there exist some W'' and V_2 such that

- $W'.\Sigma_2 \triangleright M'_2 \ \longrightarrow^*\ W''.\Sigma_2 \triangleright V_2$,
- $W'' \sqsupseteq_m W'$,

- $W''.\Sigma_1 = \Sigma_1$, and
- $(W'', V_1, V_2) \in \mathcal{V}[\![A]\!]\rho$.

Because $m < W'.n$ and $W'.n = W.n - n$, we have $n + m < W.n$. Because $(W, M_1, M_2) \in \mathcal{E}[\![A]\!]\rho$ and $W.\Sigma_1 \rhd M_1 \longrightarrow^{n+m} \Sigma_1 \rhd V_1$, there exist some W'' and V_2 such that

- $W.\Sigma_2 \rhd M_2 \longrightarrow^{*} W''.\Sigma_2 \rhd V_2$,
- $W'' \sqsupseteq_{n+m} W$,
- $W''.\Sigma_1 = \Sigma_1$,
- $(W'', V_1, V_2) \in \mathcal{V}[\![A]\!]\rho$.

Because $W.\Sigma_2 \rhd M_2 \longrightarrow^{*} \Sigma_2' \rhd M_2'$, Theorem F.1 implies that, without loss of generality, we can assume that $W'.\Sigma_2 \rhd M_2' = \Sigma_2' \rhd M_2' \longrightarrow^{*} W''.\Sigma_2 \rhd V_2$. Now, it suffices to show that $W'' \sqsupseteq_m W'$, which is implied by the following.

- $W''.n = W'.n - m$ because $W'' \sqsupseteq_{n+m} W$ implies $W''.n = W.n - (n + m)$ and $W'.n = W.n - n$;
- $W''.\Sigma_1 \supseteq W'.\Sigma_1$ by Lemma E.18 with $W'.\Sigma_1 \rhd M_1' \longrightarrow^{m} \Sigma_1 \rhd V_1 = W''.\Sigma_1 \rhd V_1$;
- $W''.\Sigma_2 \supseteq W'.\Sigma_2$ by Lemma E.18 with $W'.\Sigma_2 \rhd M_2' \longrightarrow^{*} W''.\Sigma_2 \rhd V_2$;
- $W''.\kappa \sqsupseteq \lfloor W'.\kappa \rfloor_{W''.n}$ because $\lfloor W'.\kappa \rfloor_{W''.n} = \lfloor \lfloor W.\kappa \rfloor_{W.n-n} \rfloor_{W.n-(n+m)} = \lfloor W.\kappa \rfloor_{W.n-(n+m)}$ by Lemma G.2, and $W''.\kappa \sqsupseteq \lfloor W.\kappa \rfloor_{W.n-(n+m)}$ by $W'' \sqsupseteq_{n+m} W$;
- $W'' \in \mathrm{World}$ by $W'' \sqsupseteq_{n+m} W$; and
- $W' \in \mathrm{World}$: Because $W \in \mathrm{World}$, we have $W \in \mathrm{World}_{n_0}$ for some n_0. We show $W' \in \mathrm{World}_{n_0}$ as follows.
  - $W'.n < n_0$ because $W'.n = W.n - n$ and $W \in \mathrm{World}_{n_0}$.
  - $\vdash W'.\Sigma_1$ and $\vdash W'.\Sigma_2$ by Lemma E.18 with $\vdash W.\Sigma_1$ and $\vdash W.\Sigma_2$.
  - Let $\alpha \in \mathrm{dom}(W'.\kappa)$. We show that $\lfloor W.\kappa(\alpha) \rfloor_{W.n-n} \in \mathrm{Rel}_{W.n-n}[\![W'.\Sigma_1(\alpha), W'.\Sigma_2(\alpha)]\!]$.
    * $\lfloor W.\kappa(\alpha) \rfloor_{W.n-n} \subseteq \mathrm{Atom}^{\mathrm{val}}_{W.n-n}[\![W'.\Sigma_1(\alpha), W'.\Sigma_2(\alpha)]\!]$: Let $(W''', M_1''', M_2''') \in \lfloor W.\kappa(\alpha) \rfloor_{W.n-n}$. By definition, $(W''', M_1''', M_2''') \in W.\kappa(\alpha)$ and $W'''.n < W.n - n$. Since $W \in \mathrm{World}_{n_0}$, we have $(W''', M_1''', M_2''') \in W.\kappa(\alpha) \subseteq \mathrm{Atom}^{\mathrm{val}}_{W.n}[\![W.\Sigma_1(\alpha), W.\Sigma_2(\alpha)]\!]$. Since $W'''.n < W.n - n$ and $W'.\Sigma_1 \supseteq W.\Sigma_1$ and $W'.\Sigma_2 \supseteq W.\Sigma_2$ by Lemma E.18, we have the conclusion.
    * Let $(W_1''', V_1''', V_2''') \in \lfloor W.\kappa(\alpha) \rfloor_{W.n-n}$ and $W_2''' \sqsupseteq W_1'''$. Then, we show that $(W_2''', V_1''', V_2''') \in \lfloor W.\kappa(\alpha) \rfloor_{W.n-n}$. By definition, $(W_1''', V_1''', V_2''') \in W.\kappa(\alpha)$ and $W_1'''.n < W.n - n$. Since $W \in \mathrm{World}_{n_0}$, $W.\kappa(\alpha)$ satisfies monotonicity, so $(W_2''', V_1''', V_2''') \in W.\kappa(\alpha)$. Since $W_2'''.n \leq W_1'''.n < W.n - n$, we have the conclusion $(W_2''', V_1''', V_2''') \in \lfloor W.\kappa(\alpha) \rfloor_{W.n-n}$.

Case $\exists\, m, \Sigma_1, p.\ m < W'.n \wedge W'.\Sigma_1 \rhd M_1' \longrightarrow^{m} \Sigma_1 \rhd \mathsf{blame}\, p$: We must show that there exist some Σ_2 such that $W'.\Sigma_2 \rhd M_2' \longrightarrow^{*} \Sigma_2 \rhd \mathsf{blame}\, p$. Because $m < W'.n$ and $W'.n = W.n - n$, we have $n + m < W.n$. Because $(W, M_1, M_2) \in \mathcal{E}[\![A]\!]\rho$ and $W.\Sigma_1 \rhd M_1 \longrightarrow^{n+m} \Sigma_1 \rhd \mathsf{blame}\, p$, there exist some Σ_2 such that $W.\Sigma_2 \rhd M_2 \longrightarrow^{*} \Sigma_2 \rhd \mathsf{blame}\, p$. Because $W.\Sigma_2 \rhd M_2 \longrightarrow^{*} \Sigma_2' \rhd M_2'$, Theorem F.1 implies that, without loss of generality, we can assume that $W'.\Sigma_2 \rhd M_2' = \Sigma_2' \rhd M_2' \longrightarrow^{*} \Sigma_2 \rhd \mathsf{blame}\, p$.

Otherwise: No proof obligation.

□

Lemma G.21 (Related Coercion Applications)**.** If $\Sigma \mid \Gamma \Vdash c : A \rightsquigarrow B$ and $W \in \mathcal{S}[\![\Sigma]\!]$ and $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$ and $(W, M_1, M_2) \in \mathcal{E}[\![A]\!]\rho$, then $(W, M_1\langle\rho(c)\rangle, M_2\langle\rho(c)\rangle) \in \mathcal{E}[\![B]\!]\rho$.

Proof. By induction on $W.n$.

We first show that

$$(W, M_1\langle\rho(c)\rangle, M_2\langle\rho(c)\rangle) \in \mathrm{Atom}[\![B]\!]\rho\ . \qquad (1)$$

Because $(W, M_1, M_2) \in \mathcal{E}[\![A]\!]\rho$, we have

- $W.\Sigma_1 \mid \emptyset \vdash M_1 : \rho(A)$ and

- $W.\Sigma_2 \mid \emptyset \vdash M_2 : \rho(A)$.

By Lemma G.19,

- $W.\Sigma_1 \mid \emptyset \vdash \rho(c) : \rho(A) \rightsquigarrow \rho(B)$ and
- $W.\Sigma_2 \mid \emptyset \vdash \rho(c) : \rho(A) \rightsquigarrow \rho(B)$.

Then, by (T_Crc_C), we have

- $W.\Sigma_1 \mid \emptyset \vdash M_1\langle\rho(c)\rangle : \rho(B)$ and
- $W.\Sigma_2 \mid \emptyset \vdash M_2\langle\rho(c)\rangle : \rho(B)$.

Therefore, we have (1).

Let W', V_1, V_2 such that

- $W' \sqsupseteq W$ and
- $(W', V_1, V_2) \in \mathcal{V}[\![A]\!]\rho$.

By Lemma G.16 with (1), it suffices to show that

$$(W', V_1\langle\rho(c)\rangle, V_2\langle\rho(c)\rangle) \in \mathcal{E}[\![B]\!]\rho .$$

We proceed by case analysis on the rule applied last to derive $\Sigma \mid \Gamma \Vdash c : A \rightsquigarrow B$.

Case (Ct0_Id_C): We are given $c = \mathsf{id}_A$ and $A = B$. It suffices to show that

$$(W', V_1\langle\mathsf{id}_{\rho(A)}\rangle, V_2\langle\mathsf{id}_{\rho(A)}\rangle) \in \mathcal{E}[\![A]\!]\rho .$$

By (R_Id_C),

- $W'.\Sigma_1 \triangleright V_1\langle\mathsf{id}_{\rho(A)}\rangle \longrightarrow W'.\Sigma_1 \triangleright V_1$ and
- $W'.\Sigma_2 \triangleright V_2\langle\mathsf{id}_{\rho(A)}\rangle \longrightarrow W'.\Sigma_2 \triangleright V_2$.

Supposing that $1 < W'.n$, we can prove the conclusion by Lemma G.15 with the following.

- $\blacktriangleright W' \sqsupseteq_1 W'$: By Lemma G.7 (2).
- $(\blacktriangleright W').\Sigma_1 = W'.\Sigma_1$: By definition.
- $(\blacktriangleright W').\Sigma_2 = W'.\Sigma_2$: By definition.
- $(\blacktriangleright W', V_1, V_2) \in \mathcal{E}[\![A]\!]\rho$: By Lemma G.14, it suffices to show that $(\blacktriangleright W', V_1, V_2) \in \mathcal{V}[\![A]\!]\rho$, which is implied by Lemma G.10 with $(W', V_1, V_2) \in \mathcal{V}[\![A]\!]\rho$ and $\blacktriangleright W' \sqsupseteq W'$.

Case (Ct0_Fail_C): We are given $c = \perp^p_{A \rightsquigarrow B}$ for some p. It suffices to show that

$$(W', V_1\langle\perp^p_{\rho(A)\rightsquigarrow\rho(B)}\rangle, V_2\langle\perp^p_{\rho(A)\rightsquigarrow\rho(B)}\rangle) \in \mathcal{E}[\![B]\!]\rho .$$

By (R_Fail_C),

- $W'.\Sigma_1 \triangleright V_1\langle\perp^p_{\rho(A)\rightsquigarrow\rho(B)}\rangle \longrightarrow W'.\Sigma_1 \triangleright \mathsf{blame}\, p$ and
- $W'.\Sigma_2 \triangleright V_2\langle\perp^p_{\rho(A)\rightsquigarrow\rho(B)}\rangle \longrightarrow W'.\Sigma_2 \triangleright \mathsf{blame}\, p$.

Supposing $1 < W'.n$, we can prove the conclusion by Lemma G.15 with the following.

- $\blacktriangleright W' \sqsupseteq_1 W'$ by Lemma G.7 (2).
- $(\blacktriangleright W').\Sigma_1 = W'.\Sigma_1$.
- $(\blacktriangleright W').\Sigma_2 = W'.\Sigma_2$.
- $(\blacktriangleright W', \mathsf{blame}\, p, \mathsf{blame}\, p) \in \mathcal{E}[\![B]\!]\rho$: Obvious.

Case (CT0_INJ_C): We are given $c = G!$ and $A = G$ and $B = \star$ for some G. It suffices to show that

$$(W', V_1\langle\rho(G)!\rangle, V_2\langle\rho(G)!\rangle) \in \mathcal{E}[\![\star]\!]\,\rho\ .$$

$V_1\langle\rho(G)!\rangle$ and $V_2\langle\rho(G)!\rangle$ are values. Thus, Lemma G.14 implies that it suffices to show that

$$(W', V_1\langle\rho(G)!\rangle, V_2\langle\rho(G)!\rangle) \in \mathcal{V}[\![\star]\!]\,\rho\ .$$

By definition, it suffices to show that

$$(W', V_1, V_2) \in \blacktriangleright\mathcal{V}[\![\rho(G)]\!]\,\emptyset\ .$$

Suppose that $0 < W'.n$. Then, it suffices to show that

$$(\blacktriangleright W', V_1, V_2) \in \mathcal{V}[\![\rho(G)]\!]\,\emptyset\ .$$

By Lemma G.17 (1), it suffices to show that

$$(\blacktriangleright W', V_1, V_2) \in \mathcal{V}[\![G]\!]\,\rho\ .$$

Because $(W', V_1, V_2) \in \mathcal{V}[\![A]\!]\,\rho = \mathcal{V}[\![G]\!]\,\rho$, and $\blacktriangleright W' \sqsupseteq W'$ by Lemma G.7 (2), we have the conclusion by Lemma G.10.

Case (CT0_PROJ_C): We are given $c = G?^p$ and $A = \star$ and $B = G$ for some G and p. It suffices to show that

$$(W', V_1\langle\rho(G)?^p\rangle, V_2\langle\rho(G)?^p\rangle) \in \mathcal{E}[\![G]\!]\,\rho\ .$$

By case analysis on the reduction of $W'.\Sigma_1 \triangleright V_1\langle\rho(G)?^p\rangle$ (such reduction is always possible by Theorem E.4 (Progress)).

Case (R_COLLAPSE_C): We are given V_1' such that

- $V_1 = V_1'\langle\rho(G)!\rangle$ and
- $W'.\Sigma_1 \triangleright V_1\langle\rho(G)?^p\rangle = W'.\Sigma_1 \triangleright V_1'\langle\rho(G)!\rangle\langle\rho(G)?^p\rangle \longrightarrow W'.\Sigma_1 \triangleright V_1'$.

Because $(W', V_1, V_2) \in \mathcal{V}[\![A]\!]\,\rho$, i.e., $(W', V_1'\langle\rho(G)!\rangle, V_2) \in \mathcal{V}[\![\star]\!]\,\rho$, there exists some V_2' such that

- $V_2 = V_2'\langle\rho(G)!\rangle$ and
- $(W', V_1', V_2') \in \blacktriangleright\mathcal{V}[\![\rho(G)]\!]\,\emptyset$.

Supposing that $1 < W'.n$, we can prove the conclusion by Lemma G.15 with the following.

- $W'.\Sigma_1 \triangleright V_1\langle\rho(G)?^p\rangle \longrightarrow W'.\Sigma_1 \triangleright V_1'$.
- $W'.\Sigma_2 \triangleright V_2\langle\rho(G)?^p\rangle = W'.\Sigma_2 \triangleright V_2'\langle\rho(G)!\rangle\langle\rho(G)?^p\rangle \longrightarrow W'.\Sigma_2 \triangleright V_2'$.
- $\blacktriangleright W' \sqsupseteq_1 W'$: By Lemma G.7 (2).
- $(\blacktriangleright W').\Sigma_1 = W'.\Sigma_1$.
- $(\blacktriangleright W').\Sigma_2 = W'.\Sigma_2$.
- $(\blacktriangleright W', V_1', V_2') \in \mathcal{E}[\![G]\!]\,\rho$: By Lemma G.14, it suffices to show that $(\blacktriangleright W', V_1', V_2') \in \mathcal{V}[\![G]\!]\,\rho$. By Lemma G.17 (1), it suffices to show that $(\blacktriangleright W', V_1', V_2') \in \mathcal{V}[\![\rho(G)]\!]\,\emptyset$, which is implied by $(W', V_1', V_2') \in \blacktriangleright\mathcal{V}[\![\rho(G)]\!]\,\emptyset$ and $1 < W'.n$.

Case (R_CONFLICT_C): We are given V_1' and H such that

- $V_1 = V_1'\langle H!\rangle$,
- $H \neq \rho(G)$, and
- $W'.\Sigma_1 \triangleright V_1\langle\rho(G)?^p\rangle = W'.\Sigma_1 \triangleright V_1'\langle H!\rangle\langle\rho(G)?^p\rangle \longrightarrow W'.\Sigma_1 \triangleright \mathsf{blame}\,p$.

Because $(W', V_1, V_2) \in \mathcal{V}[\![A]\!]\,\rho$, i.e., $(W', V_1'\langle H!\rangle, V_2) \in \mathcal{V}[\![\star]\!]\,\rho$, there exists some V_2' such that

- $V_2 = V_2'\langle H!\rangle$ and
- $(W', V_1', V_2') \in \blacktriangleright\mathcal{V}[\![H]\!]\,\emptyset$.

Supposing that $1 < W'.n$, we can prove the conclusion by Lemma G.15 with the following.

- $W'.\Sigma_1 \triangleright V_1\langle \rho(G)?^p\rangle \longrightarrow W'.\Sigma_1 \triangleright \mathsf{blame}\, p$.
- $W'.\Sigma_2 \triangleright V_2\langle \rho(G)?^p\rangle = W'.\Sigma_2 \triangleright V_2'\langle H!\rangle\langle \rho(G)?^p\rangle \longrightarrow W'.\Sigma_2 \triangleright \mathsf{blame}\, p$.
- $\blacktriangleright W' \sqsupseteq_1 W'$: By Lemma G.7 (2).
- $(\blacktriangleright W').\Sigma_1 = W'.\Sigma_1$.
- $(\blacktriangleright W').\Sigma_2 = W'.\Sigma_2$.
- $(\blacktriangleright W', \mathsf{blame}\, p, \mathsf{blame}\, p) \in \mathcal{E}\,[\![G]\!]\,\rho$: Obvious.

Otherwise: Contradiction.

Case (Ct0_Conceal_C): We are given $c = \alpha^-$ and $A = \mathbb{C}$ and $B = \alpha$ for some $\mathbb{C}$ and α such that $\alpha := \mathbb{C} \in \Sigma$. It suffices to show that

$$(W', V_1\langle\alpha^-\rangle, V_2\langle\alpha^-\rangle) \in \mathcal{E}\,[\![\alpha]\!]\,\rho\ .$$

Because $V_1\langle\alpha^-\rangle$ and $V_2\langle\alpha^-\rangle$ are values, Lemma G.14 implies that it suffices to show that

$$(W', V_1\langle\alpha^-\rangle, V_2\langle\alpha^-\rangle) \in \mathcal{V}\,[\![\alpha]\!]\,\rho\ .$$

By definition, it suffices to show that

$$(W', V_1, V_2) \in \blacktriangleright(W'.\kappa(\alpha))\ .$$

Suppose that $0 < W'.n$. Then, it suffices to show that

$$(\blacktriangleright W', V_1, V_2) \in W'.\kappa(\alpha)\ .$$

Because $W \in \mathcal{S}\,[\![\Sigma]\!]$ and $\alpha := \mathbb{C} \in \Sigma$, we have $W.\kappa(\alpha) = \lfloor \mathcal{V}\,[\![\mathbb{C}]\!]\,\emptyset \rfloor_{W.n}$. Because $W' \sqsupseteq W$, we have $W'.\kappa \sqsupseteq \lfloor W.\kappa \rfloor_{W'.n}$. Because $\alpha \in \mathrm{dom}(W.\kappa) = \mathrm{dom}(\lfloor W.\kappa \rfloor_{W'.n})$, we have $W'.\kappa(\alpha) = \lfloor W.\kappa \rfloor_{W'.n}(\alpha) = \lfloor W.\kappa(\alpha) \rfloor_{W'.n}$. Thus, it suffices to show that

$$(\blacktriangleright W', V_1, V_2) \in W.\kappa(\alpha) \wedge (\blacktriangleright W').n < W'.n\ .$$

The second conjunct is trivial. Because $W.\kappa(\alpha) = \lfloor \mathcal{V}\,[\![\mathbb{C}]\!]\,\emptyset \rfloor_{W.n}$, we can show the first by the following.

- $(\blacktriangleright W', V_1, V_2) \in \mathcal{V}\,[\![\mathbb{C}]\!]\,\emptyset$: Since
  - $\alpha := \mathbb{C} \in \Sigma$,
  - $W.\Sigma_1 \supseteq \Sigma$ by $W \in \mathcal{S}\,[\![\Sigma]\!]$, and
  - $\vdash W.\Sigma_1$ by $W \in \mathrm{World}$,

  we have $\rho(\mathbb{C}) = \mathbb{C}$. Thus, by Lemma G.17 (1), it suffices to show that $(\blacktriangleright W', V_1, V_2) \in \mathcal{V}\,[\![\mathbb{C}]\!]\,\rho$, which is implied by Lemma G.10 with $(W', V_1, V_2) \in \mathcal{V}\,[\![\mathbb{C}]\!]\,\rho$ and $\blacktriangleright W' \sqsupseteq W'$ obtained by Lemma G.7 (2).
- $(\blacktriangleright W').n < W.n$: By $(\blacktriangleright W').n < W'.n \leq W.n$; the second inequation is implied by $W' \sqsupseteq W$.

Case (Ct0_Reveal_C): We are given $c = \alpha^+$ and $A = \alpha$ and $B = \mathbb{C}$ for some α and $\mathbb{C}$ such that $\alpha := \mathbb{C} \in \Sigma$. It suffices to show that

$$(W', V_1\langle\alpha^+\rangle, V_2\langle\alpha^+\rangle) \in \mathcal{E}\,[\![\mathbb{C}]\!]\,\rho\ .$$

Because $(W', V_1, V_2) \in \mathcal{V}\,[\![A]\!]\,\rho = \mathcal{V}\,[\![\alpha]\!]\,\rho$, there exist some V_1' and V_2' such that

- $V_1 = V_1'\langle\alpha^-\rangle$,
- $V_2 = V_2'\langle\alpha^-\rangle$, and
- $(W', V_1', V_2') \in \blacktriangleright(W'.\kappa(\alpha))$.

Thus, by (R_Remove_C),

- $W'.\Sigma_1 \triangleright V_1\langle\alpha^+\rangle = W'.\Sigma_1 \triangleright V_1'\langle\alpha^-\rangle\langle\alpha^+\rangle \longrightarrow W'.\Sigma_1 \triangleright V_1'$ and
- $W'.\Sigma_2 \triangleright V_2\langle\alpha^+\rangle = W'.\Sigma_2 \triangleright V_2'\langle\alpha^-\rangle\langle\alpha^+\rangle \longrightarrow W'.\Sigma_2 \triangleright V_2'$.

Supposing $1 < W'.n$, we can prove the conclusion by Lemma G.15 with the following.

- $\blacktriangleright W' \sqsupseteq_1 W'$ by Lemma G.7 (2).
- $(\blacktriangleright W').\Sigma_1 = W'.\Sigma_1$.
- $(\blacktriangleright W').\Sigma_2 = W'.\Sigma_2$.
- $(\blacktriangleright W', V'_1, V'_2) \in \mathcal{E}[\![\mathbb{C}]\!]\,\rho$: By Lemma G.14, it suffices to show that

$$(\blacktriangleright W', V'_1, V'_2) \in \mathcal{V}[\![\mathbb{C}]\!]\,\rho \ .$$

  Because
  - $\alpha := \mathbb{C} \in \Sigma$,
  - $W.\Sigma_1 \supseteq \Sigma$ by $W \in \mathcal{S}[\![\Sigma]\!]$, and
  - $\vdash W.\Sigma_1$ by $W \in$ World,

  we have $\rho(\mathbb{C}) = \mathbb{C}$. By Lemma G.17, it suffices to show that

$$(\blacktriangleright W', V'_1, V'_2) \in \mathcal{V}[\![\mathbb{C}]\!]\,\emptyset \ .$$

  Because $W \in \mathcal{S}[\![\Sigma]\!]$ and $\alpha := \mathbb{C} \in \Sigma$, we have $W.\kappa(\alpha) = \lfloor \mathcal{V}[\![\mathbb{C}]\!]\,\emptyset \rfloor_{W.n}$. Thus, it suffices to show that

$$(\blacktriangleright W', V'_1, V'_2) \in W.\kappa(\alpha) \ .$$

  Because $W' \sqsupseteq W$, we have $W'.\kappa \sqsupseteq \lfloor W.\kappa \rfloor_{W'.n}$. Furthermore, $\alpha \in \mathrm{dom}(W.\kappa) = \mathrm{dom}(\lfloor W.\kappa \rfloor_{W'.n})$. Thus, $W'.\kappa(\alpha) = \lfloor W.\kappa(\alpha) \rfloor_{W'.n}$. Therefore, it suffices to show that

$$(\blacktriangleright W', V'_1, V'_2) \in W'.\kappa(\alpha) \ .$$

  Because $(W', V'_1, V'_2) \in \blacktriangleright(W'.\kappa(\alpha))$ and $1 < W'.n$, we have the conclusion $(\blacktriangleright W', V'_1, V'_2) \in W'.\kappa(\alpha)$.

Case (Ct0_Arrow_C): We are given $c = d_1 \rightarrow d_2$ and $A = A_1 \rightarrow A_2$ and $B = B_1 \rightarrow B_2$ for some d_1, d_2, A_1, A_2, B_1, and B_2. By inversion, $\Sigma \mid \Gamma \Vdash d_1 : B_1 \rightsquigarrow A_1$ and $\Sigma \mid \Gamma \Vdash d_2 : A_2 \rightsquigarrow B_2$. Because $V_1\langle d_1 \rightarrow d_2\rangle$ and $V_2\langle d_1 \rightarrow d_2\rangle$ are values, Lemma G.14 implies that it suffices to show that

$$(W', V_1\langle \rho(d_1 \rightarrow d_2)\rangle, V_2\langle \rho(d_1 \rightarrow d_2)\rangle) \in \mathcal{V}[\![B_1 \rightarrow B_2]\!]\,\rho \ .$$

Let $W'' \sqsupseteq W'$ and V''_1 and V''_2 be values such that $(W'', V''_1, V''_2) \in \mathcal{V}[\![B_1]\!]\,\rho$. Then, it suffices to show that

$$(W'', (V_1\langle \rho(d_1 \rightarrow d_2)\rangle)\ V''_1, V_2\langle \rho(d_1 \rightarrow d_2)\rangle\ V''_2) \in \mathcal{E}[\![B_2]\!]\,\rho \ .$$

By (R_Wrap_C),

- $W''.\Sigma_1 \triangleright (V_1\langle \rho(d_1 \rightarrow d_2)\rangle)\ V''_1 \longrightarrow W''.\Sigma_1 \triangleright (V_1\,(V''_1\langle \rho(d_1)\rangle))\langle \rho(d_2)\rangle$ and
- $W''.\Sigma_2 \triangleright (V_2\langle \rho(d_1 \rightarrow d_2)\rangle)\ V''_2 \longrightarrow W''.\Sigma_2 \triangleright (V_2\,(V''_2\langle \rho(d_1)\rangle))\langle \rho(d_2)\rangle$.

Supposing that $1 < W''.n$, we can prove the conclusion by Lemma G.15 with the following.

- $\blacktriangleright W'' \sqsupseteq_1 W''$ by Lemma G.7 (2).
- $(\blacktriangleright W'').\Sigma_1 = W''.\Sigma_1$.
- $(\blacktriangleright W'').\Sigma_2 = W''.\Sigma_2$.

Finally, Lemma G.15 requires us to prove that

$$(\blacktriangleright W'', (V_1\,(V''_1\langle \rho(d_1)\rangle))\langle \rho(d_2)\rangle, (V_2\,(V''_2\langle \rho(d_1)\rangle))\langle \rho(d_2)\rangle) \in \mathcal{E}[\![B_2]\!]\,\rho \ .$$

Because $\blacktriangleright W'' \sqsupseteq_1 W''$ by Lemma G.7 (2), and $W'' \sqsupseteq W'$ and $W' \sqsupseteq W$, we have $\blacktriangleright W'' \sqsupseteq W$ by Lemma G.5. Now, we have the following.

- $\Sigma \mid \Gamma \Vdash d_1 : B_1 \rightsquigarrow A_1$.
- $\blacktriangleright W'' \in \mathcal{S}[\![\Sigma]\!]$: By Lemma G.13 with $\blacktriangleright W'' \sqsupseteq W$ and $W \in \mathcal{S}[\![\Sigma]\!]$.
- $(\blacktriangleright W'', \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$: By Lemma G.12 with $\blacktriangleright W'' \sqsupseteq W$ and $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$.

- $(\blacktriangleright W'', V_1'', V_2'') \in \mathcal{E}[\![B_1]\!]\rho$ by Lemma G.10 with $(W'', V_1'', V_2'') \in \mathcal{V}[\![B_1]\!]\rho$ and $\blacktriangleright W'' \sqsupseteq_1 W''$, and then Lemma G.14.

Because $(\blacktriangleright W'').n < W.n$, the IH implies

$$(\blacktriangleright W'', V_1''\langle\rho(d_1)\rangle, V_2''\langle\rho(d_1)\rangle) \in \mathcal{E}[\![A_1]\!]\rho .$$

Let $W''' \sqsupseteq \blacktriangleright W''$ and V_1''' and V_2''' be values such that $(W''', V_1''', V_2''') \in \mathcal{V}[\![A_1]\!]\rho$. Then, Lemma G.16 implies that it suffices to show that

$$(W''', (V_1\ V_1''')\langle\rho(d_2)\rangle, (V_2\ V_2''')\langle\rho(d_2)\rangle) \in \mathcal{E}[\![B_2]\!]\rho .$$

Because $(W', V_1, V_2) \in \mathcal{V}[\![A]\!]\rho = \mathcal{V}[\![A_1 \to A_2]\!]\rho$, $W''' \sqsupseteq W'$ by Lemma G.5, and $(W''', V_1''', V_2''') \in \mathcal{V}[\![A_1]\!]\rho$, we have

$$(W''', V_1\ V_1''', V_2\ V_2''') \in \mathcal{V}[\![A_2]\!]\rho .$$

Let $W'''' \sqsupseteq W'''$ and V_1'''' and V_2'''' be values such that $(W'''', V_1'''', V_2'''') \in \mathcal{V}[\![A_2]\!]\rho$. Then, Lemma G.16 implies that it suffices to show that

$$(W'''', V_1''''\langle\rho(d_2)\rangle, V_2''''\langle\rho(d_2)\rangle) \in \mathcal{E}[\![B_2]\!]\rho .$$

Noting $W'''' \sqsupseteq W$ by Lemma G.5, we have the following.

- $\Sigma \mid \Gamma \Vdash d_2 : A_2 \rightsquigarrow B_2$.
- $W'''' \in \mathcal{S}[\![\Sigma]\!]$ by Lemma G.13.
- $(W'''', \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$ by Lemma G.12.
- $(W'''', V_1'''', V_2'''') \in \mathcal{E}[\![A_2]\!]\rho$ by Lemma G.14 with $(W'''', V_1'''', V_2'''') \in \mathcal{V}[\![A_2]\!]\rho$.

Because $W''''.n < W''.n \leq W.n$, the IH implies the conclusion.

Case (Ct0_Seq_C): We are given $c = c_1 \mathbin{;} c_2$ for some c_1 and c_2. By inversion, $\Sigma \mid \Gamma \Vdash c_1 : A \rightsquigarrow C$ and $\Sigma \mid \Gamma \Vdash c_2 : C \rightsquigarrow B$ for some C. It suffices to show that

$$(W', V_1\langle\rho(c_1 \mathbin{;} c_2)\rangle, V_2\langle\rho(c_1 \mathbin{;} c_2)\rangle) \in \mathcal{E}[\![B]\!]\rho .$$

By (R_Split_C), we have

- $W'.\Sigma_1 \triangleright V_1\langle\rho(c_1 \mathbin{;} c_2)\rangle \longrightarrow W'.\Sigma_1 \triangleright V_1\langle\rho(c_1)\rangle\langle\rho(c_2)\rangle$ and
- $W'.\Sigma_2 \triangleright V_2\langle\rho(c_1 \mathbin{;} c_2)\rangle \longrightarrow W'.\Sigma_2 \triangleright V_2\langle\rho(c_1)\rangle\langle\rho(c_2)\rangle$.

Supposing that $1 < W'.n$, we can prove the conclusion by Lemma G.15 with the following.

- $\blacktriangleright W' \sqsupseteq_1 W'$ by Lemma G.7 (2).
- $(\blacktriangleright W').\Sigma_1 = W'.\Sigma_1$.
- $(\blacktriangleright W').\Sigma_2 = W'.\Sigma_2$.

Finally, Lemma G.15 requires us to prove that

$$(\blacktriangleright W', V_1\langle\rho(c_1)\rangle\langle\rho(c_2)\rangle, V_2\langle\rho(c_1)\rangle\langle\rho(c_2)\rangle) \in \mathcal{E}[\![B]\!]\rho .$$

Noting $\blacktriangleright W' \sqsupseteq W$ by Lemma G.5 with $\blacktriangleright W' \sqsupseteq W'$ and $W' \sqsupseteq W$, we have the following.

- $\Sigma \mid \Gamma \Vdash c_1 : A \rightsquigarrow C$.
- $\blacktriangleright W' \in \mathcal{S}[\![\Sigma]\!]$ by Lemma G.13 with $W \in \mathcal{S}[\![\Sigma]\!]$.
- $(\blacktriangleright W', \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$ by Lemma G.12 with $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$.
- $(\blacktriangleright W', V_1, V_2) \in \mathcal{E}[\![A]\!]\rho$ by Lemmas G.10 and G.14 with $(W', V_1, V_2) \in \mathcal{V}[\![A]\!]\rho$.

Because $(\blacktriangleright W').n < W.n$, the IH implies that

$$(\blacktriangleright W', V_1\langle\rho(c_1)\rangle, V_2\langle\rho(c_1)\rangle) \in \mathcal{E}\,[\![C]\!]\,\rho\ .$$

Let $W'' \sqsupseteq \blacktriangleright W'$, and V_1' and V_2' be values such that $(W'', V_1', V_2') \in \mathcal{V}\,[\![C]\!]\,\rho$. Then, by Lemma G.16, it suffices to show that

$$(W'', V_1'\langle\rho(c_2)\rangle, V_2'\langle\rho(c_2)\rangle) \in \mathcal{E}\,[\![B]\!]\,\rho\ .$$

Noting $W'' \sqsupseteq W$ by Lemma G.5, we have the following.

- $\Sigma \mid \Gamma \Vdash c_2 : C \rightsquigarrow B$.
- $W'' \in \mathcal{S}\,[\![\Sigma]\!]$ by Lemma G.13 with $W \in \mathcal{S}\,[\![\Sigma]\!]$.
- $(W'', \theta, \rho) \in \mathcal{G}\,[\![\Gamma]\!]$ by Lemma G.12 with $(W, \theta, \rho) \in \mathcal{G}\,[\![\Gamma]\!]$.
- $(W'', V_1', V_2') \in \mathcal{E}\,[\![C]\!]\,\rho$ by Lemma G.14 with $(W'', V_1', V_2') \in \mathcal{V}\,[\![C]\!]\,\rho$.

Because $W''.n < W'.n \leq W.n$, the IH implies the conclusion.

Case (Ct0_All_C): We are given $c = \forall X.c_0$ and $A = \forall X.A_0$ and $B = \forall X.B_0$ for some X, c_0, A_0, and B_0. By inversion, $\Sigma \mid \Gamma, X \Vdash c_0 : A_0 \rightsquigarrow B_0$. Without loss of generality, we can suppose that $X \notin \mathrm{dom}(\rho)$. It suffices to show that

$$(W', V_1\langle\rho(\forall X.c_0)\rangle, V_2\langle\rho(\forall X.c_0)\rangle) \in \mathcal{E}\,[\![\forall X.B_0]\!]\,\rho\ .$$

Because $V_1\langle\rho(\forall X.c_0)\rangle$ and $V_2\langle\rho(\forall X.c_0)\rangle$ are values, Lemma G.14 implies that it suffices to show that

$$(W', V_1\langle\rho(\forall X.c_0)\rangle, V_2\langle\rho(\forall X.c_0)\rangle) \in \mathcal{V}\,[\![\forall X.B_0]\!]\,\rho\ ,$$

which is implied by the following two cases.

- Let $W'', \mathbb{C}_1, \mathbb{C}_2, R, M_1'', M_2'', \alpha$ such that
  - $W'' \sqsupseteq W'$,
  - $W''.\Sigma_1 \mid \emptyset \vdash \mathbb{C}_1$,
  - $W''.\Sigma_2 \mid \emptyset \vdash \mathbb{C}_2$,
  - $R \in \mathrm{Rel}_{W''.n}\,[\![\mathbb{C}_1, \mathbb{C}_2]\!]$,
  - $W''.\Sigma_1 \rhd V_1\langle\rho(\forall X.c_0)\rangle\,\mathbb{C}_1 \longrightarrow W''.\Sigma_1, \alpha := \mathbb{C}_1 \rhd M_1''\langle \mathit{coerce}^+_\alpha(\rho(B_0)[X := \alpha])\rangle$, and
  - $W''.\Sigma_2 \rhd V_2\langle\rho(\forall X.c_0)\rangle\,\mathbb{C}_2 \longrightarrow W''.\Sigma_2, \alpha := \mathbb{C}_2 \rhd M_2''\langle \mathit{coerce}^+_\alpha(\rho(B_0)[X := \alpha])\rangle$.

  Furthermore, let $W''' = W'' \boxplus (\alpha, \mathbb{C}_1, \mathbb{C}_2, R)$. Then, we must show that

  $$(W''', M_1'', M_2'') \in \blacktriangleright\mathcal{E}\,[\![B_0]\!]\,\rho\{X \mapsto \alpha\}\ .$$

  Suppose that $0 < W'''.n$. Then, it suffices to show that

  $$(\blacktriangleright W''', M_1'', M_2'') \in \mathcal{E}\,[\![B_0]\!]\,\rho\{X \mapsto \alpha\}\ .$$

  Because $(W', V_1, V_2) \in \mathcal{V}\,[\![A]\!]\,\rho = \mathcal{V}\,[\![\forall X.A_0]\!]\,\rho$, we have
  - $W'.\Sigma_1 \mid \emptyset \vdash V_1 : \forall X.\rho(A_0)$ and
  - $W'.\Sigma_2 \mid \emptyset \vdash V_2 : \forall X.\rho(A_0)$.

  Thus, Lemma E.3 implies:
  - $V_1 = (\Lambda X.(M_1' : A_1'))\overline{\langle\forall X.c_1\rangle}$,
  - $W'.\Sigma_1 \vdash \overline{\langle\forall X.c_1\rangle} : \forall X.A_1' \rightsquigarrow \forall X.\rho(A_0)$,
  - $V_2 = (\Lambda X.(M_2' : A_2'))\overline{\langle\forall X.c_2\rangle}$, and
  - $W'.\Sigma_2 \vdash \overline{\langle\forall X.c_2\rangle} : \forall X.A_2' \rightsquigarrow \forall X.\rho(A_0)$

  for some $M_1', M_2', A_1', A_2', \overline{\langle c_1\rangle}, \overline{\langle c_2\rangle}$.
  Let
  - $M_1''' = M_1'[X := \alpha]\overline{\langle c_1[X := \alpha]\rangle}$ and

– $M_2''' = M_2'[X := \alpha]\overline{\langle c_2[X := \alpha]\rangle}$.

In what follows, we show the conclusion by proving first that

$$(\blacktriangleright W''', M_1'''\langle \rho\{X \mapsto \alpha\}(c_0)\rangle, M_2'''\langle \rho\{X \mapsto \alpha\}(c_0)\rangle) \in \mathcal{E}\,[\![B_0]\!]\,\rho\{X \mapsto \alpha\}\ , \tag{2}$$

and then that

$$M_1'' = M_1'''\langle \rho\{X \mapsto \alpha\}(c_0)\rangle \wedge M_2'' = M_2'''\langle \rho\{X \mapsto \alpha\}(c_0)\rangle\ . \tag{3}$$

– We start with proving (2). We have
  * $(W', V_1, V_2) \in \mathcal{V}\,[\![A]\!]\,\rho$, i.e., $(W', (\Lambda X.(M_1' : A_1'))\overline{\langle \forall X.c_1\rangle}, (\Lambda X.(M_2' : A_2'))\overline{\langle \forall X.c_2\rangle}) \in \mathcal{V}\,[\![\forall X.A_0]\!]\,\rho$,
  * $W'' \sqsupseteq W'$,
  * $W''.\Sigma_1 \mid \emptyset \vdash \mathbb{C}_1$,
  * $W''.\Sigma_2 \mid \emptyset \vdash \mathbb{C}_2$, and
  * $R \in \mathrm{Rel}_{W''.n}\,[\![\mathbb{C}_1, \mathbb{C}_2]\!]$.

  Because
  * $W'' \sqsupseteq W'$ implies $W'' \in \mathrm{World}$, $W''.\Sigma_1 \supseteq W'.\Sigma_1$, and $W''.\Sigma_2 \supseteq W'.\Sigma_2$, and
  * $W'' \in \mathrm{World}$ implies $\vdash W''.\Sigma_1$ and $\vdash W''.\Sigma_2$,

  we have
  * $W''.\Sigma_1 \vdash \overline{\langle \forall X.c_1\rangle} : \forall X.A_1' \rightsquigarrow \forall X.\rho(A_0)$ and
  * $W''.\Sigma_2 \vdash \overline{\langle \forall X.c_2\rangle} : \forall X.A_2' \rightsquigarrow \forall X.\rho(A_0)$

  by Lemma E.6. Therefore, noting that α is fresh, we have

$$\begin{array}{ll} & W''.\Sigma_1 \rhd V_1\,\mathbb{C}_1 \\ = & W''.\Sigma_1 \rhd (\Lambda X.(M_1' : A_1'))\overline{\langle \forall X.c_1\rangle}\,\mathbb{C}_1 \\ \longrightarrow & W''.\Sigma_1, \alpha := \mathbb{C}_1 \rhd M_1'[X := \alpha]\overline{\langle c_1[X := \alpha]\rangle}\langle \mathit{coerce}_\alpha^+(\rho(A_0)[X := \alpha])\rangle \quad \text{(by (R\_TYBETA\_C))} \\ = & W''.\Sigma_1, \alpha := \mathbb{C}_1 \rhd M_1'''\langle \mathit{coerce}_\alpha^+(\rho(A_0)[X := \alpha])\rangle \end{array}$$

  and

$$\begin{array}{ll} & W''.\Sigma_2 \rhd V_2\,\mathbb{C}_2 \\ = & W''.\Sigma_2 \rhd (\Lambda X.(M_2' : A_2'))\overline{\langle \forall X.c_2\rangle}\,\mathbb{C}_2 \\ \longrightarrow & W''.\Sigma_2, \alpha := \mathbb{C}_2 \rhd M_2'[X := \alpha]\overline{\langle c_2[X := \alpha]\rangle}\langle \mathit{coerce}_\alpha^+(\rho(A_0)[X := \alpha])\rangle \quad \text{(by (R\_TYBETA\_C))} \\ = & W''.\Sigma_2, \alpha := \mathbb{C}_2 \rhd M_2'''\langle \mathit{coerce}_\alpha^+(\rho(A_0)[X := \alpha])\rangle\ . \end{array}$$

  Thus, we have

$$(W''', M_1''', M_2''') \in \blacktriangleright\mathcal{E}\,[\![A_0]\!]\,\rho\{X \mapsto \alpha\}\ .$$

  Because $0 < W'''.n$, we have

$$(\blacktriangleright W''', M_1''', M_2''') \in \mathcal{E}\,[\![A_0]\!]\,\rho\{X \mapsto \alpha\}\ .$$

  Because
  * $\blacktriangleright W''' \sqsupseteq W'''$ by Lemma G.7 (2),
  * $W''' = W'' \boxplus (\alpha, \mathbb{C}_1, \mathbb{C}_2, R) \sqsupseteq W''$ by Lemma G.6,
  * $W'' \sqsupseteq W'$, and
  * $W' \sqsupseteq W$,

  we have $\blacktriangleright W''' \sqsupseteq W$ by Lemma G.5. Then, we have
  * $\Sigma \mid \Gamma, X \Vdash c_0 : A_0 \rightsquigarrow B_0$,
  * $\blacktriangleright W''' \in \mathcal{S}\,[\![\Sigma]\!]$ by Lemma G.13 with $\blacktriangleright W''' \sqsupseteq W$ and $W \in \mathcal{S}\,[\![\Sigma]\!]$.
  * $(\blacktriangleright W''', \theta, \rho\{X \mapsto \alpha\}) \in \mathcal{G}\,[\![\Gamma, X]\!]$ because:
    · $(\blacktriangleright W''', \theta, \rho) \in \mathcal{G}\,[\![\Gamma]\!]$ by Lemma G.12 with $\blacktriangleright W''' \sqsupseteq W$ and $(W, \theta, \rho) \in \mathcal{G}\,[\![\Gamma]\!]$, and
    · $\alpha \in \mathrm{dom}((\blacktriangleright W''').\kappa)$.

Because $(\blacktriangleright W''').n < W'''.n \leq W.n$, the IH implies the conclusion (2)

$$(\blacktriangleright W''', M_1'''\langle \rho\{X \mapsto \alpha\}(c_0)\rangle, M_2'''\langle \rho\{X \mapsto \alpha\}(c_0)\rangle) \in \mathcal{E}\,[\![B_0]\!]\,\rho\{X \mapsto \alpha\} \ .$$

– We show (3). Because
  * $\Sigma \mid \Gamma \Vdash \forall X.c_0 : \forall X.A_0 \rightsquigarrow \forall X.B_0$,
  * $W' \in \mathcal{S}\,[\![\Sigma]\!]$ by Lemma G.13 with $W' \sqsupseteq W$ and $W \in \mathcal{S}\,[\![\Sigma]\!]$, and
  * $(W', \theta, \rho) \in \mathcal{G}\,[\![\Gamma]\!]$ by Lemma G.12 with $W' \sqsupseteq W$ and $(W, \theta, \rho) \in \mathcal{G}\,[\![\Gamma]\!]$,

  we have
  * $W'.\Sigma_1 \mid \emptyset \vdash \forall X.\rho(c_0) : \forall X.\rho(A_0) \rightsquigarrow \forall X.\rho(B_0)$ and
  * $W'.\Sigma_2 \mid \emptyset \vdash \forall X.\rho(c_0) : \forall X.\rho(A_0) \rightsquigarrow \forall X.\rho(B_0)$

  by Lemma G.19. Then, by (Ct_Cons_C),
  * $W'.\Sigma_1 \vdash \overline{\langle \forall X.c_1 \rangle}, \langle \forall X.\rho(c_0)\rangle : \forall X.A_1' \rightsquigarrow \forall X.\rho(B_0)$ and
  * $W'.\Sigma_2 \vdash \overline{\langle \forall X.c_2 \rangle}, \langle \forall X.\rho(c_0)\rangle : \forall X.A_2' \rightsquigarrow \forall X.\rho(B_0)$.

  Because
  * $W'' \sqsupseteq W'$ implies $W'' \in$ World, $W''.\Sigma_1 \supseteq W'.\Sigma_1$ and $W''.\Sigma_2 \supseteq W'.\Sigma_2$, and
  * $W'' \in$ World implies $\vdash W''.\Sigma_1$ and $\vdash W''.\Sigma_2$,

  Lemma E.6 implies
  * $W''.\Sigma_1 \vdash \overline{\langle \forall X.c_1 \rangle}, \langle \forall X.\rho(c_0)\rangle : \forall X.A_1' \rightsquigarrow \forall X.\rho(B_0)$ and
  * $W''.\Sigma_2 \vdash \overline{\langle \forall X.c_2 \rangle}, \langle \forall X.\rho(c_0)\rangle : \forall X.A_2' \rightsquigarrow \forall X.\rho(B_0)$.

  Therefore, noting that α is fresh, we have

$$\begin{array}{ll} & W''.\Sigma_1 \triangleright V_1\langle \rho(\forall X.c_0)\rangle\,\mathbb{C}_1 \\ = & W''.\Sigma_1 \triangleright (\Lambda X.(M_1' : A_1'))\overline{\langle \forall X.c_1\rangle}\langle \forall X.\rho(c_0)\rangle\,\mathbb{C}_1 \\ \longrightarrow & W''.\Sigma_1, \alpha := \mathbb{C}_1 \triangleright (M_1'\overline{\langle c_1\rangle}\langle \rho(c_0)\rangle)[X := \alpha]\langle \mathit{coerce}^+_\alpha(\rho(B_0)[X := \alpha])\rangle \quad \text{(by (R\_Tybeta\_C))} \\ = & W''.\Sigma_1, \alpha := \mathbb{C}_1 \triangleright M_1'''\langle \rho(c_0[X := \alpha])\rangle\langle \mathit{coerce}^+_\alpha(\rho(B_0)[X := \alpha])\rangle \end{array}$$

  and

$$\begin{array}{ll} & W''.\Sigma_2 \triangleright V_2\langle \rho(\forall X.c_0)\rangle\,\mathbb{C}_2 \\ = & W''.\Sigma_2 \triangleright (\Lambda X.(M_2' : A_2'))\overline{\langle \forall X.c_2\rangle}\langle \forall X.\rho(c_0)\rangle\,\mathbb{C}_2 \\ \longrightarrow & W''.\Sigma_2, \alpha := \mathbb{C}_2 \triangleright (M_2'\overline{\langle c_2\rangle}\langle \rho(c_0)\rangle)[X := \alpha]\langle \mathit{coerce}^+_\alpha(\rho(B_0)[X := \alpha])\rangle \quad \text{(by (R\_Tybeta\_C))} \\ = & W''.\Sigma_2, \alpha := \mathbb{C}_2 \triangleright M_2'''\langle \rho(c_0[X := \alpha])\rangle\langle \mathit{coerce}^+_\alpha(\rho(B_0)[X := \alpha])\rangle \ . \end{array}$$

  Because
  * $W''.\Sigma_1 \triangleright V_1\langle \rho(\forall X.c_0)\rangle\,\mathbb{C}_1 \longrightarrow W''.\Sigma_1, \alpha := \mathbb{C}_1 \triangleright M_1''\langle \mathit{coerce}^+_\alpha(\rho(B_0)[X := \alpha])\rangle$ and
  * $W''.\Sigma_2 \triangleright V_2\langle \rho(\forall X.c_0)\rangle\,\mathbb{C}_2 \longrightarrow W''.\Sigma_2, \alpha := \mathbb{C}_2 \triangleright M_2''\langle \mathit{coerce}^+_\alpha(\rho(B_0)[X := \alpha])\rangle$,

  Theorem F.1 implies (3).

- Let $W'' \sqsupseteq W'$. We show that

$$(W'', V_1\langle \rho(\forall X.c_0)\rangle \star, V_2\langle \rho(\forall X.c_0)\rangle \star) \in \mathcal{E}\,[\![B_0]\!]\,\rho\{X \mapsto \star\} \ .$$

  Because $(W', V_1, V_2) \in \mathcal{V}\,[\![A]\!]\,\rho = \mathcal{V}\,[\![\forall X.A_0]\!]\,\rho$, we have
  – $W'.\Sigma_1 \mid \emptyset \vdash V_1 : \forall X.\rho(A_0)$ and
  – $W'.\Sigma_2 \mid \emptyset \vdash V_2 : \forall X.\rho(A_0)$.

  Thus, by Lemma E.3,
  – $V_1 = (\Lambda X.(M_1' : A_1'))\overline{\langle \forall X.c_1\rangle}$,
  – $W'.\Sigma_1 \vdash \overline{\langle \forall X.c_1\rangle} : \forall X.A_1' \rightsquigarrow \forall X.\rho(A_0)$,
  – $V_2 = (\Lambda X.(M_2' : A_2'))\overline{\langle \forall X.c_2\rangle}$, and
  – $W'.\Sigma_2 \vdash \overline{\langle \forall X.c_2\rangle} : \forall X.A_2' \rightsquigarrow \forall X.\rho(A_0)$

for some $M'_1, M'_2, A'_1, A'_2, \overline{\langle c_1 \rangle}, \overline{\langle c_2 \rangle}$. By (R_TybetaDyn_C),

- $W''.\Sigma_1 \triangleright V_1 \langle \rho(\forall X.c_0) \rangle \star = W''.\Sigma_1 \triangleright (\Lambda X.(M'_1 : A'_1))\overline{\langle \forall X.c_1 \rangle}\langle \rho(\forall X.c_0) \rangle \star \longrightarrow W''.\Sigma_1 \triangleright (M'_1 \overline{\langle c_1 \rangle}\langle \rho(c_0) \rangle)[X := \star]$ and
- $W''.\Sigma_2 \triangleright V_2 \langle \rho(\forall X.c_0) \rangle \star = W''.\Sigma_2 \triangleright (\Lambda X.(M'_2 : A'_2))\overline{\langle \forall X.c_2 \rangle}\langle \rho(\forall X.c_0) \rangle \star \longrightarrow W''.\Sigma_2 \triangleright (M'_2 \overline{\langle c_2 \rangle}\langle \rho(c_0) \rangle)[X := \star]$.

Supposing that $1 < W''.n$, we can prove the conclusion by Lemma G.15 with the following.

- $\blacktriangleright W'' \sqsupseteq_1 W''$ by Lemma G.7 (2).
- $(\blacktriangleright W'').\Sigma_1 = W''.\Sigma_1$.
- $(\blacktriangleright W'').\Sigma_2 = W''.\Sigma_2$.

Finally, Lemma G.15 requires us to prove that

$$(\blacktriangleright W'', (M'_1 \overline{\langle c_1 \rangle}\langle \rho(c_0) \rangle)[X := \star], (M'_2 \overline{\langle c_2 \rangle}\langle \rho(c_0) \rangle)[X := \star]) \in \mathcal{E}[\![B_0]\!]\rho\{X \mapsto \star\} .$$

Because $(W', V_1, V_2) \in \mathcal{V}[\![\forall X.A_0]\!]\rho$ and $W'' \sqsupseteq W'$, we have $(W'', V_1 \star, V_2 \star) \in \mathcal{E}[\![A_0]\!]\rho\{X \mapsto \star\}$. Because, by (R_TybetaDyn_C),

- $W''.\Sigma_1 \triangleright V_1 \star \longrightarrow W''.\Sigma_1 \triangleright (M'_1 \overline{\langle c_1 \rangle})[X := \star]$ and
- $W''.\Sigma_2 \triangleright V_2 \star \longrightarrow W''.\Sigma_2 \triangleright (M'_2 \overline{\langle c_2 \rangle})[X := \star]$,

we have

$$(\blacktriangleright W'', (M'_1 \overline{\langle c_1 \rangle})[X := \star], (M'_2 \overline{\langle c_2 \rangle})[X := \star]) \in \mathcal{E}[\![A_0]\!]\rho\{X \mapsto \star\}$$

by Lemma G.20. We have $\blacktriangleright W'' \sqsupseteq W$ by Lemma G.5. Therefore, we have the following:

- $\blacktriangleright W'' \in \mathcal{S}[\![\Sigma]\!]$ by Lemma G.13, and
- $(\blacktriangleright W'', \theta, \rho\{X \mapsto \star\}) \in \mathcal{G}[\![\Gamma, X]\!]$ by Lemma G.12.

Because $\Sigma \mid \Gamma, X \Vdash c_0 : A_0 \rightsquigarrow B_0$ and $(\blacktriangleright W'').n < W.n$, the IH implies the conclusion

$$(\blacktriangleright W'', (M'_1 \overline{\langle c_1 \rangle}\langle \rho(c_0) \rangle)[X := \star], (M'_2 \overline{\langle c_2 \rangle}\langle \rho(c_0) \rangle)[X := \star]) \in \mathcal{E}[\![B_0]\!]\rho\{X \mapsto \star\} .$$

□

Lemma G.22 (Compatibility: Coercion Application). If $\Sigma \mid \Gamma \vdash M_1 \preceq M_2 : A$ and $\Sigma \mid \Gamma \vdash c : A \rightsquigarrow B$, then $\Sigma \mid \Gamma \vdash M_1\langle c \rangle \preceq M_2\langle c \rangle : B$.

Proof. We have $\Sigma \mid \Gamma \vdash M_1\langle c \rangle : B$ and $\Sigma \mid \Gamma \vdash M_2\langle c \rangle : B$ by (T_Crc_C).

Let W, θ, ρ such that $W \in \mathcal{S}[\![\Sigma]\!]$ and $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$. Then, we must show that

$$(W, \rho(\theta^1(M_1\langle c \rangle)), \rho(\theta^2(M_2\langle c \rangle))) \in \mathcal{E}[\![B]\!]\rho .$$

Because $\Sigma \mid \Gamma \vdash M_1 \preceq M_2 : A$, we have $(W, \rho(\theta^1(M_1)), \rho(\theta^2(M_2))) \in \mathcal{E}[\![A]\!]\rho$. Then, because $\Sigma \mid \Gamma \vdash c : A \rightsquigarrow B$ implies $\Sigma \mid \Gamma \Vdash c : A \rightsquigarrow B$, Lemma G.21 implies the conclusion. □

Lemma G.23 (Compatibility: Constants). If $\vdash \Sigma$ and $\Sigma \vdash \Gamma$ and $ty(k) = A$, then $\Sigma \mid \Gamma \vdash k \preceq k : A$.

Proof. By induction on A. Note that $\Sigma \mid \Gamma \vdash k : A$ by (T_Const_C).

Let W, θ, ρ such that $W \in \mathcal{S}[\![\Sigma]\!]$ and $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$. Then, we must show that

$$(W, k, k) \in \mathcal{E}[\![A]\!]\rho .$$

By Lemma G.14, it suffices to show that $(W, k, k) \in \mathcal{V}[\![A]\!]\rho$. We proceed by case analysis on A.

Case $A = \iota$: Obvious.

Case $A = \iota \to A'$: Let $W' \sqsupseteq W$, and V_1 and V_2 be values such that $(W', V_1, V_2) \in \mathcal{V}[\![\iota]\!]\rho$. Then, it suffices to show that $(W', k\, V_1, k\, V_2) \in \mathcal{E}[\![A']\!]\rho$. Because $(W', V_1, V_2) \in \mathcal{V}[\![\iota]\!]\rho$, we have $V_1 = V_2 = k'$ for some k'. Therefore, it suffices to show that $(W', k\, k', k\, k') \in \mathcal{E}[\![A']\!]\rho$. By (R_Delta_C), $W'.\Sigma_1 \triangleright k\, k' \longrightarrow W'.\Sigma_1 \triangleright k''$ and $W'.\Sigma_2 \triangleright k\, k' \longrightarrow W'.\Sigma_2 \triangleright k''$ for some k''. Supposing that $1 < W'.n$, we can prove the conclusion by Lemma G.15 with the following.

- $\blacktriangleright W' \sqsupseteq_1 W'$ by Lemma G.7 (2).
- $(\blacktriangleright W').\Sigma_1 = W'.\Sigma_1$ and $(\blacktriangleright W').\Sigma_2 = W'.\Sigma_2$ by definition,
- We show that $(\blacktriangleright W', k'', k'') \in \mathcal{E}[\![A']\!]\rho$. By the IH, $\Sigma \mid \Gamma \vdash k'' \preceq k'' : A'$. Because $\blacktriangleright W' \sqsupseteq W$ by Lemma G.5, we have $\blacktriangleright W' \in \mathcal{S}[\![\Sigma]\!]$ by Lemma G.13 with $W \in \mathcal{S}[\![\Sigma]\!]$, and $(\blacktriangleright W', \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$ by Lemma G.12 with $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$. Therefore, we have the conclusion.

□

Lemma G.24 (Compatibility: Variables)**.** If $\vdash \Sigma$ and $\Sigma \vdash \Gamma$ and $x : A \in \Gamma$, then $\Sigma \mid \Gamma \vdash x \preceq x : A$.

Proof. By (T_VAR_C), $\Sigma \mid \Gamma \vdash x : A$.

Let W, θ, ρ such that $W \in \mathcal{S}[\![\Sigma]\!]$ and $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$. Then, we must show that

$$(W, \theta^1(x), \theta^2(x)) \in \mathcal{E}[\![A]\!]\rho\ .$$

Because $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$ and $x : A \in \Gamma$, we have the conclusion by Lemma G.17. □

Lemma G.25 (Compatibility: Abstractions)**.** If $\Sigma \mid \Gamma, x : A \vdash M_1 \preceq M_2 : B$, then $\Sigma \mid \Gamma \vdash \lambda x{:}A.M_1 \preceq \lambda x{:}A.M_2 : A \rightarrow B$.

Proof. By (T_ABS_C), $\Sigma \mid \Gamma \vdash \lambda x{:}A.M_1 : A \rightarrow B$ and $\Sigma \mid \Gamma \vdash \lambda x{:}A.M_2 : A \rightarrow B$.

Let W, θ, ρ such that $W \in \mathcal{S}[\![\Sigma]\!]$ and $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$. Without loss of generality, we can suppose that $x \notin \mathrm{dom}(\theta)$. Then, we must show that

$$(W, \rho(\theta^1(\lambda x{:}A.M_1)), \rho(\theta^2(\lambda x{:}A.M_2))) \in \mathcal{E}[\![A \rightarrow B]\!]\rho\ .$$

By Lemma G.14, it suffices to show that

$$(W, \rho(\theta^1(\lambda x{:}A.M_1)), \rho(\theta^2(\lambda x{:}A.M_2))) \in \mathcal{V}[\![A \rightarrow B]\!]\rho\ .$$

Let W', V_1, and V_2 such that $W' \sqsupseteq W$ and $(W', V_1, V_2) \in \mathcal{V}[\![A]\!]\rho$. Then, it suffices to show that

$$(W', \rho(\theta^1(\lambda x{:}A.M_1))\ V_1, \rho(\theta^2(\lambda x{:}A.M_2))\ V_2) \in \mathcal{E}[\![B]\!]\rho\ .$$

By (R_BETA_C),

- $W'.\Sigma_1 \triangleright \rho(\theta^1(\lambda x{:}A.M_1))\ V_1 \longrightarrow W'.\Sigma_1 \triangleright \rho(\theta^1(M_1))[x := V_1]$ and
- $W'.\Sigma_2 \triangleright \rho(\theta^2(\lambda x{:}A.M_2))\ V_2 \longrightarrow W'.\Sigma_2 \triangleright \rho(\theta^2(M_2))[x := V_2]$.

Supposing that $1 < W'.n$, we can prove the conclusion by Lemma G.15 with the following.

- $\blacktriangleright W' \sqsupseteq_1 W'$ by Lemma G.7 (2).
- $(\blacktriangleright W').\Sigma_1 = W'.\Sigma_1$ and $(\blacktriangleright W').\Sigma_2 = W'.\Sigma_2$ by definition,
- We show that
$$(\blacktriangleright W', \rho(\theta^1(M_1))[x := V_1], \rho(\theta^2(M_2))[x := V_2]) \in \mathcal{E}[\![B]\!]\rho\ .$$
By Lemma G.5, $\blacktriangleright W' \sqsupseteq W$. Therefore, we have
  - $\blacktriangleright W' \in \mathcal{S}[\![\Sigma]\!]$ by Lemma G.13 with $W \in \mathcal{S}[\![\Sigma]\!]$ and $\blacktriangleright W' \sqsupseteq W$,
  - $(\blacktriangleright W', \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$ by Lemma G.12 with $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$ and $\blacktriangleright W' \sqsupseteq W$, and
  - $(\blacktriangleright W', V_1, V_2) \in \mathcal{V}[\![A]\!]\rho$ by Lemma G.10 with $(W', V_1, V_2) \in \mathcal{V}[\![A]\!]\rho$ and $\blacktriangleright W' \sqsupseteq W'$.

  The last two imply $(\blacktriangleright W', \theta\{x \mapsto (V_1, V_2)\}, \rho) \in \mathcal{G}[\![\Gamma, x : A]\!]$. Because $\Sigma \mid \Gamma, x : A \vdash M_1 \preceq M_2 : B$, we have the conclusion
$$(\blacktriangleright W', \rho(\theta^1(M_1))[x := V_1], \rho(\theta^2(M_2))[x := V_2]) \in \mathcal{E}[\![B]\!]\rho\ .$$

□

Lemma G.26 (Compatibility: Applications)**.** If $\Sigma \mid \Gamma \vdash M_{11} \preceq M_{21} : A \rightarrow B$ and $\Sigma \mid \Gamma \vdash M_{12} \preceq M_{22} : A$, then $\Sigma \mid \Gamma \vdash M_{11}\, M_{12} \preceq M_{21}\, M_{22} : B$.

Proof. By (T_App_C), we have $\Sigma \mid \Gamma \vdash M_{11}\, M_{12} : B$ and $\Sigma \mid \Gamma \vdash M_{21}\, M_{22} : B$.

Let W, θ, ρ such that $W \in \mathcal{S}[\![\Sigma]\!]$ and $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$. Then, we must show that

$$(W, \rho(\theta^1(M_{11}\, M_{12})), \rho(\theta^2(M_{21}\, M_{22}))) \in \mathcal{E}[\![B]\!]\,\rho\ .$$

Because $\Sigma \mid \Gamma \vdash M_{11} \preceq M_{21} : A \rightarrow B$, we have $(W, \rho(\theta^1(M_{11})), \rho(\theta^2(M_{12}))) \in \mathcal{E}[\![A \rightarrow B]\!]\,\rho$. Let $W' \sqsupseteq W$, and V_{11} and V_{21} be values such that $(W', V_{11}, V_{21}) \in \mathcal{V}[\![A \rightarrow B]\!]\,\rho$. Then, by Lemma G.16, it suffices to show that

$$(W', V_{11}\, \rho(\theta^1(M_{12})), V_{21}\, \rho(\theta^2(M_{22}))) \in \mathcal{E}[\![B]\!]\,\rho\ .$$

Because

- $\Sigma \mid \Gamma \vdash M_{12} \preceq M_{22} : A$,
- $W' \in \mathcal{S}[\![\Sigma]\!]$ by Lemma G.13 with $W \in \mathcal{S}[\![\Sigma]\!]$ and $W' \sqsupseteq W$, and
- $(W', \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$ by Lemma G.12 with $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$ and $W' \sqsupseteq W$,

we have $(W', \rho(\theta^1(M_{12})), \rho(\theta^2(M_{22}))) \in \mathcal{E}[\![A]\!]\,\rho$. Let $W'' \sqsupseteq W'$, and V_{12} and V_{22} be values such that $(W'', V_{12}, V_{22}) \in \mathcal{V}[\![A]\!]\,\rho$. Then, by Lemma G.16, it suffices to show that

$$(W'', V_{11}\, V_{12}, V_{21}\, V_{22}) \in \mathcal{E}[\![B]\!]\,\rho\ .$$

Because $(W', V_{11}, V_{21}) \in \mathcal{V}[\![A \rightarrow B]\!]\,\rho$ and $W'' \sqsupseteq W'$ and $(W'', V_{12}, V_{22}) \in \mathcal{V}[\![A]\!]\,\rho$, we have the conclusion. □

Lemma G.27 (Compatibility: Type Abstractions)**.** If $\Sigma \mid \Gamma, X \vdash M_1 \preceq M_2 : A$, then $\Sigma \mid \Gamma \vdash \Lambda X.(M_1 : A) \preceq \Lambda X.(M_2 : A) : \forall X.A$.

Proof. By (T_TyAbs_C), $\Sigma \mid \Gamma \vdash \Lambda X.(M_1 : A) : \forall X.A$ and $\Sigma \mid \Gamma \vdash \Lambda X.(M_2 : A) : \forall X.A$.

Let W, θ, ρ such that $W \in \mathcal{S}[\![\Sigma]\!]$ and $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$. Without loss of generality, we can suppose that $X \notin \mathrm{dom}(\rho)$. Then, we must show that

$$(W, \rho(\theta^1(\Lambda X.(M_1 : A))), \rho(\theta^2(\Lambda X.(M_2 : A)))) \in \mathcal{E}[\![\forall X.A]\!]\,\rho\ .$$

By Lemma G.14, it suffices to show that

$$(W, \rho(\theta^1(\Lambda X.(M_1 : A))), \rho(\theta^2(\Lambda X.(M_2 : A)))) \in \mathcal{V}[\![\forall X.A]\!]\,\rho\ .$$

This is implied by the following two cases.

- Let W', $\mathbb{B}_1$, $\mathbb{B}_2$, R, M'_1, M'_2, and α such that
  - $W' \sqsupseteq W$,
  - $W'.\Sigma_1 \mid \emptyset \vdash \mathbb{B}_1$,
  - $W'.\Sigma_2 \mid \emptyset \vdash \mathbb{B}_2$,
  - $R \in \mathrm{Rel}_{W'.n}[\![\mathbb{B}_1, \mathbb{B}_2]\!]$,
  - $W'.\Sigma_1 \triangleright \rho(\theta^1(\Lambda X.(M_1 : A)))\, \mathbb{B}_1 \longrightarrow W'.\Sigma_1, \alpha := \mathbb{B}_1 \triangleright M'_1\langle \mathit{coerce}^{+}_{\alpha}(\rho(A)[X := \alpha])\rangle$, and
  - $W'.\Sigma_2 \triangleright \rho(\theta^2(\Lambda X.(M_2 : A)))\, \mathbb{B}_2 \longrightarrow W'.\Sigma_2, \alpha := \mathbb{B}_2 \triangleright M'_2\langle \mathit{coerce}^{+}_{\alpha}(\rho(A)[X := \alpha])\rangle$.

  Let $W'' = W' \boxplus (\alpha, \mathbb{B}_1, \mathbb{B}_2, R)$. Then, it suffices to show that

  $$(W'', M'_1, M'_2) \in \blacktriangleright\mathcal{E}[\![A]\!]\,\rho\{X \mapsto \alpha\}\ .$$

  By (R_Tybeta_C) and Theorem F.1, $M'_1 = \rho(\theta^1(M_1))[X := \alpha]$ and $M'_2 = \rho(\theta^2(M_2))[X := \alpha]$. Therefore, it suffices to show that

  $$(W'', \rho(\theta^1(M_1))[X := \alpha], \rho(\theta^2(M_2))[X := \alpha]) \in \blacktriangleright\mathcal{E}[\![A]\!]\,\rho\{X \mapsto \alpha\}\ .$$

  Suppose that $0 < W''.n$. Then, it suffices to show that

  $$(\blacktriangleright W'', \rho(\theta^1(M_1))[X := \alpha], \rho(\theta^2(M_2))[X := \alpha]) \in \mathcal{E}[\![A]\!]\,\rho\{X \mapsto \alpha\}\ .$$

  Because $\blacktriangleright W'' \sqsupseteq W$ by Lemmas G.6, G.7 (2), and G.5, we have

– $\blacktriangleright W'' \in \mathcal{S}[\![\Sigma]\!]$ by Lemma G.13 with $W \in \mathcal{S}[\![\Sigma]\!]$ and $\blacktriangleright W'' \sqsupseteq W$ and
– $(\blacktriangleright W'', \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$ by Lemma G.12 with $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$ and $\blacktriangleright W'' \sqsupseteq W$.

The last one implies $(\blacktriangleright W'', \theta, \rho\{X \mapsto \alpha\}) \in \mathcal{G}[\![\Gamma, X]\!]$. Because $\Sigma \mid \Gamma, X \vdash M_1 \preceq M_2 : A$, we have the conclusion.

- Let $W' \sqsupseteq W$. We show that

$$(W', \rho(\theta^1(\Lambda X.(M_1 : A)))\,\star, \rho(\theta^2(\Lambda X.(M_2 : A)))\,\star) \in \mathcal{E}[\![A]\!]\,\rho\{X \mapsto \star\} .$$

By (R_TybetaDyn_C),

– $W'.\Sigma_1 \triangleright \rho(\theta^1(\Lambda X.(M_1 : A)))\,\star \longrightarrow W'.\Sigma_1 \triangleright \rho(\theta^1(M_1))[X := \star]$ and
– $W'.\Sigma_2 \triangleright \rho(\theta^2(\Lambda X.(M_2 : A)))\,\star \longrightarrow W'.\Sigma_2 \triangleright \rho(\theta^2(M_2))[X := \star]$.

Supposing that $1 < W'.n$, we can prove the conclusion by Lemma G.15 with the following.

– $\blacktriangleright W' \sqsupseteq_1 W'$ by Lemma G.7 (2).
– $(\blacktriangleright W').\Sigma_1 = W'.\Sigma_1$ and $(\blacktriangleright W').\Sigma_2 = W'.\Sigma_2$ by definition.
– We show that

$$(\blacktriangleright W', \rho(\theta^1(M_1))[X := \star], \rho(\theta^2(M_2))[X := \star]) \in \mathcal{E}[\![A]\!]\,\rho\{X \mapsto \star\} .$$

By Lemma G.5, $\blacktriangleright W' \sqsupseteq W$. Therefore, we have
 ∗ $\blacktriangleright W' \in \mathcal{S}[\![\Sigma]\!]$ by Lemma G.13 with $W \in \mathcal{S}[\![\Sigma]\!]$ and $\blacktriangleright W' \sqsupseteq W$, and
 ∗ $(\blacktriangleright W', \theta, \rho\{X \mapsto \star\}) \in \mathcal{G}[\![\Gamma, X]\!]$ by Lemma G.12 with $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$ and $\blacktriangleright W' \sqsupseteq W$.

Because $\Sigma \mid \Gamma, X \vdash M_1 \preceq M_2 : A$, we have the conclusion.

□

Lemma G.28 (Loosely Typing Sealing and Unsealing)**.** Assume that $\alpha := \mathbb{B} \in \Sigma$ and $\Sigma \vdash \Gamma_1, X, \Gamma_2$ and $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash A$ and α does not occur in A. Then, the following holds:

- $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \Vdash \mathit{coerce}^+_\alpha(A[X := \alpha]) : A[X := \alpha] \rightsquigarrow A[X := \mathbb{B}]$ and
- $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \Vdash \mathit{coerce}^-_\alpha(A[X := \alpha]) : A[X := \mathbb{B}] \rightsquigarrow A[X := \alpha]$.

Proof. The proof is almost the same as that of Lemma E.17. We proceed by induction on A. Note that $\Sigma \mid \Gamma_1, \Gamma_2[X := \alpha] \vdash A[X := \alpha]$ and $\Sigma \vdash \Gamma_1, \Gamma_2[X := \alpha]$ by Lemma D.5.

Case $A = \iota$: By (Ct0_Id_C).

Case $A = X$: By (Ct0_Reveal_C) and (Ct0_Conceal_C).

Case $A = Y \wedge Y \neq X$: By (Ct0_Id_C)

Case $A = \beta$: Because α does not occur in A, $\beta \neq \alpha$. Then, we have the conclusion by (Ct0_Id_C).

Case $A = \star$: By (Ct0_Id_C).

Case $A = C_1 \to C_2$: By the IHs and (Ct0_Arrow_C).

Case $A = \forall X.C$: By the IH and (Ct0_All_C).

□

Lemma G.29 (Compatibility: Type Applications)**.** If $\Sigma \mid \Gamma \vdash M_1 \preceq M_2 : \forall X.B$ and $\Sigma \mid \Gamma \vdash A$, then $\Sigma \mid \Gamma \vdash M_1\,A \preceq M_2\,A : B[X := A]$.

Proof. By (T_TYAPP_C), $\Sigma \mid \Gamma \vdash M_1\, A : B[X := A]$ and $\Sigma \mid \Gamma \vdash M_2\, A : B[X := A]$.

Let W, θ, ρ such that $W \in \mathcal{S}[\![\Sigma]\!]$ and $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$. Without loss of generality, we can suppose that $X \notin \mathrm{dom}(\rho)$. Then, we must show that

$$(W, \rho(\theta^1(M_1\, A)), \rho(\theta^2(M_2\, A))) \in \mathcal{E}[\![B[X := A]]\!]\,\rho\ .$$

Because $\Sigma \mid \Gamma \vdash M_1 \preceq M_2 : \forall X.B$, we have $(W, \rho(\theta^1(M_1)), \rho(\theta^2(M_2))) \in \mathcal{E}[\![\forall X.B]\!]\,\rho$. Let $W' \sqsupseteq W$, and V_1 and V_2 be values such that $(W', V_1, V_2) \in \mathcal{V}[\![\forall X.B]\!]\,\rho$. Then, by Lemma G.16, it suffices to show that

$$(W', V_1\,\rho(A), V_2\,\rho(A)) \in \mathcal{E}[\![B[X := A]]\!]\,\rho\ .$$

Because $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$ and $\Sigma \mid \Gamma \vdash A$, we have $\rho(\rho(A)) = \rho(A)$. Therefore, by Lemma G.17 (2),

$$\mathcal{E}[\![B[X := A]]\!]\,\rho = \mathcal{E}[\![\rho(B)[X := \rho(A)]]\!]\,\emptyset = \mathcal{E}[\![\rho(B)[X := \rho(\rho(A))]]\!]\,\emptyset = \mathcal{E}[\![B[X := \rho(A)]]\!]\,\rho\ .$$

Hence, it suffices to show that

$$(W', V_1\,\rho(A), V_2\,\rho(A)) \in \mathcal{E}[\![B[X := \rho(A)]]\!]\,\rho\ .$$

By Lemma E.3, there exist some M_1', M_2', C_1, C_2, $\overline{\langle c_1 \rangle}$, and $\overline{\langle c_2 \rangle}$ such that

- $V_1 = (\Lambda X.(M_1' : C_1))\overline{\langle \forall X.c_1 \rangle}$ and $W'.\Sigma_1 \vdash \overline{\langle \forall X.c_1 \rangle} : \forall X.C_1 \rightsquigarrow \forall X.\rho(B)$, and
- $V_2 = (\Lambda X.(M_2' : C_2))\overline{\langle \forall X.c_2 \rangle}$ and $W'.\Sigma_2 \vdash \overline{\langle \forall X.c_2 \rangle} : \forall X.C_2 \rightsquigarrow \forall X.\rho(B)$.

By case analysis on $\rho(A)$.

Case $\rho(A) = \mathbb{A}_0$: Let $\alpha \notin \mathrm{dom}(W'.\Sigma_1) \cup \mathrm{dom}(W'.\Sigma_2)$. By (R_TYBETA_C),

- $W'.\Sigma_1 \triangleright V_1\,\rho(A) \longrightarrow W'.\Sigma_1, \alpha := \mathbb{A}_0 \triangleright M_1''\langle \mathit{coerce}^+_\alpha(\rho(B)[X := \alpha])\rangle$ and
- $W'.\Sigma_2 \triangleright V_2\,\rho(A) \longrightarrow W'.\Sigma_2, \alpha := \mathbb{A}_0 \triangleright M_2''\langle \mathit{coerce}^+_\alpha(\rho(B)[X := \alpha])\rangle$

where

- $M_1'' = M_1'[X := \alpha]\overline{\langle c_1[X := \alpha] \rangle}$ and
- $M_2'' = M_2'[X := \alpha]\overline{\langle c_2[X := \alpha] \rangle}$.

Suppose that $1 < W'.n$. By Lemma G.15, it suffices to show that there exists some W'' such that

- $W''.\Sigma_1 = W'.\Sigma_1, \alpha := \mathbb{A}_0$,
- $W''.\Sigma_2 = W'.\Sigma_2, \alpha := \mathbb{A}_0$,
- $W'' \sqsupseteq_1 W'$, and
- $(W'', M_1''\langle \mathit{coerce}^+_\alpha(\rho(B)[X := \alpha])\rangle, M_2''\langle \mathit{coerce}^+_\alpha(\rho(B)[X := \alpha])\rangle) \in \mathcal{E}[\![B[X := \mathbb{A}_0]]\!]\,\rho$.

Let $W_0'' = W' \boxplus (\alpha, \mathbb{A}_0, \mathbb{A}_0, \lfloor \mathcal{V}[\![\mathbb{A}_0]\!]\,\emptyset \rfloor_{W'.n})$, and $W'' = \blacktriangleright W_0''$. Because

- $W' \in \mathrm{World}$,
- $W'.\Sigma_1 \mid \emptyset \vdash \mathbb{A}_0$ and $W'.\Sigma_2 \mid \emptyset \vdash \mathbb{A}_0$ by Lemmas D.2 (1) and D.5 (1), and
- $\lfloor \mathcal{V}[\![\mathbb{A}_0]\!]\,\emptyset \rfloor_{W'.n} \in \mathrm{Rel}_{W'.n}[\![\mathbb{A}_0, \mathbb{A}_0]\!]$ by Lemma G.10,

we have $W_0'' \sqsupseteq_0 W'$ by Lemma G.6. Because $W'' \sqsupseteq_1 W_0''$ by Lemma G.7 (2), we have $W'' \sqsupseteq_1 W'$ by Lemma G.5.

Now, it suffices to show that

$$(W'', M_1''\langle \mathit{coerce}^+_\alpha(\rho(B)[X := \alpha])\rangle, M_2''\langle \mathit{coerce}^+_\alpha(\rho(B)[X := \alpha])\rangle) \in \mathcal{E}[\![B[X := \mathbb{A}_0]]\!]\,\rho\ .$$

By Lemma G.21, it suffices to show the following.

- We show that

$$coerce^+_\alpha(\rho(B)[X := \alpha]) = \rho(coerce^+_\alpha(B[X := \alpha])) \ .$$

It suffices to show that, for any $Y \in \mathrm{dom}(\rho)$, $\alpha \neq \rho(Y)$. Suppose that there exists some $Y \in \mathrm{dom}(\rho)$ such that $\alpha = \rho(Y)$. Because $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$, we have $(W', \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$ by Lemma G.12 with $W' \sqsupseteq W$. Then, because $Y \in \mathrm{dom}(\rho)$ and $\rho(Y) = \alpha$, we have $\alpha \in \mathrm{dom}(W'.\kappa)$. Because $W' \in$ World, it implies $\alpha \in \mathrm{dom}(W'.\Sigma_1) \cap \mathrm{dom}(W'.\Sigma_2)$. However, it is contradictory with the assumption $\alpha \notin \mathrm{dom}(W'.\Sigma_1) \cup \mathrm{dom}(W'.\Sigma_2)$. Therefore, $\forall\, Y \in \mathrm{dom}(\rho).\ \alpha \neq \rho(Y)$.

- We show that

$$\Sigma, \alpha := \mathbb{A}_0 \mid \Gamma \Vdash coerce^+_\alpha(B[X := \alpha]) : B[X := \alpha] \rightsquigarrow B[X := \mathbb{A}_0] \ .$$

Because $\Sigma \mid \Gamma \vdash M_1 \preceq M_2 : \forall X.B$, we have $\Sigma \mid \Gamma \vdash \forall X.B$ and $\Sigma \vdash \Gamma$ by Lemma E.10. By Lemma D.2, $\Sigma, \alpha := \mathbb{A}_0 \mid \Gamma \vdash \forall X.B$ and $\Sigma, \alpha := \mathbb{A}_0 \vdash \Gamma$. By inversion of the first, $\Sigma, \alpha := \mathbb{A}_0 \mid \Gamma, X \vdash B$. By (TEW_TYVAR), $\Sigma, \alpha := \mathbb{A}_0 \vdash \Gamma, X$. Therefore, by Lemma G.28, we have the conclusion.

- We show that

$$W'' \in \mathcal{S}[\![\Sigma, \alpha := \mathbb{A}_0]\!] \ .$$

Because $W'' \sqsupseteq W$ by Lemma G.5 with $W'' \sqsupseteq W'$ and $W' \sqsupseteq W$, we have $W'' \in \mathcal{S}[\![\Sigma]\!]$ by Lemma G.13 with $W \in \mathcal{S}[\![\Sigma]\!]$. By Lemma G.2, we have the conclusion.

- We show that

$$(W'', \theta, \rho) \in \mathcal{G}[\![\Gamma]\!] \ ,$$

which is proven by Lemma G.12 with $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$ and $W'' \sqsupseteq W$.

- We show that

$$(W'', M''_1, M''_2) \in \mathcal{E}[\![B[X := \alpha]]\!]\,\rho \ .$$

Because $W'' = \blacktriangleright W''_0$, it suffices to show that

$$(W''_0, M''_1, M''_2) \in \blacktriangleright\mathcal{E}[\![B[X := \alpha]]\!]\,\rho \ .$$

Because

  - $(W', V_1, V_2) \in \mathcal{V}[\![\forall X.B]\!]\,\rho$,
  - $W' \sqsupseteq W'$ by Lemma G.5,
  - $W'.\Sigma_1 \mid \emptyset \vdash \mathbb{A}_0$ and $W'.\Sigma_2 \mid \emptyset \vdash \mathbb{A}_0$,
  - $\lfloor \mathcal{V}[\![\mathbb{A}_0]\!]\,\emptyset \rfloor_{W'.n} \in \mathrm{Rel}_{W'.n}[\![\mathbb{A}_0, \mathbb{A}_0]\!]$,

we have

$$(W''_0, M''_1, M''_2) \in \blacktriangleright\mathcal{E}[\![B]\!]\,\rho\{X \mapsto \alpha\} \ .$$

By Lemma G.17 (2), we have the conclusion.

Case $\rho(A) = \star$: By Lemma G.17 (2), it suffices to show that

$$(W', V_1\,\star, V_2\,\star) \in \mathcal{E}[\![B]\!]\,\rho\{X \mapsto \star\} \ ,$$

which is implied by $(W', V_1, V_2) \in \mathcal{V}[\![\forall X.B]\!]\,\rho$ and $W' \sqsupseteq W'$ obtained by Lemma G.5.

□

Lemma G.30 (Compatibility: Blame). If $\vdash \Sigma$ and $\Sigma \vdash \Gamma$ and $\Sigma \mid \Gamma \vdash A$, then $\Sigma \mid \Gamma \vdash \mathsf{blame}\, p \preceq \mathsf{blame}\, p : A$.

Proof. By (T_BLAME_C), $\Sigma \mid \Gamma \vdash \mathsf{blame}\, p : A$.

Let W, θ, ρ such that $W \in \mathcal{S}[\![\Sigma]\!]$ and $(W, \theta, \rho) \in \mathcal{G}[\![\Gamma]\!]$. Then, we must show that

$$(W, \mathsf{blame}\, p, \mathsf{blame}\, p) \in \mathcal{E}[\![A]\!]\,\rho \ ,$$

which holds by definition. □

[TS: Is the title of the next theorem right?] ⟵

Theorem G.31 (Fundamental Property (Theorem 3.3 of the paper))**.** If $\Sigma \mid \Gamma \vdash M : A$, then $\Sigma \mid \Gamma \vdash M \approx M : A$.

Proof. It suffices to show that $\Sigma \mid \Gamma \vdash M \preceq M : A$, which is proven by induction on the typing derivation with the compatibility lemmas (Lemmas G.23, G.24, G.25, G.26, G.27, G.29, G.30, and G.22). □

Lemma G.32 (Congruence of the Logical Relation)**.** If $\Sigma \vdash_C \mathcal{C}_C : (\Gamma_1 \vdash A_1) \Rightarrow (\Gamma_2 \vdash A_2)$ and $\Sigma \mid \Gamma_1 \vdash M_1 \approx M_2 : A_1$, then $\Sigma \mid \Gamma_2 \vdash \mathcal{C}_C[M_1] \approx \mathcal{C}_C[M_2] : A_2$.

Proof. By induction on the derivation of $\Sigma \vdash_C \mathcal{C}_C : (\Gamma_1 \vdash A_1) \Rightarrow (\Gamma_2 \vdash A_2)$.

Case (Ctxt_Hole_C): Obvious.

Case (Ctxt_Abs_C): We are given

$$\mathcal{C}_C = \lambda x\!:\!A_{21}.\mathcal{C}'_C, \quad A_2 = A_{21} \rightarrow A_{22}, \quad \Sigma \vdash_C \mathcal{C}'_C : (\Gamma_1 \vdash A_1) \Rightarrow (\Gamma_2, x : A_{21} \vdash A_{22})\ (\exists x, A_{21}, A_{22}, \mathcal{C}'_C)\ .$$

By the IH, $\Sigma \mid \Gamma_2, x : A_{21} \vdash \mathcal{C}'_C[M_1] \approx \mathcal{C}'_C[M_2] : A_{22}$. By Lemma G.25, $\Sigma \mid \Gamma_2 \vdash \lambda x\!:\!A_{21}.\mathcal{C}'_C[M_1] \approx \lambda x\!:\!A_{21}.\mathcal{C}'_C[M_2] : A_{21} \rightarrow A_{22}$, which is what we have to prove.

Case (Ctxt_App1_C): We are given

$$\mathcal{C}_C = \mathcal{C}'_C\, M, \quad \Sigma \vdash_C \mathcal{C}'_C : (\Gamma_1 \vdash A_1) \Rightarrow (\Gamma_2 \vdash B \rightarrow A_2), \quad \Sigma \mid \Gamma_2 \vdash M : B\ (\exists \mathcal{C}'_C, M, B)\ .$$

By the IH, $\Sigma \mid \Gamma_2 \vdash \mathcal{C}'_C[M_1] \approx \mathcal{C}'_C[M_2] : B \rightarrow A_2$. By Theorem G.31, $\Sigma \mid \Gamma_2 \vdash M \approx M : B$. By Lemma G.26, $\Sigma \mid \Gamma_2 \vdash \mathcal{C}'_C[M_1]\, M \approx \mathcal{C}'_C[M_2]\, M : A_2$, which is what we have to prove.

Case (Ctxt_App2_C): We are given

$$\mathcal{C}_C = M\, \mathcal{C}'_C, \quad \Sigma \mid \Gamma_2 \vdash M : B \rightarrow A_2, \quad \Sigma \vdash_C \mathcal{C}'_C : (\Gamma_1 \vdash A_1) \Rightarrow (\Gamma_2 \vdash B)\ (\exists \mathcal{C}'_C, M, B)\ .$$

By the IH, $\Sigma \mid \Gamma_2 \vdash \mathcal{C}'_C[M_1] \approx \mathcal{C}'_C[M_2] : B$. By Theorem G.31, $\Sigma \mid \Gamma_2 \vdash M \approx M : B \rightarrow A_2$. By Lemma G.26, $\Sigma \mid \Gamma_2 \vdash M\, \mathcal{C}'_C[M_1] \approx M\, \mathcal{C}'_C[M_2] : A_2$, which is what we have to prove.

Case (Ctxt_Tyabs_C): We are given

$$\mathcal{C}_C = \Lambda X.(\mathcal{C}'_C : A'_2), \quad \Sigma \vdash_C \mathcal{C}'_C : (\Gamma_1 \vdash A_1) \Rightarrow (\Gamma_2, X \vdash A'_2), \quad A_2 = \forall X.A'_2\ (\exists X, \mathcal{C}'_C, A'_2)\ .$$

By the IH, $\Sigma \mid \Gamma_2, X \vdash \mathcal{C}'_C[M_1] \approx \mathcal{C}'_C[M_2] : A'_2$. By Lemma G.27, $\Sigma \mid \Gamma_2 \vdash \Lambda X.(\mathcal{C}'_C[M_1] : A'_2) \approx \Lambda X.(\mathcal{C}'_C[M_2] : A'_2) : \forall X.A'_2$, which is what we have to prove.

Case (Ctxt_Tyapp_C): We are given

$$\mathcal{C}_C = \mathcal{C}'_C\, B, \quad \Sigma \vdash_C \mathcal{C}'_C : (\Gamma_1 \vdash A_1) \Rightarrow (\Gamma_2 \vdash \forall X.C), \quad \Sigma \mid \Gamma_2 \vdash B, \quad A_2 = C[X := B]\ (\exists \mathcal{C}'_C, X, B, C)\ .$$

By the IH, $\Sigma \mid \Gamma_2 \vdash \mathcal{C}'_C[M_1] \approx \mathcal{C}'_C[M_2] : \forall X.C$. By Lemma G.29, $\Sigma \mid \Gamma_2 \vdash \mathcal{C}'_C[M_1]\, B \approx \mathcal{C}'_C[M_2]\, B : C[X := B]$, which is what we have to prove.

Case (Ctxt_Crc_C): We are given

$$\mathcal{C}_C = \mathcal{C}'_C\, \langle c \rangle, \quad \Sigma \vdash_C \mathcal{C}'_C : (\Gamma_1 \vdash A_1) \Rightarrow (\Gamma_2 \vdash B), \quad \Sigma \mid \Gamma_2 \vdash c : B \rightsquigarrow A_2\ (\exists \mathcal{C}'_C, c, B)\ .$$

By the IH, $\Sigma \mid \Gamma_2 \vdash \mathcal{C}'_C[M_1] \approx \mathcal{C}'_C[M_2] : B$. By Lemma G.22, $\Sigma \mid \Gamma_2 \vdash \mathcal{C}'_C[M_1]\langle c \rangle \approx \mathcal{C}'_C[M_2]\langle c \rangle : A_2$, which is what we have to prove.

□

Lemma G.33 (Adequacy of the Logical Relation)**.** Assume that $\Sigma \mid \emptyset \vdash M_1 \approx M_2 : A$.

1. $\Sigma \triangleright M_1 \longrightarrow \Sigma_1 \triangleright V_1$ if and only if $\Sigma \triangleright M_2 \longrightarrow \Sigma_2 \triangleright V_2$.

2. $\Sigma \triangleright M_1 \longrightarrow \Sigma_1 \triangleright \mathsf{blame}\, p$ if and only if $\Sigma \triangleright M_2 \longrightarrow \Sigma_2 \triangleright \mathsf{blame}\, p$.

3. $\Sigma \triangleright M_1 \Uparrow$ if and only if $\Sigma \triangleright M_2 \Uparrow$.

Proof. Let W be a tuple $(0, \Sigma, \Sigma, \{\alpha \in \mathrm{dom}(\Sigma) \mapsto \lfloor \mathcal{V}[\![\Sigma(\alpha)]\!]\,\emptyset \rfloor_0\})$. We first show that $W \in \mathrm{World}_1 \subseteq \mathrm{World}$. Because $0 < 1$ and $\vdash \Sigma$ by Lemma E.10, it suffices to show that, for any $\alpha \in \mathrm{dom}(\Sigma)$, $\lfloor \mathcal{V}[\![\Sigma(\alpha)]\!]\,\emptyset \rfloor_0 \in \mathrm{Rel}_0[\![\Sigma(\alpha), \Sigma(\alpha)]\!]$. By definition, $\lfloor \mathcal{V}[\![\Sigma(\alpha)]\!]\,\emptyset \rfloor_0 = \emptyset \in \mathrm{Rel}_0[\![\Sigma(\alpha), \Sigma(\alpha)]\!]$. Therefore, $W \in \mathrm{World}$.

Then, we have $W \in \mathcal{S}[\![\Sigma]\!]$ and $(W, \emptyset, \emptyset) \in \mathcal{G}[\![\emptyset]\!]$ by definition. Because $\Sigma \mid \emptyset \vdash M_1 \approx M_2 : A$, we have $(W, M_1, M_2) \in \mathcal{E}[\![A]\!]\,\emptyset$ and $(W, M_2, M_1) \in \mathcal{E}[\![A]\!]\,\emptyset$. By them, Theorem F.1, and Theorem E.21, we have the conclusion. □

[TS: Is the title of the next theorem right?] ⟵

Theorem G.34 (Soundness w.r.t. Contextual Equivalence (Theorem 3.4 of the paper))**.** If $\Sigma \mid \Gamma \vdash M_1 \approx M_2 : A$, then $\Sigma \mid \Gamma \vdash M_1 \overset{\mathrm{ctx}}{=} M_2 : A$.

Proof. By Lemmas G.32 and G.33 and Theorem E.21. □

Definition G.2 (Identity Coercion Generation)**.** Given a type A, we define coercion $coerce(A)$ as follows:

$$
\begin{array}{rcl}
coerce(\iota) & \overset{\mathrm{def}}{=} & \mathsf{id}_\iota \\
coerce(\star) & \overset{\mathrm{def}}{=} & \mathsf{id}_\star \\
coerce(A \to B) & \overset{\mathrm{def}}{=} & coerce(A) \to coerce(B) \\
coerce(\forall X.A) & \overset{\mathrm{def}}{=} & \forall X.coerce(A) \\
coerce(X) & \overset{\mathrm{def}}{=} & \mathsf{id}_X \\
coerce(\alpha) & \overset{\mathrm{def}}{=} & \mathsf{id}_\alpha\ .
\end{array}
$$

Lemma G.35 (Identity Coercion Generation)**.**

- If α does not occur in type A, then $coerce^{\pm}_{\alpha}(A) = coerce(A)$.
- For any V and A, there exist some n and V' such that $\Sigma \triangleright V\langle coerce(A)\rangle \longrightarrow^n \Sigma \triangleright V'$ for any Σ.
- If $\vdash \Sigma$ and $\Sigma \vdash \Gamma$ and $\Sigma \mid \Gamma \vdash A$, then $\Sigma \mid \Gamma \vdash coerce(A) : A \rightsquigarrow A$.

Proof.

- By induction on A.
- By case analysis on A.
- By induction on A.

□

Lemma G.36 (Commutativity of Coercion Generation with Type Name Substitution)**.**

- For any A, X, and α, $coerce(A)[X := \alpha] = coerce(A[X := \alpha])$.
- For any A and X, $coerce(A)[X := \star] = coerce(A[X := \star])$.

Proof. Straightforward by induction on A. □

Lemma G.37 (Identity Coercion Produces Logically Related Values)**.** Assume that $(W, V_1, V_2) \in \mathcal{V}[\![A]\!]\,\rho$.

1. If $W.\Sigma_1 \triangleright V_1\langle coerce(\rho(A))\rangle \longrightarrow^* W.\Sigma_1 \triangleright V_1'$, then $(W, V_1', V_2) \in \mathcal{V}[\![A]\!]\,\rho$.
2. If $W.\Sigma_2 \triangleright V_2\langle coerce(\rho(A))\rangle \longrightarrow^* W.\Sigma_2 \triangleright V_2'$, then $(W, V_1, V_2') \in \mathcal{V}[\![A]\!]\,\rho$.

Proof. By induction on the size of A. Let $(W, V_1, V_2) \in \mathcal{V}[\![A]\!]\,\rho$.

1. Let V_1' be a value such that $W.\Sigma_1 \triangleright V_1\langle coerce(\rho(A))\rangle \longrightarrow^* W.\Sigma_1 \triangleright V_1'$. By case analysis on A.

Case $A = \iota\,(\exists\iota)$, $\star$, $\alpha\,(\exists\alpha)$, or $X\,(\exists X)$: Obvious by the assumption because $V_1 = V_1'$.

Case $\exists\, A_1, A_2.\ A = A_1 \to A_2$: We have $V_1' = V_1\langle \mathit{coerce}(\rho(A_1)) \to \mathit{coerce}(\rho(A_2))\rangle$ and $n = 0$. We have to show that

$$(W,\, V_1\langle \mathit{coerce}(\rho(A_1)) \to \mathit{coerce}(\rho(A_2))\rangle,\, V_2) \in \mathcal{V}[\![A_1 \to A_2]\!]\,\rho\ .$$

Let $W' \sqsupseteq W$ and V_{01}' and V_{02}' be values such that $(W',\, V_{01}',\, V_{02}') \in \mathcal{V}[\![A_1]\!]\,\rho$. It suffices to show that

$$(W',\, V_1\langle \mathit{coerce}(\rho(A_1)) \to \mathit{coerce}(\rho(A_2))\rangle\ V_{01}',\, V_2\ V_{02}') \in \mathcal{E}[\![A_2]\!]\,\rho\ .$$

By (R_Wrap_C),

$$W'.\Sigma_1 \rhd V_1\langle \mathit{coerce}(\rho(A_1)) \to \mathit{coerce}(\rho(A_2))\rangle\ V_{01}' \longrightarrow W'.\Sigma_1 \rhd (V_1\,(V_{01}'\langle \mathit{coerce}(\rho(A_1))\rangle))\langle \mathit{coerce}(\rho(A_2))\rangle\ .$$

By Lemma G.35, $W'.\Sigma_1 \rhd V_{01}'\langle \mathit{coerce}(\rho(A_1))\rangle \longrightarrow^n W'.\Sigma_1 \rhd V_1''$ for some n and V_1''. Therefore, by (R_Ctx_C),

$$W'.\Sigma_1 \rhd V_1\langle \mathit{coerce}(\rho(A_1)) \to \mathit{coerce}(\rho(A_2))\rangle\ V_{01}' \longrightarrow^{n+1} W'.\Sigma_1 \rhd (V_1\ V_1'')\langle \mathit{coerce}(\rho(A_2))\rangle\ .$$

Assume that $n+1 < W'.n$. Let $W'' = (W'.n - (n+1),\, W'.\Sigma_1,\, W'.\Sigma_2,\, \lfloor W'.\kappa \rfloor_{W'.n-(n+1)})$. We have $W'' \sqsupseteq_{n+1} W'$ by Lemma G.7 (2). Therefore, by Lemma G.15, it suffices to show that

$$(W'',\, (V_1\ V_1'')\langle \mathit{coerce}(\rho(A_2))\rangle,\, V_2\ V_{02}') \in \mathcal{E}[\![A_2]\!]\,\rho\ .$$

Because $(W',\, V_{01}',\, V_{02}') \in \mathcal{V}[\![A_1]\!]\,\rho$, we have $(W',\, V_1'',\, V_{02}') \in \mathcal{V}[\![A_1]\!]\,\rho$ by the IH. By Lemma G.10 with $W'' \sqsupseteq W'$, we have $(W'',\, V_1'',\, V_{02}')$. By Lemma G.5 with $W'' \sqsupseteq W'$ and $W' \sqsupseteq W$, we have $W'' \sqsupseteq W$. Therefore, by $(W,\, V_1,\, V_2) \in \mathcal{V}[\![A_1 \to A_2]\!]\,\rho$, we have $(W'',\, V_1\ V_1'',\, V_2\ V_{02}') \in \mathcal{E}[\![A_2]\!]\,\rho$. Let $W''' \sqsupseteq W''$, and V_1''' and V_2''' be values such that $(W''',\, V_1''',\, V_2''') \in \mathcal{V}[\![A_2]\!]\,\rho$. By Lemma G.16, it suffices to show that

$$(W''',\, V_1'''\langle \mathit{coerce}(\rho(A_2))\rangle,\, V_2''') \in \mathcal{E}[\![A_2]\!]\,\rho\ .$$

By Lemma G.35, $W'''.\Sigma_1 \rhd V_1'''\langle \mathit{coerce}(\rho(A_2))\rangle \longrightarrow^m W'''.\Sigma_1 \rhd V_1''''$ for some m and V_1''''. Let $W'''' = (W'''.n - m,\, W'''.\Sigma_1,\, W'''.\Sigma_2,\, \lfloor W'''.\kappa \rfloor_{W'''.n-m})$. By Lemma G.15 and the definition of the term relation, assume that $W'''.n > m$ and then it suffices to show that

$$(W'''',\, V_1'''',\, V_2''') \in \mathcal{V}[\![A_2]\!]\,\rho\ .$$

Note that $W'''' \sqsupseteq_m W'''$ and $W'''' \sqsupseteq_0 W''''$ by Lemma G.7 (2) and Lemma G.5. Because $(W''',\, V_1''',\, V_2''') \in \mathcal{V}[\![A_2]\!]\,\rho$ and $W'''.\Sigma_1 \rhd V_1'''\langle \mathit{coerce}(\rho(A_2))\rangle \longrightarrow^m W'''.\Sigma_1 \rhd V_1''''$, we have $(W''',\, V_1'''',\, V_2''') \in \mathcal{V}[\![A_2]\!]\,\rho$ by the IH. Because $W'''' \sqsupseteq W'''$, we have the conclusion by Lemma G.10.

Case $\exists\, X, A'.\ A = \forall X.A'$: Without loss of generality, we can assume that $X \notin \mathrm{dom}(\rho)$. We have $V_1' = V_1\langle \forall X.\mathit{coerce}(A')\rangle$ and $n = 0$. We have to show that

$$(W,\, V_1\langle \forall X.\mathit{coerce}(\rho(A'))\rangle,\, V_2) \in \mathcal{V}[\![\forall X.A']\!]\,\rho\ .$$

By Lemma E.3,

- $V_1 = (\Lambda X.(M_1 : C_1))\overline{\langle \forall X.c_1\rangle}$ and $\Sigma \vdash \overline{\langle \forall X.c_1\rangle} : \forall X.C_1 \rightsquigarrow \forall X.A'$, and
- $V_2 = (\Lambda X.(M_2 : C_2))\overline{\langle \forall X.c_2\rangle}$ and $\Sigma \vdash \overline{\langle \forall X.c_2\rangle} : \forall X.C_2 \rightsquigarrow \forall X.A'$

for some M_1, M_2, C_1, C_2, $\overline{\langle c_1\rangle}$, and $\overline{\langle c_2\rangle}$. We have the conclusion by the definition of the value relation with the following.

- Let W', $\mathbb{B}_1$, $\mathbb{B}_2$, R, M_1', M_2', and α such that
  - $W' \sqsupseteq W$,
  - $W'.\Sigma_1 \mid \emptyset \vdash \mathbb{B}_1$,
  - $W'.\Sigma_2 \mid \emptyset \vdash \mathbb{B}_2$,
  - $R \in \mathrm{Rel}_{W'.n}[\![\mathbb{B}_1, \mathbb{B}_2]\!]$,
  - $W'.\Sigma_1 \rhd V_1\langle \forall X.\mathit{coerce}(\rho(A'))\rangle\,\mathbb{B}_1 \longrightarrow W'.\Sigma_1, \alpha := \mathbb{B}_1 \rhd M_1'\langle \mathit{coerce}^+_\alpha(\rho(A')[X := \alpha])\rangle$, and
  - $W'.\Sigma_2 \rhd V_2\,\mathbb{B}_2 \longrightarrow W'.\Sigma_2, \alpha := \mathbb{B}_2 \rhd M_2'\langle \mathit{coerce}^+_\alpha(\rho(A')[X := \alpha])\rangle$.

Then, we show that

$$(W' \boxplus (\alpha, \mathbb{B}_1, \mathbb{B}_2, R), M_1', M_2') \in \blacktriangleright \mathcal{E}\,[\![A']\!]\,\rho\{X \mapsto \alpha\}\ .$$

By (R_TYBETA_C), Theorem F.1, and Lemma G.36, $M_1' = (M_1\overline{\langle c_1 \rangle})[X := \alpha]\langle \mathit{coerce}(\rho\{X \mapsto \alpha\}(A'))\rangle$ and $M_2' = (M_2\overline{\langle c_2 \rangle})[X := \alpha]$. Assume that $W'.n > 0$. Then, it suffices to show that

$$(\blacktriangleright(W' \boxplus (\alpha, \mathbb{B}_1, \mathbb{B}_2, R)), M_1', M_2') \in \mathcal{E}\,[\![A']\!]\,\rho\{X \mapsto \alpha\}\ .$$

Because

- $(W, V_1, V_2) \in \mathcal{V}\,[\![\forall X.A']\!]\,\rho$,
- $W'.\Sigma_1 \triangleright V_1\,\mathbb{B}_1 \longrightarrow W'.\Sigma_1, \alpha := \mathbb{B}_1 \triangleright (M_1\overline{\langle c_1 \rangle})[X := \alpha]\langle \mathit{coerce}^+_\alpha(\rho(A')[X := \alpha])\rangle$ by (R_TYBETA_C), and
- $W'.\Sigma_2 \triangleright V_2\,\mathbb{B}_2 \longrightarrow W'.\Sigma_2, \alpha := \mathbb{B}_2 \triangleright (M_2\overline{\langle c_2 \rangle})[X := \alpha]\langle \mathit{coerce}^+_\alpha(\rho(A')[X := \alpha])\rangle$ by (R_TYBETA_C),

we have

$$(W' \boxplus (\alpha, \mathbb{B}_1, \mathbb{B}_2, R), (M_1\overline{\langle c_1 \rangle})[X := \alpha], (M_2\overline{\langle c_2 \rangle})[X := \alpha]) \in \blacktriangleright \mathcal{E}\,[\![A']\!]\,\rho\{X \mapsto \alpha\}\ ,$$

which implies

$$(\blacktriangleright(W' \boxplus (\alpha, \mathbb{B}_1, \mathbb{B}_2, R)), (M_1\overline{\langle c_1 \rangle})[X := \alpha], (M_2\overline{\langle c_2 \rangle})[X := \alpha]) \in \mathcal{E}\,[\![A']\!]\,\rho\{X \mapsto \alpha\}\ .$$

Let $W'' \sqsupseteq \blacktriangleright(W' \boxplus (\alpha, \mathbb{B}_1, \mathbb{B}_2, R))$, and V_1'' and V_2'' be values such that $(W'', V_1'', V_2'') \in \mathcal{V}\,[\![A']\!]\,\rho\{X \mapsto \alpha\}$. Then, by Lemma G.16, it suffices to show that

$$(W'', V_1''\langle \mathit{coerce}(\rho\{X \mapsto \alpha\}(A'))\rangle, V_2'') \in \mathcal{E}\,[\![A']\!]\,\rho\{X \mapsto \alpha\}\ .$$

By Lemma G.35, $W''.\Sigma_1 \triangleright V_1''\langle \mathit{coerce}(\rho\{X \mapsto \alpha\}(A'))\rangle \longrightarrow^n W''.\Sigma_1 \triangleright V_1'''$ for some n and V_1'''. Assume that $W''.n > n$, and let $W''' = (W''.n - n, W''.\Sigma_1, W''.\Sigma_2, \lfloor W''.\kappa \rfloor_{W''.n-n})$. By Lemmas G.7 (2) and G.5, $W''' \sqsupseteq_n W''$ and $W''' \sqsupseteq_0 W'''$. Then, by Lemma G.15 and the definition of the term relation, it suffices to show that

$$(W''', V_1''', V_2'') \in \mathcal{V}\,[\![A']\!]\,\rho\{X \mapsto \alpha\}\ .$$

Because $(W'', V_1'', V_2'') \in \mathcal{V}\,[\![A']\!]\,\rho\{X \mapsto \alpha\}$, we have the conclusion by the IH and Lemma G.10.

- Let $W' \sqsupseteq W$. We show that

$$(W', V_1\langle \forall X.\mathit{coerce}(\rho(A'))\rangle\,\star, V_2\,\star) \in \mathcal{E}\,[\![A']\!]\,\rho\{X \mapsto \star\}\ .$$

By (R_TYBETADYN_C) and Lemma G.36,

- $W'.\Sigma_1 \triangleright V_1\langle \forall X.\mathit{coerce}(\rho(A'))\rangle\,\star \longrightarrow W'.\Sigma_1 \triangleright (M_1\overline{\langle c_1 \rangle})[X := \star]\langle \mathit{coerce}(\rho(A'[X := \star]))\rangle$ and
- $W'.\Sigma_2 \triangleright V_2\,\star \longrightarrow W'.\Sigma_2 \triangleright (M_2\overline{\langle c_2 \rangle})[X := \star]$.

Assume $W'.n > 1$ (thus, $\blacktriangleright W'$ is well defined, and $\blacktriangleright W' \sqsupseteq_1 W'$ by Lemma G.7 (2)). By Lemma G.15, it suffices to show that

$$(\blacktriangleright W', (M_1\overline{\langle c_1 \rangle})[X := \star]\langle \mathit{coerce}(\rho(A'[X := \star]))\rangle, (M_2\overline{\langle c_2 \rangle})[X := \star]) \in \mathcal{E}\,[\![A']\!]\,\rho\{X \mapsto \star\}\ .$$

Because $(W, V_1, V_2) \in \mathcal{V}\,[\![\forall X.A']\!]\,\rho$ and $W' \sqsupseteq W$, we have $(W', V_1\,\star, V_2\,\star) \in \mathcal{E}\,[\![A']\!]\,\rho\{X \mapsto \star\}$. Because $W'.\Sigma_1 \triangleright V_1\,\star \longrightarrow W'.\Sigma_1 \triangleright (M_1\overline{\langle c_1 \rangle})[X := \star]$ by (R_TYBETADYN_C), we have $(\blacktriangleright W', (M_1\overline{\langle c_1 \rangle})[X := \star], (M_2\overline{\langle c_2 \rangle})[X := \star]) \in \mathcal{E}\,[\![A']\!]\,\rho\{X \mapsto \star\}$ by Lemma G.20. Let $W'' \sqsupseteq \blacktriangleright W'$, and V_1'' and V_2'' be values such that $(W'', V_1'', V_2'') \in \mathcal{V}\,[\![A']\!]\,\rho\{X \mapsto \star\}$. By Lemma G.16, it suffices to show that

$$(W'', V_1''\langle \mathit{coerce}(\rho(A'[X := \star]))\rangle, V_2'') \in \mathcal{E}\,[\![A']\!]\,\rho\{X \mapsto \star\}\ .$$

By Lemma G.35, $W''.\Sigma_1 \triangleright V_1''\langle \mathit{coerce}(\rho(A'[X := \star]))\rangle \longrightarrow^n W''.\Sigma_1 \triangleright V_1'''$ for some n and V_1'''. Assume that $W''.n > n$, and let $W''' = (W''.n - n, W''.\Sigma_1, W''.\Sigma_2, \lfloor W''.\kappa \rfloor_{W''.n-n})$. By Lemmas G.7 (2) and G.5, $W''' \sqsupseteq_n W''$ and $W''' \sqsupseteq_0 W'''$. Then, by Lemma G.15 and the definition of the term relation, it suffices to show that

$$(W''', V_1''', V_2'') \in \mathcal{V}\,[\![A']\!]\,\rho\{X \mapsto \star\}\ .$$

Because $(W'', V_1'', V_2'') \in \mathcal{V}\,[\![A']\!]\,\rho\{X \mapsto \star\}$, we have the conclusion by the IH and Lemma G.10.

2. Let V'_2 be a value such that $W.\Sigma_2 \triangleright V_2\langle coerce(\rho(A))\rangle \longrightarrow^* W.\Sigma_2 \triangleright V'_2$. By case analysis on A.

Case $A = \iota\,(\exists\iota)$, $\star$, $\alpha\,(\exists\alpha)$, or $X\,(\exists X)$: Obvious by the assumption because $V_2 = V'_2$.

Case $\exists\, A_1, A_2.\ A = A_1 \rightarrow A_2$: We have $V'_2 = V_2\langle coerce(\rho(A_1)) \rightarrow coerce(\rho(A_2))\rangle$. We have to show that

$$(W,\, V_1,\, V_2\langle coerce(\rho(A_1)) \rightarrow coerce(\rho(A_2))\rangle) \in \mathcal{V}\,[\![A_1 \rightarrow A_2]\!]\,\rho\ .$$

Let $W' \sqsupseteq W$ and V'_{01} and V'_{02} be values such that $(W',\, V'_{01},\, V'_{02}) \in \mathcal{V}\,[\![A_1]\!]\,\rho$. It suffices to show that

$$(W',\, V_1\; V'_{01},\, V_2\langle coerce(\rho(A_1)) \rightarrow coerce(\rho(A_2))\rangle\; V'_{02}) \in \mathcal{E}\,[\![A_2]\!]\,\rho\ .$$

By (R_Wrap_C),

$$W'.\Sigma_2 \triangleright V_2\langle coerce(\rho(A_1)) \rightarrow coerce(\rho(A_2))\rangle\; V'_{02} \longrightarrow W'.\Sigma_2 \triangleright (V_2\,(V'_{02}\langle coerce(\rho(A_1))\rangle))\langle coerce(\rho(A_2))\rangle\ .$$

By Lemma G.35, $W'.\Sigma_2 \triangleright V'_{02}\langle coerce(\rho(A_1))\rangle \longrightarrow^* W'.\Sigma_2 \triangleright V''_2$ for some V''_2. Therefore, by (R_Ctx_C),

$$W'.\Sigma_2 \triangleright V_2\langle coerce(\rho(A_1)) \rightarrow coerce(\rho(A_2))\rangle\; V'_{02} \longrightarrow^* W'.\Sigma_2 \triangleright (V_2\; V''_2)\langle coerce(\rho(A_2))\rangle\ .$$

Assume that $0 < W'.n$. We have $W' \sqsupseteq_0 W'$ by Lemma G.5. Therefore, by Lemma G.15, it suffices to show that

$$(W',\, V_1\; V'_{01}, (V_1\; V''_2)\langle coerce(\rho(A_2))\rangle) \in \mathcal{E}\,[\![A_2]\!]\,\rho\ .$$

Because $(W',\, V'_{01},\, V'_{02}) \in \mathcal{V}\,[\![A_1]\!]\,\rho$, we have $(W',\, V'_{01},\, V''_2) \in \mathcal{V}\,[\![A_1]\!]\,\rho$ by the IH. By $(W,\, V_1,\, V_2) \in \mathcal{V}\,[\![A_1 \rightarrow A_2]\!]\,\rho$, we have $(W',\, V_1\; V'_{01},\, V_2\; V''_2) \in \mathcal{E}\,[\![A_2]\!]\,\rho$. Let $W''' \sqsupseteq W'$, and V'''_1 and V'''_2 be values such that $(W''',\, V'''_1,\, V'''_2) \in \mathcal{V}\,[\![A_2]\!]\,\rho$. By Lemma G.16, it suffices to show that

$$(W''',\, V'''_1,\, V'''_2\langle coerce(\rho(A_2))\rangle) \in \mathcal{E}\,[\![A_2]\!]\,\rho\ .$$

By Lemma G.35, $W'''.\Sigma_2 \triangleright V'''_2\langle coerce(\rho(A_2))\rangle \longrightarrow^* W'''.\Sigma_2 \triangleright V''''_2$ for some V''''_2. By Lemma G.5, $W''' \sqsupseteq_0 W'''$. Therefore, by Lemma G.15 and the definition of the term relation, assume that $W'''.n > 0$ and then it suffices to show that

$$(W''',\, V'''_1,\, V''''_2) \in \mathcal{V}\,[\![A_2]\!]\,\rho\ .$$

Because $(W''',\, V'''_1,\, V'''_2) \in \mathcal{V}\,[\![A_2]\!]\,\rho$ and $W'''.\Sigma_2 \triangleright V'''_2\langle coerce(\rho(A_2))\rangle \longrightarrow^* W'''.\Sigma_2 \triangleright V''''_2$, we have the conclusion by the IH.

Case $\exists\, X, A'.\ A = \forall X.A'$: Without loss of generality, we can assume that $X \notin \mathrm{dom}(\rho)$. We have $V'_2 = V_2\langle \forall X.coerce(A')\rangle$. We have to show that

$$(W,\, V_1,\, V_2\langle \forall X.coerce(\rho(A'))\rangle) \in \mathcal{V}\,[\![\forall X.A']\!]\,\rho\ .$$

By Lemma E.3,

- $V_1 = (\Lambda X.(M_1 : C_1))\overline{\langle \forall X.c_1\rangle}$ and $\Sigma \vdash \overline{\langle \forall X.c_1\rangle} : \forall X.C_1 \rightsquigarrow \forall X.A'$, and
- $V_2 = (\Lambda X.(M_2 : C_2))\overline{\langle \forall X.c_2\rangle}$ and $\Sigma \vdash \overline{\langle \forall X.c_2\rangle} : \forall X.C_2 \rightsquigarrow \forall X.A'$

for some M_1, M_2, C_1, C_2, $\overline{\langle c_1\rangle}$, and $\overline{\langle c_2\rangle}$. We have the conclusion by the definition of the value relation with the following.

- Let W', $\mathbb{B}_1$, $\mathbb{B}_2$, R, M'_1, M'_2, and α such that
  - $W' \sqsupseteq W$,
  - $W'.\Sigma_1 \mid \emptyset \vdash \mathbb{B}_1$,
  - $W'.\Sigma_2 \mid \emptyset \vdash \mathbb{B}_2$,
  - $R \in \mathrm{Rel}_{W'.n}\,[\![\mathbb{B}_1, \mathbb{B}_2]\!]$,
  - $W'.\Sigma_1 \triangleright V_1\,\mathbb{B}_1 \longrightarrow W'.\Sigma_1, \alpha := \mathbb{B}_1 \triangleright M'_1\langle coerce^+_\alpha(\rho(A')[X := \alpha])\rangle$, and
  - $W'.\Sigma_2 \triangleright V_2\langle \forall X.coerce(\rho(A'))\rangle\,\mathbb{B}_2 \longrightarrow W'.\Sigma_2, \alpha := \mathbb{B}_2 \triangleright M'_2\langle coerce^+_\alpha(\rho(A')[X := \alpha])\rangle$.

Then, we show that

$$(W' \boxplus (\alpha, \mathbb{B}_1, \mathbb{B}_2, R), M'_1, M'_2) \in \blacktriangleright\mathcal{E}\,[\![A']\!]\,\rho\{X \mapsto \alpha\} .$$

By (R_TYBETA_C), Theorem F.1, and Lemma G.36, $M'_1 = (M_1\overline{\langle c_1\rangle})[X := \alpha]$ and $M'_2 = (M_2\overline{\langle c_2\rangle})[X := \alpha]\langle \mathit{coerce}(\rho\{X \mapsto \alpha\}(A'))\rangle$. Assume that $W'.n > 0$. Then, it suffices to show that

$$(\blacktriangleright(W' \boxplus (\alpha, \mathbb{B}_1, \mathbb{B}_2, R)), M'_1, M'_2) \in \mathcal{E}\,[\![A']\!]\,\rho\{X \mapsto \alpha\} .$$

Because

- $(W, V_1, V_2) \in \mathcal{V}\,[\![\forall X.A']\!]\,\rho$,
- $W'.\Sigma_1 \rhd V_1\,\mathbb{B}_1 \longrightarrow W'.\Sigma_1, \alpha := \mathbb{B}_1 \rhd (M_1\overline{\langle c_1\rangle})[X := \alpha]\langle \mathit{coerce}^+_\alpha(\rho(A')[X := \alpha])\rangle$ by (R_TYBETA_C), and
- $W'.\Sigma_2 \rhd V_2\,\mathbb{B}_2 \longrightarrow W'.\Sigma_2, \alpha := \mathbb{B}_2 \rhd (M_2\overline{\langle c_2\rangle})[X := \alpha]\langle \mathit{coerce}^+_\alpha(\rho(A')[X := \alpha])\rangle$ by (R_TYBETA_C),

we have

$$(W' \boxplus (\alpha, \mathbb{B}_1, \mathbb{B}_2, R), (M_1\overline{\langle c_1\rangle})[X := \alpha], (M_2\overline{\langle c_2\rangle})[X := \alpha]) \in \blacktriangleright\mathcal{E}\,[\![A']\!]\,\rho\{X \mapsto \alpha\} ,$$

which implies

$$(\blacktriangleright(W' \boxplus (\alpha, \mathbb{B}_1, \mathbb{B}_2, R)), (M_1\overline{\langle c_1\rangle})[X := \alpha], (M_2\overline{\langle c_2\rangle})[X := \alpha]) \in \mathcal{E}\,[\![A']\!]\,\rho\{X \mapsto \alpha\} .$$

Let $W'' \sqsupseteq \blacktriangleright(W' \boxplus (\alpha, \mathbb{B}_1, \mathbb{B}_2, R))$, and V''_1 and V''_2 be values such that $(W'', V''_1, V''_2) \in \mathcal{V}\,[\![A']\!]\,\rho\{X \mapsto \alpha\}$. Then, by Lemma G.16, it suffices to show that

$$(W'', V''_1, V''_2\langle \mathit{coerce}(\rho\{X \mapsto \alpha\}(A'))\rangle) \in \mathcal{E}\,[\![A']\!]\,\rho\{X \mapsto \alpha\} .$$

By Lemma G.35, $W''.\Sigma_2 \rhd V''_2\langle \mathit{coerce}(\rho\{X \mapsto \alpha\}(A'))\rangle \longrightarrow^* W''.\Sigma_2 \rhd V'''_2$ for some V'''_2. Assume that $W''.n > 0$. By Lemma G.5, $W'' \sqsupseteq_0 W''$. Then, by the definition of the term relation, it suffices to show that

$$(W'', V''_1, V'''_2) \in \mathcal{V}\,[\![A']\!]\,\rho\{X \mapsto \alpha\} .$$

Because $(W'', V''_1, V''_2) \in \mathcal{V}\,[\![A']\!]\,\rho\{X \mapsto \alpha\}$, we have the conclusion by the IH.

- Let $W' \sqsupseteq W$. We show that

$$(W', V_1\,\star, V_2\langle \forall X.\mathit{coerce}(\rho(A'))\rangle\,\star) \in \mathcal{E}\,[\![A']\!]\,\rho\{X \mapsto \star\} .$$

By (R_TYBETADYN_C) and Lemma G.36,

- $W'.\Sigma_1 \rhd V_1\,\star \longrightarrow W'.\Sigma_1 \rhd (M_1\overline{\langle c_1\rangle})[X := \star]$ and
- $W'.\Sigma_2 \rhd V_2\langle \forall X.\mathit{coerce}(\rho(A'))\rangle\,\star \longrightarrow W'.\Sigma_2 \rhd (M_2\overline{\langle c_2\rangle})[X := \star]\langle \mathit{coerce}(\rho(A'[X := \star]))\rangle$.

Assume $W'.n > 1$ (thus, $\blacktriangleright W'$ is well defined, and $\blacktriangleright W' \sqsupseteq_1 W'$ by Lemma G.7 (2)). By Lemma G.15, it suffices to show that

$$(\blacktriangleright W', (M_1\overline{\langle c_1\rangle})[X := \star], (M_2\overline{\langle c_2\rangle})[X := \star]\langle \mathit{coerce}(\rho(A'[X := \star]))\rangle) \in \mathcal{E}\,[\![A']\!]\,\rho\{X \mapsto \star\} .$$

Because $(W, V_1, V_2) \in \mathcal{V}\,[\![\forall X.A']\!]\,\rho$ and $W' \sqsupseteq W$, we have $(W', V_1\,\star, V_2\,\star) \in \mathcal{E}\,[\![A']\!]\,\rho\{X \mapsto \star\}$. Because $W'.\Sigma_2 \rhd V_2\,\star \longrightarrow W'.\Sigma_2 \rhd (M_2\overline{\langle c_2\rangle})[X := \star]$ by (R_TYBETADYN_C), we have $(\blacktriangleright W', (M_1\overline{\langle c_1\rangle})[X := \star], (M_2\overline{\langle c_2\rangle})[X := \star]) \in \mathcal{E}\,[\![A']\!]\,\rho\{X \mapsto \star\}$ by Lemma G.20. Let $W'' \sqsupseteq \blacktriangleright W'$, and V''_1 and V''_2 be values such that $(W'', V''_1, V''_2) \in \mathcal{V}\,[\![A']\!]\,\rho\{X \mapsto \star\}$. By Lemma G.16, it suffices to show that

$$(W'', V''_1, V''_2\langle \mathit{coerce}(\rho(A'[X := \star]))\rangle) \in \mathcal{E}\,[\![A']\!]\,\rho\{X \mapsto \star\} .$$

By Lemma G.35, $W''.\Sigma_2 \rhd V''_2\langle \mathit{coerce}(\rho(A'[X := \star]))\rangle \longrightarrow^* W''.\Sigma_2 \rhd V'''_2$ for some V'''_2. Assume that $W''.n > 0$. By Lemma G.5, $W'' \sqsupseteq_0 W''$. Then, by the definition of the term relation, it suffices to show that

$$(W'', V''_1, V'''_2) \in \mathcal{V}\,[\![A']\!]\,\rho\{X \mapsto \star\} .$$

Because $(W'', V''_1, V''_2) \in \mathcal{V}\,[\![A']\!]\,\rho\{X \mapsto \star\}$, we have the conclusion by the IH.

□

Lemma G.38 (Identity Coercion Produces Contextually Equivalent Values)**.** If $\Sigma \mid \emptyset \vdash V : A$ and $\Sigma \triangleright V\langle coerce(A)\rangle \longrightarrow^* \Sigma \triangleright V'$, then $\Sigma \mid \emptyset \vdash V \stackrel{\text{ctx}}{=} V' : A$.

Proof. By Theorem G.34, it suffices to show that $\Sigma \mid \emptyset \vdash V \approx V' : A$. Note that $\Sigma \mid \emptyset \vdash V' : A$ by Lemmas E.10 and G.35 and Theorems E.21 and F.1. Let $W \in \mathcal{S}[\![\Sigma]\!]$ and $(W, \theta, \rho) \in \mathcal{G}[\![\emptyset]\!]$. By definition, $\rho = \emptyset$ and $\theta = \emptyset$. Therefore, by the definition of the term relation, assume that $W.n > 0$ and then it suffices to show that $(W, V, V') \in \mathcal{V}[\![A]\!]\emptyset$ and $(W, V', V) \in \mathcal{V}[\![A]\!]\emptyset$. Note that $W \sqsupseteq_0 W$ by Lemma G.5. By Theorem G.31, $\Sigma \mid \emptyset \vdash V \approx V : A$. Therefore, $(W, V, V) \in \mathcal{V}[\![A]\!]\emptyset$. By Lemma G.37, we have the conclusion. □

Lemma G.39 (Subterm Evaluation of Terminating Term)**.** If $\Sigma \triangleright F[M] \longrightarrow^n \Sigma' \triangleright V$, then $\Sigma \triangleright M \longrightarrow^m \Sigma'' \triangleright V''$ and $\Sigma'' \triangleright F[V''] \longrightarrow^{n-m} \Sigma' \triangleright V$ for some m, Σ'', and V''.

Proof. Straightforward by induction on n with Lemma G.1 and Theorem F.1. □

Theorem G.40 (Free Theorem: K-Combinator)**.** If $\Sigma \mid \emptyset \vdash V : \forall X.\forall Y.X \to Y \to X$ and $\Sigma \mid \emptyset \vdash V_1 : \mathbb{A}$ and $\Sigma \mid \emptyset \vdash V_2 : \mathbb{B}$, then one of the following holds:

- $\Sigma \triangleright V\,\mathbb{A}\,\mathbb{B}\,V_1\,V_2 \longrightarrow^* \Sigma' \triangleright V_1'$ and $\Sigma' \mid \emptyset \vdash V_1' \stackrel{\text{ctx}}{=} V_1 : \mathbb{A}$ for some Σ' and V_1';
- $\Sigma \triangleright V\,\mathbb{A}\,\mathbb{B}\,V_1\,V_2 \longrightarrow^* \Sigma' \triangleright \mathsf{blame}\,p$ for some Σ' and p; or
- $\Sigma \triangleright V\,\mathbb{A}\,\mathbb{B}\,V_1\,V_2 \Uparrow$.

Proof. By Lemma E.10, $\Sigma \mid \emptyset \vdash \mathbb{A}$ and $\Sigma \mid \emptyset \vdash \mathbb{B}$. Therefore, $\Sigma \mid \emptyset \vdash V\,\mathbb{A}\,\mathbb{B}\,V_1\,V_2 : \mathbb{A}$. By Theorem E.21, it suffices to consider the case that

$$\Sigma \triangleright V\,\mathbb{A}\,\mathbb{B}\,V_1\,V_2 \longrightarrow^n \Sigma' \triangleright V_1' \tag{4}$$

for some n, Σ', and V_1'. Then, we have to show that

$$\Sigma' \mid \emptyset \vdash V_1' \stackrel{\text{ctx}}{=} V_1 : \mathbb{A}\ .$$

By Theorem G.31, $\Sigma \mid \emptyset \vdash V \approx V : \forall X.\forall Y.X \to Y \to X$.

Let $W_0 = (n+1, \Sigma, \Sigma, \{\alpha \in \mathrm{dom}(\Sigma) \mapsto \lfloor \mathcal{V}[\![\Sigma(\alpha)]\!]\emptyset\rfloor_{n+1}\})$. We first show that $W_0 \in \mathrm{World}_{n+2} \subseteq \mathrm{World}$. We have $n+1 < n+2$ and $\vdash \Sigma$ by Lemma E.10. Let $\alpha \in \mathrm{dom}(\Sigma)$. It suffices to show that $\lfloor \mathcal{V}[\![\Sigma(\alpha)]\!]\emptyset\rfloor_{n+1} \in \mathrm{Rel}_{n+1}[\![\Sigma(\alpha), \Sigma(\alpha)]\!]$, which is proven by the following.

- We show that $\lfloor \mathcal{V}[\![\Sigma(\alpha)]\!]\emptyset\rfloor_{n+1} \subseteq \mathrm{Atom}^{\mathrm{val}}_{n+1}[\![\Sigma(\alpha), \Sigma(\alpha)]\!]$. By definition,

  $$\lfloor \mathcal{V}[\![\Sigma(\alpha)]\!]\emptyset\rfloor_{n+1} \subseteq \mathcal{V}[\![\Sigma(\alpha)]\!]\emptyset \subseteq \mathrm{Atom}^{\mathrm{val}}[\![\Sigma(\alpha)]\!]\emptyset = \bigcup_{m \geq 0} \mathrm{Atom}^{\mathrm{val}}_m[\![\Sigma(\alpha), \Sigma(\alpha)]\!] \subseteq \bigcup_{m \geq 0} \mathrm{Atom}_m[\![\Sigma(\alpha), \Sigma(\alpha)]\!]\ .$$

  Therefore, for any $(W, M_1, M_2) \in \lfloor \mathcal{V}[\![\Sigma(\alpha)]\!]\emptyset\rfloor_{n+1}$, we have $W \in \mathrm{World}_m$ and $W.\Sigma_1 \mid \emptyset \vdash M_1 : \Sigma(\alpha)$ and $W.\Sigma_2 \mid \emptyset \vdash M_2 : \Sigma(\alpha)$ for some m. Also, M_1 and M_2 are values and $W.n < n+1$, which implies $W \in \mathrm{World}_{n+1}$. Therefore, we have the conclusion.

- We show the monotonicity of $\lfloor \mathcal{V}[\![\Sigma(\alpha)]\!]\emptyset\rfloor_{n+1}$. Let $(W_1, V_1'', V_2'') \in \lfloor \mathcal{V}[\![\Sigma(\alpha)]\!]\emptyset\rfloor_{n+1}$ and $W_2 \sqsupseteq W_1$. Then, we show that $(W_2, V_1'', V_2'') \in \lfloor \mathcal{V}[\![\Sigma(\alpha)]\!]\emptyset\rfloor_{n+1}$. By definition, $(W_1, V_1'', V_2'') \in \mathcal{V}[\![\Sigma(\alpha)]\!]\emptyset$ and $W_1.n < n+1$. By Lemma G.10, $(W_2, V_1'', V_2'') \in \mathcal{V}[\![\Sigma(\alpha)]\!]\emptyset$. Because $W_2 \sqsupseteq W_1$, we have $W_2.n \leq W_1.n < n+1$. Therefore, we have the conclusion.

Because $W_0 \in \mathcal{S}[\![\Sigma]\!]$ and $(W_0, \emptyset, \emptyset) \in \mathcal{G}[\![\emptyset]\!]$ by definition, we have $(W_0, V, V) \in \mathcal{E}[\![\forall X.\forall Y.X \to Y \to X]\!]\emptyset$, which implies $(W_0', V, V) \in \mathcal{V}[\![\forall X.\forall Y.X \to Y \to X]\!]\emptyset$ for some W_0' such that $W_0' \sqsupseteq_0 W_0$ and $W_0'.\Sigma_1 = W_0'.\Sigma_2 = \Sigma$. Let V_{01} be a value and j be a natural number such that, for any Σ_0, $\Sigma_0 \triangleright V_1\langle coerce(\mathbb{A})\rangle \longrightarrow^j \Sigma_0 \triangleright V_{01}$ (by Lemma G.35, there exist such V_{01} and j). Let $R_1 = \{(W, V_{01}, V_{01}) \mid W \sqsupseteq W_0' \wedge W.n < W_0'.n - 1\}$. Then, we have the following.

- $W_0' \sqsupseteq W_0'$ by Lemma G.5.

- $W'_0.\Sigma_1 \mid \emptyset \vdash \mathbb{A}$ and $W'_0.\Sigma_2 \mid \emptyset \vdash \mathbb{A}$ by Lemma E.10.
- $R_1 \in \mathrm{Rel}_{W'_0.n}[\![\mathbb{A}, \mathbb{A}]\!]$ by the following:
  - We show that $R_1 \subseteq \mathrm{Atom}^{\mathrm{val}}_{W'_0.n}[\![\mathbb{A}, \mathbb{A}]\!]$. Let $(W', V''_1, V''_2) \in R_1$. Then, $W' \sqsupseteq W'_0$ and $W'.n < W'_0.n - 1$ and $V''_1 = V''_2 = V_{01}$. By $W' \sqsupseteq W'_0$, we have $W' \in \mathrm{World}$ and $W'.\Sigma_1 \supseteq \Sigma$ and $W'.\Sigma_2 \supseteq \Sigma$. Because $W'.n < W'_0.n - 1$, we have $W' \in \mathrm{World}_{W'_0.n}$. Because $\Sigma \mid \emptyset \vdash V_1 : \mathbb{A}$, we have $\Sigma \mid \emptyset \vdash V_{01} : \mathbb{A}$ by Lemmas E.10 and G.35 and Theorems E.21 and F.1. Because $\vdash W'.\Sigma_1$ and $\vdash W'.\Sigma_2$ by $W' \in \mathrm{World}$, we have $W'.\Sigma_1 \mid \emptyset \vdash V_{01} : \mathbb{A}$ and $W'.\Sigma_2 \mid \emptyset \vdash V_{01} : \mathbb{A}$ by Lemma E.8. Therefore, we have the conclusion.
  - We show the monotonicity of R_1. Let $(W'_1, V''_1, V''_2) \in R_1$ and $W'_2 \sqsupseteq W'_1$. We show that $(W'_2, V''_1, V''_2) \in R_1$. By the definition of R_1, it suffices to show that $W'_2 \sqsupseteq W'_0$ and $W'_2.n < W'_0.n - 1$. $W'_2 \sqsupseteq W'_0$ is derived by Lemma G.5 with $W'_2 \sqsupseteq W'_1$ and $W'_1 \sqsupseteq W'_0$. $W'_2.n < W'_0.n - 1$ is derived by $W'_2 \sqsupseteq W'_1$ and $(W'_1, V''_1, V''_2) \in R_1$ (that is, $W'_1.n < W'_0.n - 1$).
- We have $\Sigma \triangleright V\,\mathbb{A} \longrightarrow \Sigma, \alpha := \mathbb{A} \triangleright M_1\langle \forall Y.\alpha^- \to \mathsf{id}_Y \to \alpha^+\rangle$ for some α and M_1 by Lemma E.3 and (R_Tybeta_C). Note that we can assume that α is fresh without loss of generality.

Therefore, by $(W'_0, V, V) \in \mathcal{V}[\![\forall X.\forall Y.X \to Y \to X]\!]\emptyset$,

$$(W'_0 \boxplus (\alpha, \mathbb{A}, \mathbb{A}, R_1), M_1, M_1) \in \blacktriangleright\mathcal{E}[\![\forall Y.X \to Y \to X]\!]\emptyset\{X \mapsto \alpha\}\ .$$

Because $W'_0.n = W_0.n = n + 1 > 0$, we have

$$(\blacktriangleright(W'_0 \boxplus (\alpha, \mathbb{A}, \mathbb{A}, R_1)), M_1, M_1) \in \mathcal{E}[\![\forall Y.X \to Y \to X]\!]\emptyset\{X \mapsto \alpha\}\ . \tag{5}$$

Note that $W'_0 \boxplus (\alpha, \mathbb{A}, \mathbb{A}, R_1) = (W'_0.n, (\Sigma, \alpha := \mathbb{A}), (\Sigma, \alpha := \mathbb{A}), W'_0.\kappa\{\alpha \mapsto R_1\})$. Because

$$\begin{array}{lll} \Sigma \triangleright V\,\mathbb{A}\,\mathbb{B}\,V_1\,V_2 & \longrightarrow & \Sigma, \alpha := \mathbb{A} \triangleright M_1\langle \forall Y.\alpha^- \to \mathsf{id}_Y \to \alpha^+\rangle\,\mathbb{B}\,V_1\,V_2 \quad \text{(by Lemma G.1)} \\ & \longrightarrow^{n-1} & \Sigma' \triangleright V'_1 \quad \text{(by Theorem F.1 with (4))}\ , \end{array}$$

Lemma G.39 implies

$$\Sigma, \alpha := \mathbb{A} \triangleright M_1 \longrightarrow^m \Sigma_1 \triangleright V' \tag{6}$$

for some m, Σ_1, and V' such that $m < n = (\blacktriangleright(W'_0 \boxplus (\alpha, \mathbb{A}, \mathbb{A}, R_1))).n$. Therefore, by (5) and Theorem F.1, there exists some W_1 such that $W_1.\Sigma_1 = W_1.\Sigma_2 = \Sigma_1$ and $W_1 \sqsupseteq_m \blacktriangleright(W'_0 \boxplus (\alpha, \mathbb{A}, \mathbb{A}, R_1))$ and

$$(W_1, V', V') \in \mathcal{V}[\![\forall Y.X \to Y \to X]\!]\emptyset\{X \mapsto \alpha\}\ . \tag{7}$$

Let $R_2 = \{(W, V_2, V_2) \mid W \sqsupseteq W_1 \wedge W.n < W_1.n - 1\}$. Then, we have the following.

- $W_1 \sqsupseteq W_1$ by Lemma G.5.
- $W_1.\Sigma_1 \mid \emptyset \vdash \mathbb{B}$ and $W_1.\Sigma_2 \mid \emptyset \vdash \mathbb{B}$ by Lemma D.2 (1) with $\Sigma \mid \emptyset \vdash \mathbb{B}$. Note that $W_1.\Sigma_1 = W_1.\Sigma_2 = \Sigma_1 \supseteq \Sigma$, which can be easily proven by induction on the derivation of (6).
- $R_2 \in \mathrm{Rel}_{W_1.n}[\![\mathbb{B}, \mathbb{B}]\!]$ by the following:
  - We show that $R_2 \subseteq \mathrm{Atom}^{\mathrm{val}}_{W_1.n}[\![\mathbb{B}, \mathbb{B}]\!]$. Let $(W', V''_1, V''_2) \in R_2$. Then, $W' \sqsupseteq W_1$ and $W'.n < W_1.n - 1$ and $V''_1 = V''_2 = V_2$. By $W' \sqsupseteq W_1$, we have $W' \in \mathrm{World}$ and $W'.\Sigma_1 \supseteq \Sigma_1$ and $W'.\Sigma_2 \supseteq \Sigma_1$. Because $W'.n < W_1.n - 1$, we have $W' \in \mathrm{World}_{W_1.n}$. Because $\Sigma \mid \emptyset \vdash V_2 : \mathbb{B}$, and $\vdash W'.\Sigma_1$ and $\vdash W'.\Sigma_2$ (by $W' \in \mathrm{World}$), and $\Sigma_1 \supseteq \Sigma$ (discussed above), we have $W'.\Sigma_1 \mid \emptyset \vdash V_2 : \mathbb{B}$ and $W'.\Sigma_2 \mid \emptyset \vdash V_2 : \mathbb{B}$ by Lemma E.8. Therefore, we have the conclusion.
  - We show the monotonicity of R_2. Let $(W'_1, V''_1, V''_2) \in R_2$ and $W'_2 \sqsupseteq W'_1$. We show that $(W'_2, V''_1, V''_2) \in R_2$. By the definition of R_2, it suffices to show that $W'_2 \sqsupseteq W_1$ and $W'_2.n < W_1.n - 1$. $W'_2 \sqsupseteq W_1$ is derived by Lemma G.5 with $W'_2 \sqsupseteq W'_1$ and $W'_1 \sqsupseteq W_1$. $W'_2.n < W_1.n - 1$ is derived by $W'_2 \sqsupseteq W'_1$ and $(W'_1, V''_1, V''_2) \in R_2$ (that is, $W'_1.n < W_1.n - 1$).
- We have $\Sigma_1 \triangleright V'\,\mathbb{B} \longrightarrow \Sigma_1, \beta := \mathbb{B} \triangleright M_2\langle \mathsf{id}_\alpha \to \beta^- \to \mathsf{id}_\alpha\rangle$ for some β and M_2 by Lemma E.3 and (R_Tybeta_C). Note that we can assume that β is fresh without loss of generality.

Therefore, by (7), we have

$$(W_1 \boxplus (\beta, \mathbb{B}, \mathbb{B}, R_2), M_2, M_2) \in \blacktriangleright\mathcal{E}\,[\![X \to Y \to X]\!]\,\emptyset\{X \mapsto \alpha\}\{Y \mapsto \beta\}\ .$$

Note that $W_1 \boxplus (\beta, \mathbb{B}, \mathbb{B}, R_2) = (W_1.n, (\Sigma_1, \beta := \mathbb{B}), (\Sigma_1, \beta := \mathbb{B}), W_1.\kappa\{\beta \mapsto R_2\})$. Because $W_1 \sqsupseteq_m \blacktriangleright(W_0' \boxplus (\alpha, \mathbb{A}, \mathbb{A}, R_1))$, we have $W_1.n + m = W_0'.n - 1 = n$, that is, $W_1.n = n - m > 0$. Therefore,

$$(\blacktriangleright(W_1 \boxplus (\beta, \mathbb{B}, \mathbb{B}, R_2)), M_2, M_2) \in \mathcal{E}\,[\![X \to Y \to X]\!]\,\emptyset\{X \mapsto \alpha\}\{Y \mapsto \beta\}\ . \tag{8}$$

We have

$$\begin{array}{lll}
\Sigma \triangleright V\,\mathbb{A}\,\mathbb{B}\,V_1\,V_2 & \longrightarrow & \Sigma, \alpha := \mathbb{A} \triangleright M_1\langle \forall Y.\alpha^- \to \mathsf{id}_Y \to \alpha^+\rangle\,\mathbb{B}\,V_1\,V_2 \\
& \longrightarrow^m & \Sigma_1 \triangleright V'\langle \forall Y.\alpha^- \to \mathsf{id}_Y \to \alpha^+\rangle\,\mathbb{B}\,V_1\,V_2 \\
& & \quad \text{(by Lemma G.1 with (6))} \\
& \longrightarrow & \Sigma_1, \beta := \mathbb{B} \triangleright M_2\langle \alpha^- \to \mathsf{id}_\beta \to \alpha^+\rangle\langle \mathit{coerce}^-_\beta(\mathbb{A}) \to \beta^- \to \mathit{coerce}^+_\beta(\mathbb{A})\rangle\,V_1\,V_2 \\
& & \quad \text{(by (R_TYBETA_C))} \\
& \longrightarrow^{n-(m+2)} & \Sigma' \triangleright V_1' \\
& & \quad \text{(by Theorem F.1 with (4))}\ .
\end{array}$$

Lemma G.39 implies

$$\Sigma_1, \beta := \mathbb{B} \triangleright M_2 \longrightarrow^i \Sigma_2 \triangleright V'' \tag{9}$$

for some i, Σ_2, and V'' such that $i < n - (m + 2)$. Therefore,

$$\Sigma \triangleright V\,\mathbb{A}\,\mathbb{B}\,V_1\,V_2 \longrightarrow^{m+i+2} \Sigma_2 \triangleright V''\langle \alpha^- \to \mathsf{id}_\beta \to \alpha^+\rangle\langle \mathit{coerce}^-_\beta(\mathbb{A}) \to \beta^- \to \mathit{coerce}^+_\beta(\mathbb{A})\rangle\,V_1\,V_2 \tag{10}$$

by Lemma G.1. Because $(\blacktriangleright(W_1 \boxplus (\beta, \mathbb{B}, \mathbb{B}, R_2))).n = W_1.n - 1 = n - m - 1 > i$, we have that, by (8) and Theorem F.1, there exists some W_2 such that $W_2.\Sigma_1 = W_2.\Sigma_2 = \Sigma_2$ and $W_2 \sqsupseteq_i \blacktriangleright(W_1 \boxplus (\beta, \mathbb{B}, \mathbb{B}, R_2))$ and

$$(W_2, V'', V'') \in \mathcal{V}\,[\![X \to Y \to X]\!]\,\emptyset\{X \mapsto \alpha\}\{Y \mapsto \beta\}\ . \tag{11}$$

Now, we have the following.

- $W_2 \sqsupseteq W_2$ by Lemma G.5.
- $(W_2, V_{01}\langle\alpha^-\rangle, V_{01}\langle\alpha^-\rangle) \in \mathcal{V}\,[\![X]\!]\,\emptyset\{X \mapsto \alpha\}\{Y \mapsto \beta\}$, which is proven as follows. Because
  - $W_2 \sqsupseteq_i \blacktriangleright(W_1 \boxplus (\beta, \mathbb{B}, \mathbb{B}, R_2))$,
  - $\blacktriangleright(W_1 \boxplus (\beta, \mathbb{B}, \mathbb{B}, R_2)) \sqsupseteq_1 W_1$ and $\blacktriangleright(W_0' \boxplus (\alpha, \mathbb{A}, \mathbb{A}, R_1)) \sqsupseteq_1 W_0'$ by Lemmas G.6, G.7 (2), and G.5, and
  - $W_1 \sqsupseteq_m \blacktriangleright(W_0' \boxplus (\alpha, \mathbb{A}, \mathbb{A}, R_1))$,

  we have

  $$W_2 \sqsupseteq \blacktriangleright(W_0' \boxplus (\alpha, \mathbb{A}, \mathbb{A}, R_1)) \text{ and } W_2 \sqsupseteq W_0'$$

  by Lemma G.5. Therefore, $W_2.\kappa(\alpha) = \lfloor R_1 \rfloor_{W_2.n}$. Because $W_2.n = n - (m + i + 1)$ and $i < n - m - 1$, we have $W_2.n > 0$. Therefore, $\blacktriangleright W_2$ is well defined, and it suffices to show that

  $$(\blacktriangleright W_2, V_{01}, V_{01}) \in \lfloor R_1 \rfloor_{W_2.n}\ .$$

  Because $(\blacktriangleright W_2).n < W_2.n$, it suffices to show that

  $$(\blacktriangleright W_2, V_{01}, V_{01}) \in R_1\ .$$

  By definition, it suffices to show the following.

  - $\blacktriangleright W_2 \sqsupseteq W_0'$, which is derived by Lemma G.5 with $\blacktriangleright W_2 \sqsupseteq W_2$ (implied by Lemma G.7 (2)) and $W_2 \sqsupseteq W_0'$.
  - $(\blacktriangleright W_2).n < W_0'.n - 1$, which holds because $W_2.n = n - (m + i + 1) = W_0'.n - 1 - (m + i + 1)$.

Therefore, by (11),

$$(W_2, V'' (V_{01} \langle \alpha^- \rangle), V'' (V_{01} \langle \alpha^- \rangle)) \in \mathcal{E} \llbracket Y \to X \rrbracket \emptyset \{ X \mapsto \alpha \} \{ Y \mapsto \beta \} . \tag{12}$$

Furthermore, we have

$$\begin{array}{ll}
& \Sigma \triangleright V \, \mathbb{A} \mathbb{B} \, V_1 \, V_2 \\
\longrightarrow^{m+2} & \Sigma_1, \beta := \mathbb{B} \triangleright M_2 \langle \alpha^- \to \mathsf{id}_\beta \to \alpha^+ \rangle \langle \mathit{coerce}^-_\beta(\mathbb{A}) \to \beta^- \to \mathit{coerce}^+_\beta(\mathbb{A}) \rangle \, V_1 \, V_2 \\
\longrightarrow^{i} & \Sigma_2 \triangleright V'' \langle \alpha^- \to \mathsf{id}_\beta \to \alpha^+ \rangle \langle \mathit{coerce}^-_\beta(\mathbb{A}) \to \beta^- \to \mathit{coerce}^+_\beta(\mathbb{A}) \rangle \, V_1 \, V_2 \\
& \quad \text{(by Lemma G.1 with (9))} \\
\longrightarrow & \Sigma_2 \triangleright (V'' \langle \alpha^- \to \mathsf{id}_\beta \to \alpha^+ \rangle \, (V_1 \langle \mathit{coerce}^-_\beta(\mathbb{A}) \rangle)) \langle \beta^- \to \mathit{coerce}^+_\beta(\mathbb{A}) \rangle \, V_2 \\
& \quad \text{(by (R_WRAP_C)/(R_CTX_C))} \\
\longrightarrow^{j} & \Sigma_2 \triangleright (V'' \langle \alpha^- \to \mathsf{id}_\beta \to \alpha^+ \rangle \, V_{01}) \langle \beta^- \to \mathit{coerce}^+_\beta(\mathbb{A}) \rangle \, V_2 \\
& \quad \text{(by Lemmas G.35 and G.1)} \\
\longrightarrow & \Sigma_2 \triangleright (V'' \, (V_{01} \langle \alpha^- \rangle)) \langle \mathsf{id}_\beta \to \alpha^+ \rangle \langle \beta^- \to \mathit{coerce}^+_\beta(\mathbb{A}) \rangle \, V_2 \\
& \quad \text{(by (R_WRAP_C)/(R_CTX_C))} \\
\longrightarrow^{n-(m+i+j+4)} & \Sigma' \triangleright V'_1 \\
& \quad \text{(by Theorem F.1)} .
\end{array}$$

By Lemma G.39,

$$\Sigma_2 \triangleright V'' \, (V_{01} \langle \alpha^- \rangle) \longrightarrow^k \Sigma_3 \triangleright V''' \tag{13}$$

for some k, Σ_3, and V'' such that $k < n - (m + i + j + 4)$. Because $W_2.n = n - (m + i + 1) > k$, we have that, by (12) and Theorem F.1, there exists some W_3 such that $W_3.\Sigma_1 = W_3.\Sigma_2 = \Sigma_3$ and $W_3 \sqsupseteq_k W_2$ and

$$(W_3, V''', V''') \in \mathcal{V} \llbracket Y \to X \rrbracket \emptyset \{ X \mapsto \alpha \} \{ Y \mapsto \beta \} . \tag{14}$$

Now, we have the following.

- $W_3 \sqsupseteq W_3$ by Lemma G.5.
- $(W_3, V_2 \langle \beta^- \rangle, V_2 \langle \beta^- \rangle) \in \mathcal{V} \llbracket Y \rrbracket \emptyset \{ X \mapsto \alpha \} \{ Y \mapsto \beta \}$, which is proven as follows. Because
  - $W_3 \sqsupseteq_k W_2$,
  - $W_2 \sqsupseteq_i \blacktriangleright(W_1 \boxplus (\beta, \mathbb{B}, \mathbb{B}, R_2))$, and
  - $\blacktriangleright(W_1 \boxplus (\beta, \mathbb{B}, \mathbb{B}, R_2)) \sqsupseteq W_1$,

  we have

  $$W_3 \sqsupseteq \blacktriangleright(W_1 \boxplus (\beta, \mathbb{B}, \mathbb{B}, R_2)) \text{ and } W_3 \sqsupseteq W_1$$

  by Lemma G.5. Therefore, $W_3.\kappa(\beta) = \lfloor R_2 \rfloor_{W_3.n}$. Because $W_3.n = W_2.n - k = n - (m + i + k + 1)$ and $k < n - (m + i + j + 4)$, we have $W_3.n > 0$. Therefore, $\blacktriangleright W_3$ is well defined, and it suffices to show that

  $$(\blacktriangleright W_3, V_2, V_2) \in \lfloor R_2 \rfloor_{W_3.n} .$$

  Because $(\blacktriangleright W_3).n < W_3.n$, it suffices to show that

  $$(\blacktriangleright W_3, V_2, V_2) \in R_2 .$$

  By definition, it suffices to show the following.

  - $\blacktriangleright W_3 \sqsupseteq W_1$, which is derived by Lemma G.5 with $\blacktriangleright W_3 \sqsupseteq W_3$ (implied by Lemma G.7 (2)) and $W_3 \sqsupseteq W_1$.
  - $(\blacktriangleright W_3).n < W_1.n - 1$, which holds because $W_3.n = n - (m + i + k + 1) \leq n - m - 1 = W_1.n - 1$.

Therefore, by (14),

$$(W_3, V''' \ V_2\langle\beta^-\rangle, V''' \ V_2\langle\beta^-\rangle) \in \mathcal{E}\,[\![X]\!]\,\emptyset\{X \mapsto \alpha\}\{ Y \mapsto \beta\} \ . \tag{15}$$

Furthermore, we have the following.

$$\begin{array}{ll}
 & \Sigma \triangleright V\ \mathbb{A}\,\mathbb{B}\ V_1\ V_2 \\
\longrightarrow^{m+i+j+4} & \Sigma_2 \triangleright (\,V''\,(\,V_{01}\langle\alpha^-\rangle))\langle \mathsf{id}_\beta \to \alpha^+\rangle\langle\beta^- \to \mathit{coerce}^+_\beta(\mathbb{A})\rangle\ V_2 \\
\longrightarrow^{k} & \Sigma_3 \triangleright V'''\langle \mathsf{id}_\beta \to \alpha^+\rangle\langle\beta^- \to \mathit{coerce}^+_\beta(\mathbb{A})\rangle\ V_2 \\
 & \quad \text{(by Lemma G.1 with (13))} \\
\longrightarrow^{3} & \Sigma_3 \triangleright (\,V'''\,(\,V_2\langle\beta^-\rangle))\langle\alpha^+\rangle\langle \mathit{coerce}^+_\beta(\mathbb{A})\rangle \\
 & \quad \text{(by (R_WRAP_C) and (R_ID_C) with (R_CTX_C))} \\
\longrightarrow^{n-(m+i+j+k+7)} & \Sigma' \triangleright V_1' \\
 & \quad \text{(by Theorem F.1 with (4))}\ .
\end{array}$$

By Lemma G.39,

$$\Sigma_3 \triangleright V'''\,(\,V_2\langle\beta^-\rangle) \longrightarrow^{l} \Sigma_4 \triangleright V'''' \tag{16}$$

for some l, Σ_4, and V'''' such that $l < n-(m+i+j+k+7)$. Because $W_3.n = n-(m+i+k+1) > l$, we have that, by (15) and Theorem F.1, there exists some W_4 such that $W_4.\Sigma_1 = W_4.\Sigma_2 = \Sigma_4$ and $W_4 \sqsupseteq_l W_3$ and

$$(\,W_4, V'''', V'''') \in \mathcal{V}\,[\![X]\!]\,\emptyset\{X \mapsto \alpha\}\{ Y \mapsto \beta\} \ . \tag{17}$$

By definition, $V'''' = V_{01}'\langle\alpha^-\rangle$ for some V_{01}' such that $(\,W_4, V_{01}', V_{01}') \in \blacktriangleright(\,W_4.\kappa(\alpha))$. Because $W_4.n = W_3.n - l = n - (m+i+k+l+1)$ and $l < n-(m+i+j+k+7)$, we have $W_4.n > 0$. Therefore, $(\blacktriangleright W_4, V_{01}', V_{01}') \in W_4.\kappa(\alpha)$. Because

- $W_4 \sqsupseteq W_3$,
- $W_3 \sqsupseteq W_2$, and
- $W_2 \sqsupseteq \blacktriangleright(\,W_0' \boxplus (\alpha, \mathbb{A}, \mathbb{A}, R_1))$,

we have $W_4 \sqsupseteq \blacktriangleright(\,W_0' \boxplus (\alpha, \mathbb{A}, \mathbb{A}, R_1))$ by Lemma G.5. Therefore, $W_4.\kappa(\alpha) = \lfloor R_1 \rfloor_{W_4.n}$, so $V_{01}' = V_{01}$. Now, we have the following.

$$\begin{array}{ll}
 & \Sigma \triangleright V\ \mathbb{A}\,\mathbb{B}\ V_1\ V_2 \\
\longrightarrow^{*} & \Sigma_3 \triangleright (\,V'''\,(\,V_2\langle\beta^-\rangle))\langle\alpha^+\rangle\langle \mathit{coerce}^+_\beta(\mathbb{A})\rangle \\
\longrightarrow^{*} & \Sigma_4 \triangleright V''''\langle\alpha^+\rangle\langle \mathit{coerce}^+_\beta(\mathbb{A})\rangle \quad \text{(by Lemma G.1 with (16))} \\
= & \Sigma_4 \triangleright V_{01}\langle\alpha^-\rangle\langle\alpha^+\rangle\langle \mathit{coerce}^+_\beta(\mathbb{A})\rangle \\
\longrightarrow & \Sigma_4 \triangleright V_{01}\langle \mathit{coerce}^+_\beta(\mathbb{A})\rangle \quad \text{(by (R_REMOVE_C)/(R_CTX_C))} \\
\longrightarrow^{*} & \Sigma' \triangleright V_1' \quad \text{(by Theorem F.1 with (4))}\ .
\end{array}$$

By Lemma G.35 and Theorem F.1, $\Sigma' = \Sigma_4$ and $\Sigma_4 \triangleright V_{01}\langle \mathit{coerce}(\mathbb{A})\rangle \longrightarrow^* \Sigma_4 \triangleright V_1'$.

Because $\Sigma \mid \emptyset \vdash V\ \mathbb{A}\,\mathbb{B}\ V_1\ V_2 : \mathbb{A}$ and $\Sigma \triangleright V\ \mathbb{A}\,\mathbb{B}\ V_1\ V_2 \longrightarrow^* \Sigma' \triangleright V_1'$, we have $\Sigma \subseteq \Sigma'$ and $\Sigma' \mid \emptyset \vdash V_1' : \mathbb{A}$ by Theorems E.21 and F.1. By Lemma E.10, $\vdash \Sigma'$. By Lemma E.8, $\Sigma' \mid \emptyset \vdash V_1 : \mathbb{A}$. By Lemma G.35, $\Sigma' \triangleright V_1\langle \mathit{coerce}(\mathbb{A})\rangle \longrightarrow^* \Sigma' \triangleright V_{01}$. Therefore,

$$\Sigma' \mid \emptyset \vdash V_1 \overset{\mathrm{ctx}}{=} V_{01} : \mathbb{A}$$

by Lemma G.38.

By the definition of the contextual equivalence, $\Sigma' \mid \emptyset \vdash V_{01} : \mathbb{A}$. Because $\Sigma \mid \emptyset \vdash \mathbb{A}$, we have $\Sigma' \mid \emptyset \vdash \mathbb{A}$ by Lemma D.2 (1). Therefore, $\Sigma' \mid \emptyset \vdash V_{01}\langle \mathit{coerce}(\mathbb{A})\rangle : \mathbb{A}$ by Lemma G.35 and (T_CRC_C). Because $\Sigma' \triangleright V_{01}\langle \mathit{coerce}(\mathbb{A})\rangle \longrightarrow^* \Sigma' \triangleright V_1'$, we have

$$\Sigma' \mid \emptyset \vdash V_{01} \overset{\mathrm{ctx}}{=} V_1' : \mathbb{A}$$

by Lemma G.38.

Because the contextual equivalence is transitive (which can be easily proven using Theorem F.1) and symmetric, we have $\Sigma' \mid \emptyset \vdash V_1' \overset{\mathrm{ctx}}{=} V_1 : \mathbb{A}$. □

H Space-efficiency

H.1 $\lambda C^{\forall}_{mp}$

Lemma H.1 (Only Type Application Generates Coercions)**.** If $\Sigma \triangleright M \longrightarrow \Sigma' \triangleright M'$ is derived without (R_TYBETA_C), then, for any c' occurring in M', there exists some c in M such that c' is a sub-coercion of c.

Proof. By induction on the derivation of $\Sigma \triangleright M \longrightarrow \Sigma' \triangleright M'$.

Case (R_DELTA_C), (R_FAIL_C), (R_CONFLICT_C), (R_BLAME_C): Obvious because there is no c' in M'.

Case (R_BETA_C), (R_ID_C), (R_COLLAPSE_C), (R_REMOVE_C): Obvious by letting $c = c'$ because any c' in M' also occurs in M.

Case (R_WRAP_C): We are given

$$M = (V\langle c'' \to d''\rangle)\, V', \quad M' = (V\,(V'\langle c''\rangle))\langle d''\rangle \quad (\exists c'', d'', V, V')\ .$$

By case analysis on c'.

Case $c' = c''$ or $c' = d''$: We have the conclusion by letting $c = c'' \to d''$.

Case c' occurs in V or V': We have the conclusion by letting $c = c'$ because c' also occurs in M.

Case (R_SPLIT_C): Similarly to the case of (R_WRAP_C).

Case (R_CTX_C): We are given

$$M = E[M_1], \quad M' = E[M_1'], \quad \Sigma \triangleright M_1 \longrightarrow \Sigma' \triangleright M_1' \quad (\exists E, M_1, M_1')\ .$$

By case analysis on c'.

Case c' occurs in M_1': By the assumption, $\Sigma \triangleright M_1 \longrightarrow \Sigma' \triangleright M_1'$ is derived without (R_TYBETA_C). Therefore, by the IH, there exists some c_1 in M_1 such that c' is a sub-coercion of c_1. Because M_1 is a subterm of M, c_1 occurs in M. Therefore, we have the conclusion by letting $c = c_1$.

Case c' occurs in E: Because c' also occurs in M, we have the conclusion by letting $c = c'$.

□

Lemma H.2 (Only Type Application Generates Coercions: Multi-Step Evaluation)**.** If $\Sigma \triangleright M \longrightarrow^* \Sigma' \triangleright M'$ is derived without (R_TYBETA_C), then, for any c' occurring in M', there exists some c in M such that c' is a sub-coercion of c.

Proof. Straightforward by induction on the length of the reduction sequence of $\Sigma \triangleright M \longrightarrow^* \Sigma' \triangleright M'$ with Lemma H.1.

□

Theorem H.3 ($\lambda C^{\forall}_{mp}$ Cannot Be Made Space-Efficient (Theorem **??** of the paper))**.** There exists a closed well-typed term M such that, for any natural number n, there exist some store Σ and term M' satisfying the following:

(1) $\emptyset \triangleright M \longrightarrow^* \Sigma \triangleright M'$; and

(2) there exist some types A, B and coercion sequence $\overline{\langle c' \rangle}$ that appears in M' such that

  (i) $\Sigma \vdash \overline{\langle c' \rangle} : A \rightsquigarrow B$,

  (ii) $\mathrm{size}(\overline{\langle c' \rangle}) > n$, and

  (iii) there does not exist a coercion c such that

   (a) $\Sigma \mid \emptyset \vdash \langle c \rangle \overset{\mathrm{ctx}}{=} \overline{\langle c' \rangle} : A \rightsquigarrow B$ and

   (b) $\mathrm{size}(c) < \mathrm{size}(\overline{\langle c' \rangle})$.

Proof. Let

- $f \overset{\text{def}}{=} \lambda f_0 : \star . \Lambda X.((\lambda x : X.(f_0 \langle \star \to \star ?^{p_1} \rangle\, f_0) \langle (\forall X.\star)?^{p_2} \rangle \langle \forall X.(\star \to \star ?^{p_3} \,;\, X! \to \mathsf{id}_\star) \rangle \star (x \langle X! \rangle)) : X \to \star)$,
- $V \overset{\text{def}}{=} f \langle \mathsf{id}_\star \to ((\forall X.((X?^{p_4} \to \mathsf{id}_\star) \,;\, \star \to \star !)) \,;\, (\forall X.\star)!) \,;\, \star \to \star ! \rangle$, and
- $M \overset{\text{def}}{=} f\ V \star (0 \langle \mathsf{Int}! \rangle)$.

First, we show that $\emptyset \mid \emptyset \vdash M : \star$.

- We start with showing that $\emptyset \mid \emptyset \vdash f : \star \to (\forall X.X \to \star)$. Let $\Gamma = \emptyset, f_0 : \star, X, x : X$. By (T_Abs_C) and (T_TyAbs_C), it suffices to show that

  $$\emptyset \mid \Gamma \vdash (f_0 \langle \star \to \star ?^{p_1} \rangle\, f_0) \langle (\forall X.\star)?^{p_2} \rangle \langle \forall X.(\star \to \star ?^{p_3} \,;\, X! \to \mathsf{id}_\star) \rangle \star (x \langle X! \rangle) : \star\ .$$

  We have the following derivation.

  $$,\quad \dfrac{\dfrac{\vdash \emptyset \qquad \emptyset \vdash \Gamma \qquad x : X \in \Gamma}{\emptyset \mid \Gamma \vdash x : X}\,\text{(T\_Var\_C)} \qquad \dfrac{\vdash \emptyset \qquad \emptyset \vdash \Gamma \qquad \emptyset \mid \Gamma \vdash X}{\emptyset \mid \Gamma \vdash X! : X \rightsquigarrow \star}\,\text{(Ct\_Inj\_C)}}{\emptyset \mid \Gamma \vdash x \langle X! \rangle : \star}\,\text{(T\_Crc\_C)}$$

  Therefore, by (T_Tyapp_C) and (T_App_C), it suffices to show that

  $$\emptyset \mid \Gamma \vdash (f_0 \langle \star \to \star ?^{p_1} \rangle\, f_0) \langle (\forall X.\star)?^{p_2} \rangle \langle \forall X.(\star \to \star ?^{p_3} \,;\, X! \to \mathsf{id}_\star) \rangle : \forall X.(X \to \star)\ .$$

  We have the following derivations.

  $$\dfrac{\vdash \emptyset \qquad \emptyset \vdash \Gamma \qquad \emptyset \mid \Gamma \vdash \forall X.\star}{\emptyset \mid \Gamma \vdash (\forall X.\star)?^{p_2} : \star \rightsquigarrow \forall X.\star}\,\text{(Ct\_Proj\_C)}$$

  $$\dfrac{\dfrac{\dfrac{\vdash \emptyset \qquad \emptyset \vdash \Gamma, Y \qquad \emptyset \mid \Gamma, Y \vdash \star \to \star}{\emptyset \mid \Gamma, Y \vdash \star \to \star ?^{p_3} : \star \rightsquigarrow \star \to \star}\,\text{(Ct\_Proj\_C)} \qquad \dfrac{\emptyset \mid \Gamma, Y \vdash Y! : Y \rightsquigarrow \star \qquad \emptyset \mid \Gamma, Y \vdash \mathsf{id}_\star : \star \rightsquigarrow \star}{\emptyset \mid \Gamma, Y \vdash Y! \to \mathsf{id}_\star : \star \to \star \rightsquigarrow Y \to \star}\,\text{(Ct\_Arrow\_C)}}{\emptyset \mid \Gamma, Y \vdash \star \to \star ?^{p_3} \,;\, Y! \to \mathsf{id}_\star : \star \rightsquigarrow Y \to \star}\,\text{(Ct\_Seq\_C)}}{\emptyset \mid \Gamma \vdash \forall X.(\star \to \star ?^{p_3} \,;\, X! \to \mathsf{id}_\star) : \forall X.\star \rightsquigarrow \forall X.(X \to \star)}\,\text{(Ct\_All\_C)}$$

  where $\emptyset \mid \Gamma, Y \vdash Y! : Y \rightsquigarrow \star$ is derived by (Ct_Inj_C), and $\emptyset \mid \Gamma, Y \vdash \mathsf{id}_\star : \star \rightsquigarrow \star$ is by (Ct_Id_C). Therefore, by (T_Crc_C), it suffices to show that

  $$\emptyset \mid \Gamma \vdash f_0 \langle \star \to \star ?^{p_1} \rangle\, f_0 : \star\ ,$$

  which is derived by the following derivation.

  $$\dfrac{\dfrac{\dfrac{}{\emptyset \mid \Gamma \vdash f_0 : \star}\,\text{(T\_Var\_C)} \qquad \dfrac{}{\emptyset \mid \Gamma \vdash \star \to \star ?^{p_1} : \star \rightsquigarrow \star \to \star}\,\text{(Ct\_Proj\_C)}}{\emptyset \mid \Gamma \vdash f_0 \langle \star \to \star ?^{p_1} \rangle : \star \to \star}\,\text{(T\_Crc\_C)} \qquad \dfrac{}{\emptyset \mid \Gamma \vdash f_0 : \star}\,\text{(T\_Var\_C)}}{\emptyset \mid \Gamma \vdash f_0 \langle \star \to \star ?^{p_1} \rangle\, f_0 : \star}\,\text{(T\_App\_C)}$$

- Next, we show that $\emptyset \mid \emptyset \vdash V : \star$. Because $\emptyset \mid \emptyset \vdash f : \star \to (\forall X.X \to \star)$, (T_Crc_C) implies that it suffices to show that

  $$\emptyset \mid \emptyset \vdash \mathsf{id}_\star \to ((\forall X.((X?^{p_4} \to \mathsf{id}_\star) \,;\, \star \to \star !)) \,;\, (\forall X.\star)!) \,;\, \star \to \star ! : \star \to (\forall X.X \to \star) \rightsquigarrow \star\ .$$

  By (Ct_Seq_C) and (Ct_Inj_C), it suffices to show that

  $$\emptyset \mid \emptyset \vdash \mathsf{id}_\star \to ((\forall X.((X?^{p_4} \to \mathsf{id}_\star) \,;\, \star \to \star !)) \,;\, (\forall X.\star)!) : \star \to (\forall X.X \to \star) \rightsquigarrow \star \to \star\ .$$

  By (Ct_Arrow_C) and (Ct_Id_C), it suffices to show that

  $$\emptyset \mid \emptyset \vdash (\forall X.((X?^{p_4} \to \mathsf{id}_\star) \,;\, \star \to \star !)) \,;\, (\forall X.\star)! : \forall X.X \to \star \rightsquigarrow \star\ .$$

By (Ct_Seq_C) and (Ct_Inj_C), it suffices to show that

$$\emptyset \mid \emptyset \vdash \forall X.((X?^{p_4} \to \mathsf{id}_\star) \,;\, \star \to \star!) : \forall X.X \to \star \rightsquigarrow \forall X.\star \;.$$

By (Ct_All_C), it suffices to show that

$$\emptyset \mid \emptyset, X \vdash (X?^{p_4} \to \mathsf{id}_\star) \,;\, \star \to \star! : X \to \star \rightsquigarrow \star \;.$$

By (Ct_Seq_C) and (Ct_Inj_C), it suffices to show that

$$\emptyset \mid \emptyset, X \vdash X?^{p_4} \to \mathsf{id}_\star : X \to \star \rightsquigarrow \star \to \star \;,$$

which is derived by the following.

$$\dfrac{\dfrac{}{\emptyset \mid \emptyset, X \vdash X?^{p_4} : \star \rightsquigarrow X}\;(\text{Ct\_Proj\_C}) \qquad \dfrac{}{\emptyset \mid \emptyset, X \vdash \mathsf{id}_\star : \star \rightsquigarrow \star}\;(\text{Ct\_Id\_C})}{\emptyset \mid \emptyset, X \vdash X?^{p_4} \to \mathsf{id}_\star : X \to \star \rightsquigarrow \star \to \star}\;(\text{Ct\_Arrow\_C})$$

- Finally, we show that $\emptyset \mid \emptyset \vdash M : \star$, which is obvious because $\emptyset \mid \emptyset \vdash f\ V : \forall X.X \to \star$.

Let

- $n > 0$,
- $V_n \overset{\text{def}}{=} 0\langle \mathsf{Int}! \rangle \langle \alpha_1^- \rangle \langle \alpha_1 ! \rangle \cdots \langle \alpha_n^- \rangle \langle \alpha_n ! \rangle$, and
- $\Sigma_n \overset{\text{def}}{=} \emptyset, \alpha_1 := \star, \cdots, \alpha_n := \star$, and
- $F \overset{\text{def}}{=} \Box\langle (\forall X.\star)?^{p_2} \rangle \langle \forall X.(\star \to \star?^{p_3} \,;\, X! \to \mathsf{id}_\star) \rangle$.

We show that

$$\emptyset \rhd M \;\longrightarrow^*\; \Sigma_n \rhd F'[F[\,V\langle \star \to \star?^{p_1} \rangle\ V\,] \star V_n]$$

for some F'. We prove it by induction on n. Note that $f \;=\; \lambda f_0 \mathbin{:} \star.\Lambda X.((\lambda x \mathbin{:} X.F[f_0\langle \star \to \star?^{p_1} \rangle\, f_0] \star (x\langle X! \rangle)) : X \to \star)$.

Case $n = 1$: By (R_Beta_C), (R_Tybeta_C), and (R_Wrap_C), we have the following evaluation sequence:

$$\begin{array}{ll}
 & \emptyset \rhd M \\
= & \emptyset \rhd f\ V \star (0\langle \mathsf{Int}! \rangle) \\
\longrightarrow & \emptyset \rhd (\Lambda X.((\lambda x \mathbin{:} X.F[\,V\langle \star \to \star?^{p_1} \rangle\ V\,] \star (x\langle X! \rangle)) : X \to \star)) \star (0\langle \mathsf{Int}! \rangle) \\
\longrightarrow & \emptyset, \alpha_1 := \star \rhd (\lambda x \mathbin{:} \alpha_1.F[\,V\langle \star \to \star?^{p_1} \rangle\ V\,] \star (x\langle \alpha_1 ! \rangle))\langle \alpha_1^- \to \mathsf{id}_\star \rangle\, (0\langle \mathsf{Int}! \rangle) \\
\longrightarrow & \emptyset, \alpha_1 := \star \rhd ((\lambda x \mathbin{:} \alpha_1.F[\,V\langle \star \to \star?^{p_1} \rangle\ V\,] \star (x\langle \alpha_1 ! \rangle))\, (0\langle \mathsf{Int}! \rangle \langle \alpha_1^- \rangle))\langle \mathsf{id}_\star \rangle \\
\longrightarrow & \emptyset, \alpha_1 := \star \rhd (F[\,V\langle \star \to \star?^{p_1} \rangle\ V\,] \star (0\langle \mathsf{Int}! \rangle \langle \alpha_1^- \rangle \langle \alpha_1 ! \rangle))\langle \mathsf{id}_\star \rangle \;.
\end{array}$$

Therefore, we have the conclusion for the case of $n = 1$ by letting $F' = \Box\langle \mathsf{id}_\star \rangle$.

Case $\exists\, m.\ n = m + 1 \wedge m > 0$: By the IH,

$$\emptyset \rhd M \;\longrightarrow^*\; \Sigma_m \rhd F_0[F[\,V\langle \star \to \star?^{p_1} \rangle\ V\,] \star V_m]$$

for some F_0. By (R_Split_C), (R_Collapse_C), (R_Wrap_C), (R_Id_C), (R_Beta_C), We have

$$\begin{array}{ll}
 & \Sigma_m \rhd V\langle \star \to \star?^{p_1} \rangle\ V \\
\longrightarrow^* & \Sigma_m \rhd f\langle \mathsf{id}_\star \to ((\forall X.((X?^{p_4} \to \mathsf{id}_\star) \,;\, \star \to \star!)) \,;\, (\forall X.\star)!) \rangle\ V \\
\longrightarrow & \Sigma_m \rhd (f\ (V\langle \mathsf{id}_\star \rangle))\langle (\forall X.((X?^{p_4} \to \mathsf{id}_\star) \,;\, \star \to \star!)) \,;\, (\forall X.\star)! \rangle \\
\longrightarrow & \Sigma_m \rhd (f\ V)\langle (\forall X.((X?^{p_4} \to \mathsf{id}_\star) \,;\, \star \to \star!)) \,;\, (\forall X.\star)! \rangle \\
\longrightarrow & \Sigma_m \rhd V'\langle (\forall X.((X?^{p_4} \to \mathsf{id}_\star) \,;\, \star \to \star!)) \,;\, (\forall X.\star)! \rangle
\end{array}$$

where

$$V' \stackrel{\text{def}}{=} \Lambda X.((\lambda x : X.F[\,V\langle \star \to \star ?^{p_1}\rangle\ V\,] \star (x\langle X!\rangle)) : X \to \star)\ .$$

Therefore, by (R_Split_C), (R_Collapse_C), (R_Tybeta_C), (R_Wrap_C), (R_Id_C), (R_Beta_C),

$$\begin{array}{ll}
 & \Sigma_m \triangleright F[\,V\langle \star \to \star ?^{p_1}\rangle\ V\,] \star V_m \\
\longrightarrow^* & \Sigma_m \triangleright F[\,V'\langle(\forall X.((X?^{p_4} \to \mathsf{id}_\star)\,;\star \to \star!))\,;(\forall X.\star)!\rangle] \star V_m \quad \text{(by Lemma G.1)} \\
\longrightarrow^* & \Sigma_m \triangleright V'\langle(\forall X.((X?^{p_4} \to \mathsf{id}_\star)\,;\star \to \star!))\rangle\langle \forall X.(\star \to \star ?^{p_3}\,; X! \to \mathsf{id}_\star)\rangle \star V_m \\
\longrightarrow & \Sigma_n \triangleright (\lambda x : \alpha_n.F[\,V\langle \star \to \star ?^{p_1}\rangle\ V\,] \star (x\langle \alpha_n!\rangle))\langle \alpha_n?^{p_4} \to \mathsf{id}_\star\,; \star \to \star!\rangle\langle \star \to \star ?^{p_3}\,; \alpha_n! \to \mathsf{id}_\star\rangle\langle \alpha_n^- \to \mathsf{id}_\star\rangle\ V_m \\
\longrightarrow^* & \Sigma_n \triangleright (\lambda x : \alpha_n.F[\,V\langle \star \to \star ?^{p_1}\rangle\ V\,] \star (x\langle \alpha_n!\rangle))\langle \alpha_n?^{p_4} \to \mathsf{id}_\star\rangle\langle \alpha_n! \to \mathsf{id}_\star\rangle\langle \alpha_n^- \to \mathsf{id}_\star\rangle\ V_m \\
\longrightarrow^* & \Sigma_n \triangleright ((\lambda x : \alpha_n.F[\,V\langle \star \to \star ?^{p_1}\rangle\ V\,] \star (x\langle \alpha_n!\rangle))\,(\,V_m\langle \alpha_n^-\rangle\langle \alpha_n!\rangle\langle \alpha_n?^{p_4}\rangle))\langle \mathsf{id}_\star\rangle\langle \mathsf{id}_\star\rangle\langle \mathsf{id}_\star\rangle \\
\longrightarrow & \Sigma_n \triangleright ((\lambda x : \alpha_n.F[\,V\langle \star \to \star ?^{p_1}\rangle\ V\,] \star (x\langle \alpha_n!\rangle))\,(\,V_m\langle \alpha_n^-\rangle))\langle \mathsf{id}_\star\rangle\langle \mathsf{id}_\star\rangle\langle \mathsf{id}_\star\rangle \\
\longrightarrow & \Sigma_n \triangleright (F[\,V\langle \star \to \star ?^{p_1}\rangle\ V\,] \star (\,V_m\langle \alpha_n^-\rangle\langle \alpha_n!\rangle))\langle \mathsf{id}_\star\rangle\langle \mathsf{id}_\star\rangle\langle \mathsf{id}_\star\rangle \\
= & \Sigma_n \triangleright (F[\,V\langle \star \to \star ?^{p_1}\rangle\ V\,] \star V_n)\langle \mathsf{id}_\star\rangle\langle \mathsf{id}_\star\rangle\langle \mathsf{id}_\star\rangle
\end{array}$$

Then, we have the conclusion by Lemma G.1 when we take $F_0[\Box\langle \mathsf{id}_\star\rangle\langle \mathsf{id}_\star\rangle\langle \mathsf{id}_\star\rangle]$ as F'.

Therefore, it suffices to show the item (2) by taking Int as A, $\star$ as B, and $\langle \mathsf{Int}!\rangle, \langle \alpha_1^-\rangle, \langle \alpha_1!\rangle, \ldots, \langle \alpha_n^-\rangle, \langle \alpha_n!\rangle$ as $\overline{\langle c'\rangle}$. Note that we can take $\langle \mathsf{Int}!\rangle$ as $\overline{\langle c'\rangle}$ for the case of $n = 0$ because $\langle \mathsf{Int}!\rangle$ occurs in M. In what follows, let $\Sigma_0 = \emptyset$.

(2-i) $\Sigma_n \vdash \overline{\langle c'\rangle} : \mathsf{Int} \rightsquigarrow \star$ because $\Sigma_n \mid \emptyset \vdash \mathsf{Int}! : \mathsf{Int} \rightsquigarrow \star$ by (Ct_Inj_C), $\Sigma_n \mid \emptyset \vdash \alpha_i^- : \star \rightsquigarrow \alpha_i$ by (Ct_Conceal_C), and $\Sigma_n \mid \emptyset \vdash \alpha_i! : \alpha_i \rightsquigarrow \star$ by (Ct_Inj_C).

(2-ii) $\mathrm{size}\,(\overline{\langle c'\rangle}) > n$ because the length of $\overline{\langle c'\rangle}$ is $2n+1$, and the size of every coercion is larger than zero.

(2-iii) Let c be an arbitrary coercion such that $\Sigma_n \mid \emptyset \vdash \langle c\rangle \stackrel{\text{ctx}}{=} \overline{\langle c'\rangle} : \mathsf{Int} \rightsquigarrow \star$. Then, it suffices to show that

$$\mathrm{size}(c) \geq \mathrm{size}\,(\overline{\langle c'\rangle})\ .$$

Because $\Sigma_n \mid \emptyset \vdash \langle c\rangle \stackrel{\text{ctx}}{=} \overline{\langle c'\rangle} : \mathsf{Int} \rightsquigarrow \star$, we have

$$\Sigma \mid \emptyset \vdash \lambda x : \mathsf{Int}.x\langle c\rangle \stackrel{\text{ctx}}{=} \lambda x : \mathsf{Int}.x\langle \mathsf{Int}!\rangle\langle \alpha_1^-\rangle\langle \alpha_1!\rangle \cdots \langle \alpha_n^-\rangle\langle \alpha_n!\rangle : \mathsf{Int} \to \star\ .$$

Let $\mathcal{C}_C \stackrel{\text{def}}{=} (\Box\,0)\,\langle \alpha_n?^{q_n}\rangle\,\langle \alpha_n^+\rangle \cdots \langle \alpha_1?^{q_1}\rangle\,\langle \alpha_1^+\rangle\,\langle \mathsf{Int}?^{q}\rangle$. We can prove $\Sigma \vdash \mathcal{C}_C : (\emptyset \vdash \mathsf{Int} \to \star) \Rightarrow (\emptyset \vdash \mathsf{Int})$ easily. Furthermore, by (R_Beta_C), (R_Collapse_C), (R_Remove_C), (R_Ctx_C), we have

$$\begin{array}{ll}
 & \Sigma_n \triangleright \mathcal{C}_C[\lambda x : \mathsf{Int}.x\langle \mathsf{Int}!\rangle\langle \alpha_1^-\rangle\langle \alpha_1!\rangle \cdots \langle \alpha_n^-\rangle\langle \alpha_n!\rangle] \\
= & \Sigma_n \triangleright ((\lambda x : \mathsf{Int}.x\langle \mathsf{Int}!\rangle\langle \alpha_1^-\rangle\langle \alpha_1!\rangle \cdots \langle \alpha_n^-\rangle\langle \alpha_n!\rangle)\,0)\langle \alpha_n?^{q_n}\rangle\langle \alpha_n^+\rangle \cdots \langle \alpha_1?^{q_1}\rangle\langle \alpha_1^+\rangle\langle \mathsf{Int}?^{q}\rangle \\
\longrightarrow & \Sigma_n \triangleright 0\langle \mathsf{Int}!\rangle\langle \alpha_1^-\rangle\langle \alpha_1!\rangle \cdots \langle \alpha_n^-\rangle\langle \alpha_n!\rangle\langle \alpha_n?^{q_n}\rangle\langle \alpha_n^+\rangle \cdots \langle \alpha_1?^{q_1}\rangle\langle \alpha_1^+\rangle\langle \mathsf{Int}?^{q}\rangle \\
\longrightarrow & \Sigma_n \triangleright 0\langle \mathsf{Int}!\rangle\langle \alpha_1^-\rangle\langle \alpha_1!\rangle \cdots \langle \alpha_n^-\rangle\langle \alpha_n^+\rangle \cdots \langle \alpha_1?^{q_1}\rangle\langle \alpha_1^+\rangle\langle \mathsf{Int}?^{q}\rangle \\
\longrightarrow & \Sigma_n \triangleright 0\langle \mathsf{Int}!\rangle\langle \alpha_1^-\rangle\langle \alpha_1!\rangle \cdots \langle \alpha_{n-1}^-\rangle\langle \alpha_{n-1}!\rangle\langle \alpha_{n-1}?^{q_{n-1}}\rangle\langle \alpha_{n-1}^+\rangle \cdots \langle \alpha_1?^{q_1}\rangle\langle \alpha_1^+\rangle\langle \mathsf{Int}?^{q}\rangle \\
\longrightarrow^* & \Sigma_n \triangleright 0\langle \mathsf{Int}!\rangle\langle \mathsf{Int}?^{q}\rangle \\
\longrightarrow & \Sigma_n \triangleright 0\ .
\end{array}$$

Because 0 is a value, and $\Sigma \mid \emptyset \vdash \lambda x : \mathsf{Int}.x\langle c\rangle \stackrel{\text{ctx}}{=} \lambda x : \mathsf{Int}.x\langle \mathsf{Int}!\rangle\langle \alpha_1^-\rangle\langle \alpha_1!\rangle \cdots \langle \alpha_n^-\rangle\langle \alpha_n!\rangle : \mathsf{Int} \to \star$, Corollary F.2 implies that $\Sigma_n \triangleright \mathcal{C}_C[\lambda x : \mathsf{Int}.x\langle c\rangle] \longrightarrow_C^* \Sigma_n' \triangleright V'$ for some Σ_n' and V'. Furthermore, by (R_Beta_C), (R_Ctx_C), and Corollary F.2, we have

$$\begin{array}{ll}
 & \Sigma_n \triangleright \mathcal{C}_C[\lambda x : \mathsf{Int}.x\langle c\rangle] \\
= & \Sigma_n \triangleright ((\lambda x : \mathsf{Int}.x\langle c\rangle)\,0)\langle \alpha_n?^{q_n}\rangle\langle \alpha_n^+\rangle \cdots \langle \alpha_1?^{q_1}\rangle\langle \alpha_1^+\rangle\langle \mathsf{Int}?^{q}\rangle \\
\longrightarrow & \Sigma_n \triangleright 0\langle c\rangle\langle \alpha_n?^{q_n}\rangle\langle \alpha_n^+\rangle \cdots \langle \alpha_1?^{q_1}\rangle\langle \alpha_1^+\rangle\langle \mathsf{Int}?^{q}\rangle \\
\longrightarrow^* & \Sigma_n' \triangleright V'\ .
\end{array}$$

Therefore, $0\langle c\rangle$ also evaluates to a value, so $\Sigma_n \triangleright 0\langle c\rangle \longrightarrow^* \Sigma_{01} \triangleright V_{01}$ for some Σ_{01} and V_{01}. Then, by Lemma G.1 and Corollary F.2,

$$\begin{aligned}\Sigma_n \triangleright 0\langle c\rangle\langle \alpha_n ?^{q_n}\rangle\langle \alpha_n^+\rangle \cdots \langle \alpha_1 ?^{q_1}\rangle\langle \alpha_1^+\rangle\langle \mathsf{Int}?^q\rangle &\longrightarrow^* \Sigma_{01} \triangleright V_{01}\langle \alpha_n ?^{q_n}\rangle\langle \alpha_n^+\rangle \cdots \langle \alpha_1 ?^{q_1}\rangle\langle \alpha_1^+\rangle\langle \mathsf{Int}?^q\rangle \\ &\longrightarrow^* \Sigma'_n \triangleright V' \ .\end{aligned}$$

Therefore, the subterm $V_{01}\langle \alpha_n ?^{q_n}\rangle$ also evaluates to a value. It indicates that $V_{01} = V_{02}\langle \alpha_n !\rangle$ for some V_{02}. Therefore,

$$\Sigma_n \triangleright 0\langle c\rangle \longrightarrow^* \Sigma_{01} \triangleright V_{02}\langle \alpha_n !\rangle \ .$$

Because $0\langle c\rangle$ does not include type application, the derivation of $\Sigma \triangleright 0\langle c\rangle \longrightarrow^* \Sigma_{01} \triangleright V_{02}\langle \alpha_n !\rangle$ does not use the rule (R_Tybeta_C). Therefore, Lemma H.2 implies that $\alpha_n!$ is a sub-coercion of c. Furthermore, by (R_Collapse_C), Lemma G.1, and Corollary F.2, we have

$$\begin{aligned}\Sigma_n \triangleright 0\langle c\rangle\langle \alpha_n ?^{q_n}\rangle\langle \alpha_n^+\rangle \cdots \langle \alpha_1 ?^{q_1}\rangle\langle \alpha_1^+\rangle\langle \mathsf{Int}?^q\rangle &\longrightarrow^* \Sigma_{01} \triangleright V_{02}\langle \alpha_n !\rangle\langle \alpha_n ?^{q_n}\rangle\langle \alpha_n^+\rangle \cdots \langle \alpha_1 ?^{q_1}\rangle\langle \alpha_1^+\rangle\langle \mathsf{Int}?^q\rangle \\ &\longrightarrow \Sigma_{01} \triangleright V_{02}\langle \alpha_n^+\rangle \cdots \langle \alpha_1 ?^{q_1}\rangle\langle \alpha_1^+\rangle\langle \mathsf{Int}?^q\rangle \\ &\longrightarrow^* \Sigma'_n \triangleright V' \ .\end{aligned}$$

Therefore, the subterm $V_2\langle \alpha_n^+\rangle$ also evaluates to a value. Then, we can find that $V_{02} = V_{03}\langle \alpha_n^-\rangle$ for some V_{03}, and

$$\Sigma_n \triangleright 0\langle c\rangle \longrightarrow^* \Sigma_{01} \triangleright V_{03}\langle \alpha_n^-\rangle\langle \alpha !\rangle \ .$$

Therefore, Lemma H.2 implies that α_n^- is a sub-coercion of c. Furthermore, by (R_Remove_C), Lemma G.1, and Corollary F.2, we have

$$\begin{aligned}\Sigma_n \triangleright 0\langle c\rangle\langle \alpha_n ?^{q_n}\rangle\langle \alpha_n^+\rangle \cdots \langle \alpha_1 ?^{q_1}\rangle\langle \alpha_1^+\rangle\langle \mathsf{Int}?^q\rangle &\longrightarrow^* \Sigma_{01} \triangleright V_{03}\langle \alpha_n^-\rangle\langle \alpha_n^+\rangle \cdots \langle \alpha_1 ?^{q_1}\rangle\langle \alpha_1^+\rangle\langle \mathsf{Int}?^q\rangle \\ &\longrightarrow \Sigma_{01} \triangleright V_{03}\langle \alpha_{n-1} ?^{q_{n-1}}\rangle\langle \alpha_{n-1}^+\rangle \cdots \langle \alpha_1 ?^{q_1}\rangle\langle \alpha_1^+\rangle\langle \mathsf{Int}?^q\rangle \\ &\longrightarrow^* \Sigma'_n \triangleright V' \ .\end{aligned}$$

Therefore, we can apply the same discussion to V_{03}. That is, we can prove that the coercions $\alpha_n!, \alpha_n^-, \ldots, \alpha_1!, \alpha_1^-$ and $\mathsf{Int}!$ are sub-coercions of c by induction on n. Therefore, the coercion c involves at least $2n+1$ sub-coercions. Then, we can easily prove that

$$\mathrm{size}(c) \geq 2(2n+1) - 1 \ .$$

Therefore,

$$\begin{aligned}\mathrm{size}(c) &\geq 2(2n+1) - 1 \\ &= \mathrm{size}\left(\langle \mathsf{Int}!\rangle, \langle \alpha_1^-\rangle, \langle \alpha_1 !\rangle, \ldots, \langle \alpha_n^-\rangle, \langle \alpha_n !\rangle\right).\end{aligned}$$

□

H.2 $\lambda S^{\forall}_{mp}$ (Proof of Theorem 4.2)

$$\begin{aligned}\mathrm{height}(G?^p \ ; b) &= \mathrm{height}(b) & \mathrm{height}(\bot^p) &= \mathrm{height}(\mathsf{id}) = 1 \\ \mathrm{height}(g \ ; G!) &= \mathrm{height}(g) & \mathrm{height}(s \to t) &= \max(\mathrm{height}(s), \mathrm{height}(t)) + 1 \\ \mathrm{height}(\forall X.s \ ,, \ t) &= \max(\mathrm{height}(s), \mathrm{height}(t)) + 1\end{aligned}$$

$$\begin{aligned}\mathrm{size}(G?^p \ ; b) &= \mathrm{size}(b) + 2 & \mathrm{size}(\bot^p) &= \mathrm{size}(\mathsf{id}) = 1 \\ \mathrm{size}(g \ ; G!) &= \mathrm{size}(g) + 2 & \mathrm{size}(s \to t) &= \mathrm{size}(s) + \mathrm{size}(t) + 1 \\ \mathrm{size}(\forall X.s \ ,, \ t) &= \mathrm{size}(s) + \mathrm{size}(t) + 1\end{aligned}$$

Lemma H.4 (Bounding Size by Height (Lemma H.4 of the paper))**.** For any space-efficient coercion s, the following holds.

- $\text{size}(s) \leq 5(2^{\text{height}(s)} - 1)$.
- If s is a possibly blaming coercion, then $\text{size}(s) \leq 5(2^{\text{height}(s)} - 1) - 2$.
- If s is an intermediate coercion, then $\text{size}(s) \leq 5(2^{\text{height}(s)} - 1) - 2$.
- If s is a ground coercion, then $\text{size}(s) \leq 5(2^{\text{height}(s)} - 1) - 4$.

Proof. By induction on s with case analysis on the form of s.

Case $s = \mathsf{id}(\exists A)$: We have

$$\begin{aligned}\text{size}(\mathsf{id}) &= 1 \\ \text{height}(\mathsf{id}) &= 1 \ .\end{aligned}$$

Therefore, it suffices to show that $1 \leq 5(2^1 - 1) - 4$, because s is a ground coercion. Because

$$\begin{aligned}(\text{the left-hand side}) &= 1 \\ (\text{the right-hand side}) &= 5(2^1 - 1) - 4 \\ &= 5 - 4 \\ &= 1 \ ,\end{aligned}$$

finishing the case.

Case $s = s' \to t'(\exists s', t')$: We have

$$\begin{aligned}\text{size}(s' \to t') &= \text{size}(s) + \text{size}(t) + 1 \\ \text{height}(s' \to t') &= \max(\text{height}(s'), \text{height}(t')) + 1 \ .\end{aligned}$$

Consider the next two cases.

Case $\text{height}(s') \geq \text{height}(t')$: We have $\max(\text{height}(s'), \text{height}(t')) = \text{height}(s')$. Because $s' \to t'$ is a ground coercion, it suffices to show that $\text{size}(s') + \text{size}(t') + 1 \leq 5(2^{\text{height}(s')+1} - 1) - 4$. By the IH,

$$\begin{aligned}\text{size}(s') &\leq 5(2^{\text{height}(s')} - 1) \\ \text{size}(t') &\leq 5(2^{\text{height}(t')} - 1) \ .\end{aligned}$$

Therefore,

$$\begin{aligned}\text{size}(s') + \text{size}(t') + 1 &\leq 5(2^{\text{height}(s')} - 1) + 5(2^{\text{height}(t')} - 1) + 1 \\ &\leq 2 \cdot 5(2^{\text{height}(s')} - 1) + 1 \\ &= 5(2^{\text{height}(s')+1} - 1) - 4 \ .\end{aligned}$$

Case $\text{height}(s') < \text{height}(t')$: Similar.

Case $s = \forall X.s' \,,, t'(\exists X, s', t')$: Similar to the case for $s = s' \to t'$.

Case $s = g \,;\, G!(\exists g, G)$: We have

$$\begin{aligned}\text{size}(g \,;\, G!) &= \text{size}(g) + 1 \\ \text{height}(g \,;\, G!) &= \text{height}(g) \ .\end{aligned}$$

Because $g \,;\, G!$ is an intermediate coercion, it suffices to show that $\text{size}(g) + 1 \leq 5(2^{\text{height}(g)} - 1) - 2$. By the IH, $\text{size}(g) \leq 5(2^{\text{height}(g)} - 1) - 4$. Therefore,

$$\begin{aligned}\text{size}(g) + 1 &\leq 5(2^{\text{height}(g)} - 1) - 4 + 1 \\ &\leq 5(2^{\text{height}(g)} - 1) - 2 \ .\end{aligned}$$

Case $s = \bot^p(\exists p)$: Similarl to the case $s = \mathsf{id}$.

Case $s = G?^p ; b(\exists p, b)$: We have

$$\begin{aligned} \text{size}(G?^p ; b) &= \text{size}(b) + 1 \\ \text{height}(G?^p ; b) &= \text{height}(b) \ . \end{aligned}$$

Therefore, it suffices to show that $\text{size}(b) + 1 \leq 5(2^{\text{height}(b)} - 1)$. By the IH, $\text{size}(b) \leq 5(2^{\text{height}(b)} - 1) - 2$. Therefore,

$$\begin{aligned} \text{size}(b) + 1 &\leq 5(2^{\text{height}(i)} - 1) - 2 + 1 \\ &\leq 5(2^{\text{height}(i)} - 1) \ . \end{aligned}$$

□

Lemma H.5 (Height of Coercion)**.** $\text{height}(s) \geq 1$.

Proof. Straightforward by induction on s. □

Lemma H.6 (Composition Does Not Increase Height)**.** If $s ⨟ t$ is well defined, then $\text{height}(s ⨟ t) \leq \max(\text{height}(s), \text{height}(t))$.

Proof. By induction on the sum of $\text{size}(s)$ and $\text{size}(t)$. We proceed by case analysis on s.

Case $s = \mathsf{id}$: We have $\text{height}(\mathsf{id}) = 1$. By case analysis on t.

Case $t = g ; G!(\exists g, G)$: We have $\mathsf{id} ⨟ (g ; G!) = (\mathsf{id} ⨟ g) ; G!$ and

$$\text{height}(g ; G!) = \text{height}(g), \quad \text{height}((\mathsf{id} ⨟ g) ; G!) = \text{height}(\mathsf{id} ⨟ g) \ .$$

Therefore, it suffices to show that $\text{height}(\mathsf{id} ⨟ g) \leq \max(\text{height}(\mathsf{id}), \text{height}(g))$, which follows from the IH.

Case $t = \bot^p(\exists p)$: We have $\mathsf{id} ⨟ \bot^p = \bot^p$ and $\text{height}(\bot^p) = 1$. Therefore, it suffices to show that $1 \leq \max(1, 1) = 1$, which holds trivially.

Otherwise: We have $\mathsf{id} ⨟ t = t$. Therefore, it suffices to show that $\text{height}(t) \leq \max(1, \text{height}(t))$, which is trivial.

Case $s = s' \to t'(\exists s', t')$: We have $\text{height}(s' \to t') = \max(\text{height}(s'), \text{height}(t')) + 1$. By case analysis on t.

Case $t = s'' \to t''(\exists s'', t'')$: We have $(s' \to t') ⨟ (s'' \to t'') = (s'' ⨟ s') \to (t' ⨟ t'')$ and

$$\begin{aligned} \text{height}(s'' \to t'') &= \max(\text{height}(s''), \text{height}(t'')) + 1, \\ \text{height}((s'' ⨟ s') \to (t' ⨟ t'')) &= \max(\text{height}(s'' ⨟ s'), \text{height}(t' ⨟ t'')) + 1 \ . \end{aligned}$$

Therefore, it suffices to show that

$$\max(\text{height}(s'' ⨟ s'), \text{height}(t' ⨟ t'')) + 1 \leq \max(\max(\text{height}(s'), \text{height}(t')) + 1, \max(\text{height}(s''), \text{height}(t'')) + 1) \ .$$

By the assumption, $(s'' ⨟ s') \to (t' ⨟ t'')$ is well defined; and so are $s'' ⨟ s'$ and $t' ⨟ t''$. Then, by the IHs,

$$\text{height}(s'' ⨟ s') \leq \max(\text{height}(s''), \text{height}(s')), \quad \text{height}(t' ⨟ t'') \leq \max(\text{height}(t'), \text{height}(t'')) \ .$$

Therefore,

$$\begin{aligned} \max(\text{height}(s'' ⨟ s'), \text{height}(t' ⨟ t'')) + 1 &\leq \max(\max(\text{height}(s''), \text{height}(s')), \max(\text{height}(t'), \text{height}(t''))) + 1 \\ &= \max(\max(\text{height}(s'), \text{height}(t')), \max(\text{height}(s''), \text{height}(t''))) + 1 \\ &= \max(\max(\text{height}(s'), \text{height}(t')) + 1, \max(\text{height}(s''), \text{height}(t'')) + 1) \ . \end{aligned}$$

Case $t = g ; G!(\exists g, G)$: Similar to the case of $s = \mathsf{id}, t = g ; G!$.

Case $t = \bot^p (\exists p)$: We have $(s' \to t') \fatsemi \bot^p = \bot^p$ and $\mathrm{height}(\bot^p) = 1$. Therefore, it suffices to show that $1 \le \max(\mathrm{height}(s' \to t'), 1)$, which follows from Lemma H.5.

Case $t = \mathsf{id}$: We have $(s' \to t') \fatsemi \mathsf{id} = s' \to t'$ and $\mathrm{height}(\mathsf{id}) = 1$. Therefore, it suffices to show that $\mathrm{height}(s' \to t') \le \max(\mathrm{height}(s' \to t'), 1)$, which is trivial.

Otherwise: Contradictory because $(s' \to t') \fatsemi t$ is not well defined.

Case $s = \forall X.s_1 \,,, s_2 (\exists X, s_1, s_2)$: We have $\mathrm{height}(\forall X.s_1 \,,, s_2) = \max(\mathrm{height}(s_1), \mathrm{height}(s_2)) + 1$. By case analysis on t.

Case $t = \forall Y.t_1 \,,, t_2 (\exists Y, t_1, t_2)$: Because $(\forall X.s_1 \,,, s_2) \fatsemi (\forall Y.t_1 \,,, t_2)$ is well defined, $Y = X$. Furthermore, $(\forall X.s_1 \,,, s_2) \fatsemi (\forall X.t_1 \,,, t_2) = \forall X.(s_1 \fatsemi t_1) \,,, (s_2 \fatsemi t_2)$, and

$$\mathrm{height}(\forall X.(s_1 \,,, s_2)) = \max(\mathrm{height}(s_1), \mathrm{height}(s_2)) + 1$$
$$\mathrm{height}(\forall X.(t_1 \,,, t_2)) = \max(\mathrm{height}(t_1), \mathrm{height}(t_2)) + 1$$
$$\mathrm{height}(\forall X.(s_1 \fatsemi t_1) \,,, (s_2 \fatsemi t_2)) = \max(\mathrm{height}(s_1 \fatsemi t_1), \mathrm{height}(s_2 \fatsemi t_2)) + 1 \ .$$

Therefore, it suffices to show that

$$\max(\mathrm{height}(s_1 \fatsemi t_1), \mathrm{height}(s_2 \fatsemi t_2)) + 1 \le \max(\mathrm{height}(s_1), \mathrm{height}(s_2), \mathrm{height}(t_1), \mathrm{height}(t_2)) + 1) \ .$$

It follows from the IH.

Case $t = g \,;\, G!(\exists g, G)$: Similar to the case of $s = \mathsf{id}, t = g \,;\, G!$.

Case $t = \bot^p (\exists p)$: Similar to the case of $s = s' \to t', t = \bot^p$.

Case $t = \mathsf{id}$: Similar to the case of $s = s' \to t', t = \mathsf{id}$.

Otherwise: Contradictory because $(\forall X.s') \fatsemi t$ is not well defined.

Case $s = g \,;\, G!(\exists g, G)$: We have $\mathrm{height}(g \,;\, G!) = \mathrm{height}(g)$. By case analysis on t.

Case $t = \mathsf{id}$: We have $(g \,;\, G!) \fatsemi \mathsf{id} = g \,;\, G!$ and $\mathrm{height}(\mathsf{id}) = 1$. Therefore, it suffices to show that

$$\mathrm{height}(g \,;\, G!) \le \max(\mathrm{height}(g \,;\, G!), 1) \ ,$$

which is trivial.

Case $t = i \,;\, H!(\exists i, H)$: We have $(g \,;\, G!) \fatsemi (i \,;\, H!) = ((g \,;\, G!) \fatsemi i) \,;\, H!$ and

$$\mathrm{height}(((g \,;\, G!) \fatsemi i) \,;\, H!) = \mathrm{height}((g \,;\, G!) \fatsemi i), \quad \mathrm{height}(i \,;\, H!) = \mathrm{height}(i) \ .$$

Therefore, it suffices to show that

$$\mathrm{height}((g \,;\, G!) \fatsemi i) \le \max(\mathrm{height}(g \,;\, G!), \mathrm{height}(i)) \ .$$

By the assumption, $((g \,;\, G!) \fatsemi i) \,;\, H!$ is well defined, and so is $(g \,;\, G!) \fatsemi i$. Then, by the IH,

$$\mathrm{height}((g \,;\, G!) \fatsemi i) \le \max(\mathrm{height}(g \,;\, G!), \mathrm{height}(i)) \ .$$

Case $t = \bot^p (\exists p)$: Provable similarly to the case of $s = s' \to t', t = \bot^p$.

Case $t = G?^p \,;\, t'(\exists p, t')$: We have $(i \,;\, G!) \fatsemi (G?^p \,;\, t') = i \fatsemi t'$ and $\mathrm{height}(G?^p \,;\, t') = \mathrm{height}(t')$. Therefore, it suffices to show that

$$\mathrm{height}(i \fatsemi t') \le \max(\mathrm{height}(i), \mathrm{height}(t')) \ .$$

By the assumption, $i \fatsemi t'$ is well defined. Then, by the IH,

$$\mathrm{height}(i \fatsemi t') \le \max(\mathrm{height}(i), \mathrm{height}(t')) \ .$$

Case $t = H?^p \; ; t'(G \neq H)(\exists p, H, t')$: We have $(i \; ; G!) \fatsemi (H?^p \; ; t') = \perp^p$ and

$$\text{height}(\perp^p) = 1, \quad \text{height}(H?^p \; ; t') = \text{height}(t') \; .$$

Therefore, it suffices to show that

$$1 \leq \max(\text{height}(i), \text{height}(t')) \; ,$$

which follows from Lemma H.5.

Otherwise: Contradictory because $(i \; ; G!) \fatsemi t$ is not well defined.

Case $s = \perp^p(\exists p)$: We have $\perp^p \fatsemi t = \perp^p$ and $\text{height}(\perp^p) = 1$. Therefore, it suffices to show that

$$1 \leq \max(1, \text{height}(t)) \; ,$$

which is trivial.

Case $s = G?^p \; ; b(\exists G, p, b)$: We have $(G?^p \; ; b) \fatsemi t = G?^p \; ; (b \fatsemi t)$ and

$$\text{height}(G?^p \; ; (b \fatsemi t)) = \text{height}(b \fatsemi t), \quad \text{height}(G?^p \; ; b) = \text{height}(b) \; .$$

Therefore, it suffices to show that

$$\text{height}(b \fatsemi t) \leq \max(\text{height}(b), \text{height}(t)) \; .$$

By the assumption, $G?^p \; ; (b \fatsemi t)$ is well defined, and so is $b \fatsemi t$. Then, by the IH,

$$\text{height}(b \fatsemi t) \leq \max(\text{height}(b), \text{height}(t)) \; .$$

□

Lemma H.7. $\text{height}(s[X := \star]) \leq \text{height}(s)$ and $\text{height}(s[X := \alpha]) \leq \text{height}(s)$.

Proof. By straightforward induction on s. □

Lemma H.8. If $\Sigma \triangleright M \longrightarrow^* \Sigma' \triangleright M'$, then for any s' that occurs in M', there exists some s that occurs in M and $\text{height}(s') \leq \text{height}(s)$.

Proof. By induction on the derivation of $\Sigma \triangleright M \longrightarrow \Sigma' \triangleright M'$. We perform case analysis on the rule applied last to derive $\Sigma \triangleright M \longrightarrow \Sigma' \triangleright M'$.

Case (R_Delta_S), (R_Fail_S), (R_BlameE_S), (R_BlameC_S): No s' occurs in M'.

Case (R_Beta_S), (R_Id_S): Obvious because any s' in M' also appears in M.

Case (R_Wrap_S): We are given

$$M \; = \; (U\langle s_1 \to t_1 \rangle) \; V, \quad M' = (U \; (V\langle s_1 \rangle))\langle t_1 \rangle, \quad \Sigma' \; = \; \Sigma \quad (\exists s_1, t_1, U, V) \; .$$

By case analysis on s'.

Case $s' = s_1$ or $s' = t_1$: Take $s = s_1 \to t_1$. By definition $\text{height}(s \to t) = \max(\text{height}(s), \text{height}(t)) + 1$.

Otherwise: s' occurs also in M.

Case (R_Merge_S): We are given

$$M = M_1\langle s_1 \rangle\langle t_1 \rangle, \quad M' = M_1\langle s_1 \fatsemi t_1 \rangle, \quad \Sigma' \; = \; \Sigma \quad (\exists s_1, t_1, M_1) \; .$$

By case analysis on s'.

Case $s' = s_1 \fatsemi t_1$: By Lemma H.6, $\text{height}(s \fatsemi t) \leq \max(\text{height}(s_1), \text{height}(t_1))$. Take $s = s_1$ if $\text{height}(s_1) > \text{height}(t_1)$ or t_1 otherwise.

Otherwise: s' also occurs in M.

Case (R_TybetaDyn_S), (R_TybetaDynC_S), (R_Tybeta_S), (R_TybetaC_S): Easily follows from Lemma H.7.

Case (R_CtxE_S): We are given

$$M = E[M_1], \quad M' = E[M_1'], \quad \Sigma \triangleright M_1 \longrightarrow \Sigma' \triangleright M_1' \quad (\exists E, M_1, M_1')\ .$$

By the IH, for any s' that occurs in M_1', there exists s that occurs in M_1 (and M, too) and $\mathrm{height}(s') \le \mathrm{height}(s)$. (Any s' that occurs in E also appears in M.)

Case (R_CtxC_S): Similar to the case of (R_CtxE_S).

□

Theorem H.9 ($\lambda S^{\forall}_{mp}$ is Space-Efficient (Theorem 4.2 of the paper))**.** If $\emptyset \mid \emptyset \vdash M : A$ and $\emptyset \triangleright M \longrightarrow^* \Sigma' \triangleright M'$, then, for any s' appearing in M', there exists some s appearing in M such that $\mathrm{height}(s') \le \mathrm{height}(s)$ and $\mathrm{size}(s') \le 5(2^{\mathrm{height}(s)} - 1)$.

Proof. By induction on the length of the evaluation sequence of $\emptyset \triangleright M \longrightarrow^* \Sigma \triangleright M'$. By case analysis on the length.

Case The length is zero: Because $M = M'$, s' appears in M. Therefore, we can let $s = s'$. $\mathrm{height}(s') \le \mathrm{height}(s')$ trivially holds. Furthermore, by Lemma H.4, $\mathrm{size}(s') \le 5(2^{\mathrm{height}(s')} - 1)$.

Case The length is greater than zero: We are given

$$\emptyset \triangleright M \longrightarrow^* \Sigma'' \triangleright M'', \quad \Sigma'' \triangleright M'' \longrightarrow \Sigma' \triangleright M' \quad (\exists \Sigma'', M'')\ .$$

By the IH, for any s'' appearing in M'', there exists some s appearing in M such that $\mathrm{height}(s'') \le \mathrm{height}(s)$. By Lemma H.8 and Lemma H.4, we conclude that there exists some s appearing in M such that $\mathrm{height}(s') \le \mathrm{height}(s)$ and $\mathrm{size}(s') \le 5(2^{\mathrm{height}(s')} - 1) \le 5(2^{\mathrm{height}(s)} - 1)$.

□

I Translation

I.1 Proof of Theorem 4.3

Lemma I.1 (Identity Coercion Translation Preserves Typing)**.** If $\Sigma \mid \Gamma \vdash_C \mathsf{id}_A : A \rightsquigarrow A$ and $\Delta = \{X_1, \ldots, X_n\} \subseteq \mathrm{dom}(\Gamma)$, then $|\mathsf{id}_A|_{\Gamma\setminus\Delta}$ is a well-defined ground coercion and $\Sigma \mid \Sigma(\Gamma) \setminus \Delta \vdash_S |\mathsf{id}_A|_{\Gamma\setminus\Delta} : \Sigma(A[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow \Sigma(A[X_1 := \star, \cdots, X_n := \star])$.

Proof. By straightforward induction on the structure of type A with Lemma D.10 (2) and Lemma D.10 (5). We show the case where $A = \forall X.B$. By the IH, $\Sigma \mid \Sigma(\Gamma, X) \setminus \Delta \vdash_S |\mathsf{id}_B|_{(\Gamma,X)\setminus\Delta} : \Sigma(B[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow \Sigma(B[X_1 := \star, \cdots, X_n := \star])$ and $\Sigma \mid \Sigma(\Gamma, X) \setminus (\Delta, X) \vdash_S |\mathsf{id}_B|_{(\Gamma,X)\setminus(\Delta,X)} : \Sigma(B[X_1 := \star, \cdots, X_n := \star][X := \star]) \rightsquigarrow \Sigma(B[X_1 := \star, \cdots, X_n := \star][X := \star])$. Since $\Sigma(\Gamma, X) \setminus \Delta = (\Sigma(\Gamma) \setminus \Delta), X$ and $\Sigma(\Gamma, X) \setminus (\Delta, X) = \Sigma(\Gamma \setminus \Delta)$, we have, by (Ct_All_S), $\Sigma \mid \Sigma(\Gamma) \setminus \Delta \vdash_S \forall X.(|\mathsf{id}_B|_{(\Gamma\setminus\Delta),X} ,, |\mathsf{id}_B|_{(\Gamma\setminus\Delta)}) : \Sigma(\forall X.B[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow \Sigma(\forall X.B[X_1 := \star, \cdots, X_n := \star])$. □

Lemma I.2 (Strengthening)**.**

1. If $\Sigma \mid \Delta_1, X, \Delta_2 \vdash A$ and $X \notin \mathrm{ftv}(A)$, then $\Sigma \mid \Delta_1, \Delta_2 \vdash A$.

2. If $\Sigma \vdash \Delta_1, X, \Delta_2$, then $\Sigma \vdash \Delta_1, \Delta_2$.

3. If $\Sigma \mid \Delta_1, X, \Delta_2 \vdash_C c : A \rightsquigarrow B$ and $X \notin \mathrm{ftv}(c)$, then $\Sigma \mid \Delta_1, \Delta_2 \vdash_C c : A \rightsquigarrow B$.

Proof. 1. By induction on A.

2. By induction on Δ_2.

3. By straightforward induction on $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_C \ c : A \rightsquigarrow B$.

□

Lemma I.3 (Coercion Translation Preserves Typing)**.** If $\Sigma \mid \Gamma \vdash_C \ c : A \rightsquigarrow B$ and $\Delta = \{X_1, \ldots, X_n\} \subseteq \mathrm{dom}(\Gamma)$, then $\Sigma \mid \Sigma(\Gamma) \setminus \Delta \vdash_S \ |c|_{\Gamma\setminus\Delta} : \Sigma(A[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow \Sigma(B[X_1 := \star, \cdots, X_n := \star])$.

Proof. The proof is by induction on the derivation of $\Sigma \mid \Gamma \vdash_C \ c : A \rightsquigarrow B$ with case analysis on the last rule applied.

Case (Ct_Id_C): By Lemma I.1.

Case (Ct_Inj_C): We are given

$$c = G!, \quad A = G, \quad B = \star, \quad \vdash \Sigma, \quad \Sigma \vdash \Gamma, \quad \Sigma \mid \Gamma \vdash G \quad (\exists G) .$$

We have two cases.

Case $G = X \notin \Gamma \setminus \Delta$: We have $|G!|_{\Gamma\setminus\Delta} = \mathsf{id}$. By $\Sigma \mid \Gamma \vdash X$, we have $X \in \Gamma$, and thus $X \in \Delta$. It suffices to show $\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S \ \mathsf{id} : (X[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow \star$, which follows from (Ct_Id_S).

Otherwise: We have $X \in \Gamma \setminus \Delta$ and $|G!|_{\Gamma\setminus\Delta} = |\mathsf{id}_G|_{\Gamma\setminus\Delta} \,;\, G!$. By (Ct_Id_C), $\Sigma \mid \Gamma \vdash_C \ \mathsf{id}_G : G \rightsquigarrow G$. Therefore, by Lemma I.1, $|\mathsf{id}_G|_{\Gamma\setminus\Delta}$ is a well-defined ground coercion and $\Sigma \mid \Sigma(\Gamma) \setminus \Delta \vdash_S \ |\mathsf{id}_G|_{\Gamma\setminus\Delta} : \Sigma(G[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow \Sigma(G[X_1 := \star, \cdots, X_n := \star])$. But, $G[X_1 := \star, \cdots, X_n := \star] = G$ because G cannot be a type variable $X_i \in \Delta$. Thus, by (Ct_Inj_S), $\Sigma \mid \Sigma(\Gamma) \setminus \Delta \vdash_S \ |\mathsf{id}_G|_{\Gamma\setminus\Delta} \,;\, G! : \Sigma(G) \rightsquigarrow \star$. Since $\Sigma(\star) = \star$, we have $\Sigma \mid \Sigma(\Gamma) \setminus \Delta \vdash_S \ |\mathsf{id}_G|_{\Gamma\setminus\Delta} \,;\, G! : \Sigma(G) \rightsquigarrow \Sigma(\star)$.

Case (Ct_Proj_C): Similarl to the case of (Ct_Inj_C).

Case (Ct_Conceal_C): We are given

$$c = \alpha^-, \quad A = \mathbb{A}, \quad B = \alpha, \quad \vdash \Sigma, \quad \Sigma \vdash \Gamma, \quad \alpha := \mathbb{A} \in \Sigma \quad (\exists \alpha) .$$

We have $|\alpha^-|_{\Gamma\setminus\Delta} = |\mathsf{id}_\alpha|_{\Gamma\setminus\Delta} = \mathsf{id}$. Then, by (Tw_Name), $\Sigma \mid \Gamma \setminus \Delta \vdash \alpha$, and so, by Lemma D.10 (2), we have $\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash \alpha$. Furthermore, Lemma D.10 (5) implies $\emptyset \vdash \Sigma(\Gamma \setminus \Delta)$. Hence, by (Ct_Id_S), we have $\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S \ \mathsf{id} : \Sigma(\alpha) \rightsquigarrow \Sigma(\alpha)$. Therefore, by Corollary D.11, $\Sigma(\alpha[X_1 := \star, \cdots, X_n := \star]) = \Sigma(\alpha) = \Sigma(\mathbb{A}) = \Sigma(\mathbb{A}[X_1 := \star, \cdots, X_n := \star]))$ (the last equality comes from $\vdash \Sigma$ and (Sw_Binding), and so we have $\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S \ \mathsf{id} : \Sigma(\mathbb{A}) \rightsquigarrow \Sigma(\alpha)$.

Case (Ct_Reveal_C): Provable similarly to the case of (Ct_Conceal_C).

Case (Ct_Fail_C): We are given

$$c = \bot^p_{A \rightsquigarrow B}, \quad \vdash \Sigma, \quad \Sigma \vdash \Gamma, \quad \Sigma \mid \Gamma \vdash A, \quad \Sigma \mid \Gamma \vdash B \quad (\exists p) .$$

By Lemma D.5 (1), we have $\Sigma \mid \Gamma \setminus \Delta \vdash A[X_1 := \star, \cdots, X_n := \star]$ and $\Sigma \mid \Gamma \setminus \Delta \vdash B[X_1 := \star, \cdots, X_n := \star]$. Then, by Lemma D.10 (5), we have $\emptyset \vdash \Sigma(\Gamma) \setminus \Delta$ and, by Lemma D.10 (3), we have $\emptyset \mid \Sigma(\Gamma) \setminus \Delta \vdash \Sigma(A[X_1 := \star, \cdots, X_n := \star])$ and $\emptyset \mid \Sigma(\Gamma) \setminus \Delta \vdash \Sigma(B[X_1 := \star, \cdots, X_n := \star])$. Because $|\bot^p_{A\rightsquigarrow B}|_{\Gamma\setminus\Delta} = \bot^p$, we have, by (Ct_Fail_S), $\Sigma \mid \Sigma(\Gamma) \setminus \Delta \vdash_S \ \bot^p : \Sigma(A[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow \Sigma(B[X_1 := \star, \cdots, X_n := \star])$.

Case (Ct_Arrow_C): We are given

$$\begin{gathered} c = c' \to d', \quad A = A' \to B', \quad B = A'' \to B'', \\ \Sigma \mid \Gamma \vdash_C \ c' : A'' \rightsquigarrow A', \quad \Sigma \mid \Gamma \vdash_C \ d' : B' \rightsquigarrow B'' \quad (\exists A', A'', B', B'', c', d') . \end{gathered}$$

We have $|c' \to d'|_{\Gamma\setminus\Delta} = |c'|_{\Gamma\setminus\Delta} \to |d'|_{\Gamma\setminus\Delta}$. By the IHs,

$$\begin{gathered} \Sigma \mid \Sigma(\Gamma) \setminus \Delta \vdash_S \ |c'|_{\Gamma\setminus\Delta} : \Sigma(A''[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow \Sigma(A'[X_1 := \star, \cdots, X_n := \star]) \\ \Sigma \mid \Sigma(\Gamma) \setminus \Delta \vdash_S \ |d'|_{\Gamma\setminus\Delta} : \Sigma(B'[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow \Sigma(B''[X_1 := \star, \cdots, X_n := \star]) . \end{gathered}$$

Thus, by (Ct_Arrow_S), $\Sigma \mid \Sigma(\Gamma) \setminus \Delta \vdash_S \ |c'|_{\Gamma\setminus\Delta} \to |d'|_{\Gamma\setminus\Delta} : (\Sigma(A'[X_1 := \star, \cdots, X_n := \star]) \to \Sigma(B'[X_1 := \star, \cdots, X_n := \star])) \rightsquigarrow (\Sigma(A''[X_1 := \star, \cdots, X_n := \star]) \to \Sigma(B''[X_1 := \star, \cdots, X_n := \star]))$. By the definiton of $\Sigma(A)$, we have $\Sigma \mid \Sigma(\Gamma) \setminus \Delta \vdash_S \ |c|_{\Gamma\setminus\Delta} : \Sigma(A[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow \Sigma(B[X_1 := \star, \cdots, X_n := \star])$.

Case (CT_SEQ_C): We are given

$$c = c' \,;\, d', \quad \Sigma \mid \Gamma \vdash_C c' : A \rightsquigarrow C, \quad \Sigma \mid \Gamma \vdash_C d' : C \rightsquigarrow B \quad (\exists C, c', d') .$$

We have $|c' \,;\, d'|_{\Gamma\setminus\Delta} = |c'|_{\Gamma\setminus\Delta} \mathbin{⨟} |d'|_{\Gamma\setminus\Delta}$. By the IH,

$$\Sigma \mid \Sigma(\Gamma) \setminus \Delta \vdash_S |c'|_{\Gamma\setminus\Delta} : \Sigma(A[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow \Sigma(C[X_1 := \star, \cdots, X_n := \star])$$
$$\Sigma \mid \Sigma(\Gamma) \setminus \Delta \vdash_S |d'|_{\Gamma\setminus\Delta} : \Sigma(C[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow \Sigma(B[X_1 := \star, \cdots, X_n := \star]) .$$

Therefore, by Lemma E.24, $\Sigma \mid \Sigma(\Gamma) \setminus \Delta \vdash_S |c'|_{\Gamma\setminus\Delta} \mathbin{⨟} |d'|_{\Gamma\setminus\Delta} : \Sigma(A[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow \Sigma(B[X_1 := \star, \cdots, X_n := \star])$.

Case (CT_ALL_C): We are given

$$c = \forall X.c', \quad A = \forall X.A', \quad B = \forall X.B', \quad \Sigma \mid \Gamma, X \vdash_C c' : A' \rightsquigarrow B' \quad (\exists X, A', B', c') .$$

We have $|\forall X.c'|_{\Gamma\setminus\Delta} = \forall X.|c'|_{(\Gamma\setminus\Delta),X} \,,, |c'|_{\Gamma\setminus\Delta} = \forall X.|c'|_{(\Gamma,X)\setminus\Delta} \,,, |c'|_{(\Gamma,X)\setminus(\Delta,X)}$ (because $(\Gamma, X) \setminus \Delta = (\Gamma \setminus \Delta), X$ and $(\Gamma, X) \setminus (\Delta, X) = \Gamma \setminus \Delta$). By the IH,

$$\Sigma \mid \Sigma(\Gamma, X) \setminus \Delta \vdash_S |c'|_{(\Gamma,X)\setminus\Delta} : \Sigma(A'[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow \Sigma(B'[X_1 := \star, \cdots, X_n := \star])$$
$$\Sigma \mid \Sigma(\Gamma, X) \setminus (\Delta, X) \vdash_S |c'|_{(\Gamma,X)\setminus(\Delta,X)} : \Sigma(A'[X_1 := \star, \cdots, X_n := \star][X := \star]) \rightsquigarrow \Sigma(B'[X_1 := \star, \cdots, X_n := \star][X := \star]) .$$

Thus, by (CT_ALL_S), $\Sigma \mid \Sigma(\Gamma) \setminus \Delta \vdash_S |\forall X.c'|_{\Gamma\setminus\Delta} : \Sigma(\forall X.A'[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow \Sigma(\forall X.B'[X_1 := \star, \cdots, X_n := \star])$.

□

Theorem I.4 (Translation Preserves Typing (Theorem 4.3 of the paper))**.** If $\Sigma \mid \Gamma \vdash_C M : A$, then $\Sigma \mid \Sigma(\Gamma) \vdash_S |M|_\Gamma : \Sigma(A)$.

Proof. By induction on the derivation of $\Sigma \mid \Gamma \vdash_C M : A$. We perform case analysis on the rule applied last to derive $\Sigma \mid \Gamma \vdash_C M : A$.

Case (T_CONST_C): We are given

$$M = k, \quad \vdash \Sigma, \quad \Sigma \vdash \Gamma, \quad ty(k) = A \quad (\exists k) .$$

We have $|k|_\Gamma = k$. By Lemma D.10 (5), $\emptyset \vdash \Sigma(\Gamma)$. Since A is a constant type, by $A = \Sigma(A)$, $ty(k) = \Sigma(A)$. Therefore, by (T_CONST_S), we have $\Sigma \mid \Sigma(\Gamma) \vdash_S k : \Sigma(A)$.

Case (T_VAR_C): We are given

$$M = x, \quad \vdash \Sigma, \quad \Sigma \vdash \Gamma, \quad x : A \in \Gamma \quad (\exists x) .$$

We have $|x|_\Gamma = x$. By Lemma D.10 (5), $\emptyset \vdash \Sigma(\Gamma)$. By the definition of $\Sigma(\Gamma)$, $x : \Sigma(A) \in \Sigma(\Gamma)$. By (T_VAR_S), $\Sigma \mid \Sigma(\Gamma) \vdash_S x : \Sigma(A)$.

Case (T_ABS_C): We are given

$$M = \lambda x\!:\!A'.M', \quad A = A' \to B, \quad \Sigma \mid \Gamma, x : A' \vdash_C M' : B \quad (\exists A', B, x, M') .$$

We have $|\lambda x\!:\!A'.M'|_\Gamma = \lambda x\!:\!A'.|M'|_{\Gamma,x:A'}$. By the IH, $\Sigma \mid \Sigma(\Gamma), x : \Sigma(A') \vdash_S |M'|_{\Gamma,x:A} : \Sigma(B)$. By (T_ABS_S), $\Sigma \mid \Sigma(\Gamma) \vdash_S \lambda x\!:\!A'.|M'|_{\Gamma,x:A'} : \Sigma(A') \to \Sigma(B)$. By the definition of $\Sigma(A \to B)$, we have $\Sigma \mid \Sigma(\Gamma) \vdash_S \lambda x\!:\!A'.|M'|_{\Gamma,x:A'} : \Sigma(A' \to B)$.

Case (T_APP_C): We are given

$$M = M_1\, M_2, \quad \Sigma \mid \Gamma \vdash_C M_1 : B \to A, \quad \Sigma \mid \Gamma \vdash_C M_2 : B \quad (\exists B, M_1, M_2) .$$

We have $|M_1\, M_2|_\Gamma = |M_1|_\Gamma\, |M_2|_\Gamma$. By the IHs, $\Sigma \mid \Sigma(\Gamma) \vdash_S |M_1|_\Gamma : \Sigma(B \to A)$ and $\Sigma \mid \Gamma \vdash_S |M_2|_\Gamma : \Sigma(B)$. By the definition of $\Sigma(B \to A)$, we have $\Sigma \mid \Sigma(\Gamma) \vdash_S |M_1|_\Gamma : \Sigma(B) \to \Sigma(A)$. Therefore, by (T_APP_S), $\Sigma \mid \Sigma(\Gamma) \vdash_S |M_1|_\Gamma\, |M_2|_\Gamma : \Sigma(A)$.

Case (T_TYABS_C): We are given

$$M = \Lambda X.(M' : A'), \quad A = \forall X.A', \quad \Sigma \mid \Gamma, X \vdash_C M' : A' \quad (\exists X, A', M') .$$

We have $|\Lambda X.(M' : A')|_\Gamma = \Lambda X.|M'|_{\Gamma,X}$. By the IH, $|M'|_{\Gamma,X}$ is well defined, and $\Sigma \mid \Sigma(\Gamma), X \vdash_S |M'|_{\Gamma,X} : \Sigma(A')$. Therefore, by (T_TYABS_S), $\Sigma \mid \Sigma(\Gamma) \vdash_S \Lambda X.|M'|_{\Gamma,X} : \forall X.\Sigma(A')$. By the definition of $\Sigma(\forall X.A)$, we have $\Sigma \mid \Sigma(\Gamma) \vdash_S \Lambda X.|M'|_{\Gamma,X} : \Sigma(\forall X.A')$.

Case (T_TYAPP_C): We are given

$$M = M' \, A', \quad A = B[X := A'], \quad \Sigma \mid \Gamma \vdash_C M' : \forall X.B, \quad \Sigma \mid \Gamma \vdash A' \quad (\exists X, A', B, M') .$$

We have $|M' \, A'|_\Gamma = |M'|_\Gamma \, A'$. By the IH, $\Sigma \mid \Sigma(\Gamma) \vdash_S |M'|_\Gamma : \Sigma(\forall X.B)$. By the definition of $\Sigma(\forall X.A)$, we have $\Sigma \mid \Sigma(\Gamma) \vdash_S |M'|_\Gamma : \forall X.\Sigma(B)$. Therefore, by (T_TYAPP_S), $\Sigma \mid \Sigma(\Gamma) \vdash_S |M'|_\Gamma \, A' : \Sigma(B)[X := \Sigma(A')]$, and so, by the definition of $\Sigma(B[X := A'])$, we have $\Sigma \mid \Sigma(\Gamma) \vdash_S |M'|_\Gamma \, A' : \Sigma(B[X := A'])$.

Case (T_BLAME_C): We are given

$$M = \mathsf{blame}\, p, \quad \vdash \Sigma, \quad \Sigma \vdash \Gamma, \quad \Sigma \mid \Gamma \vdash A \quad (\exists p) .$$

We have $|\mathsf{blame}\, p|_\Gamma = \mathsf{blame}\, p$. By Lemma D.10 (5), $\emptyset \vdash \Sigma(\Gamma)$. By Lemma D.10 (3), $\emptyset \mid \Sigma(\Gamma) \vdash \Sigma(A)$. Therefore, by (T_BLAME_S), $\Sigma \mid \Sigma(\Gamma) \vdash_S \mathsf{blame}\, p : \Sigma(A)$.

Case (T_CRC_C): We are given

$$M = M'\langle c \rangle, \quad \Sigma \mid \Gamma \vdash_C M' : B, \quad \Sigma \mid \Gamma \vdash_C c : B \rightsquigarrow A \quad (\exists B, c, M') .$$

We have $|M'\langle c \rangle|_\Gamma = |M'|_\Gamma \langle |c|_\Gamma \rangle$. By the IH, $\Sigma \mid \Sigma(\Gamma) \vdash_S |M'|_\Gamma : \Sigma(B)$. By Lemma I.3, we have $\Sigma \mid \Sigma(\Gamma) \vdash_S |c|_\Gamma : \Sigma(B) \rightsquigarrow \Sigma(A)$. Thus, by (T_CRC_S), we have $\Sigma \mid \Sigma(\Gamma) \vdash_S |M'|_\Gamma \langle |c|_\Gamma \rangle : \Sigma(A)$.

□

I.2 Proof of Theorem 4.4

Lemma I.5 (Identity Coercions as The Left Unit)**.** If $\Sigma \mid \Gamma \vdash_S t : A \rightsquigarrow B$, then $\mathsf{id} \fatsemi t = t$.

Proof. By induction on the structure of type t.

Case $t = \perp^p$ $(\exists p)$: $\mathsf{id} \fatsemi \perp^p = \perp^p$.

Case $t = \mathsf{id}$: $\mathsf{id} \fatsemi \mathsf{id} = \mathsf{id}$.

Case $t = h \,;\, H!$ $(\exists h, H)$: $\mathsf{id} \fatsemi (h \,;\, H!) = (\mathsf{id} \fatsemi h) \,;\, H! = h \,;\, H!$.

Otherwise: $\mathsf{id} \fatsemi t = t$.

□

Lemma I.6 (Identity Coercions as The Right Unit)**.** $s \fatsemi \mathsf{id} = s$.

Proof. By induction on the structure of type s.

Case $s = G?^p \,;\, b$ $(\exists G, p, b)$: By the IH, $b \fatsemi \mathsf{id} = b$. Therefore, $(G?^p \,;\, b) \fatsemi \mathsf{id} = G?^p \,;\, (b \fatsemi \mathsf{id}) = G?^p \,;\, b$.

Case $s = \perp^p$ $(\exists p)$: $\perp^p \fatsemi \mathsf{id} = \perp^p$.

Case $s = i$ $(\exists i)$: $i \fatsemi \mathsf{id} = i$.

□

Lemma I.7 (Composition is Associative)**.** If $\Sigma \mid \Gamma \vdash_S s_1 : A \rightsquigarrow B$ and $\Sigma \mid \Gamma \vdash_S s_2 : B \rightsquigarrow C$ and $\Sigma \mid \Gamma \vdash_S s_3 : C \rightsquigarrow D$, then $(s_1 \fatsemi s_2) \fatsemi s_3 = s_1 \fatsemi (s_2 \fatsemi s_3)$.

Proof. [YT: The proof implicitly requires $\fatsemi$ determinacy. Can we prove this?] By induction on the total sum of the sizes of s_1, s_2, and s_3. First, we consider the cases where either s_1, s_2, or s_3 is an identity coercion. ⟵

Case $s_1 = \mathsf{id}$: By Lemma I.5,

$$\begin{aligned}(\mathsf{id} \fatsemi s_2) \fatsemi s_3 &= s_2 \fatsemi s_3,\\ \mathsf{id} \fatsemi (s_2 \fatsemi s_3) &= s_2 \fatsemi s_3\ .\end{aligned}$$

Case $s_2 = \mathsf{id}$: By Lemma I.5 and Lemma I.6,

$$\begin{aligned}(s_1 \fatsemi \mathsf{id}) \fatsemi s_3 &= s_1 \fatsemi s_3,\\ s_1 \fatsemi (\mathsf{id} \fatsemi s_3) &= s_1 \fatsemi s_3\ .\end{aligned}$$

Case $s_3 = \mathsf{id}$: By Lemma I.6,

$$\begin{aligned}(s_1 \fatsemi s_2) \fatsemi \mathsf{id} &= s_1 \fatsemi s_2,\\ s_1 \fatsemi (s_2 \fatsemi \mathsf{id}) &= s_1 \fatsemi s_2\ .\end{aligned}$$

Next, we consider the cases where neither s_1, s_2, or s_3 is an identity coercion. By case analysis on whether s_1 is an intermediate coercion or not.

Case $s_1 = G_1?^{p_1} ; b_1$ $(\exists G_1, p_1, b_1)$: Because $\Sigma \mid \Gamma \vdash_S G_1?^{p_1} ; b_1 : A \rightsquigarrow B$ is derived by (Ct_Proj_S), we have

$$A = \star, \quad \Sigma \mid \Gamma \vdash_S b_1 : \Sigma(G_1) \rightsquigarrow B\ .$$

Then, by the IH, $(b_1 \fatsemi s_2) \fatsemi s_3 = b_1 \fatsemi (s_2 \fatsemi s_3)$. Therefore,

$$\begin{aligned}((G_1?^{p_1} ; b_1) \fatsemi s_2) \fatsemi s_3 &= (G_1?^{p_1} ; (b_1 \fatsemi s_2)) \fatsemi s_3\\ &= G_1?^{p_1} ; ((b_1 \fatsemi s_2) \fatsemi s_3),\\ (G_1?^{p_1} ; b_1) \fatsemi (s_2 \fatsemi s_3) &= G_1?^{p_1} ; (b_1 \fatsemi (s_2 \fatsemi s_3))\\ &= G_1?^{p_1} ; ((b_1 \fatsemi s_2) \fatsemi s_3)\ .\end{aligned}$$

Case $s_1 = \bot^{p_1}$ $(\exists p_1)$:

$$\begin{aligned}(\bot^{p_1} \fatsemi s_2) \fatsemi s_3 &= \bot^{p_1} \fatsemi s_3\\ &= \bot^{p_1}\ ,\\ \bot^{p_1} \fatsemi (s_2 \fatsemi s_3) &= \bot^{p_1}\ .\end{aligned}$$

Case $s_1 = i_1$ $(\exists i_1)$: By case analysis on whether s_2 is an intermediate coercion or not.

Case $s_2 = G_2?^{p_2} ; b_2$: Because $\Sigma \mid \Gamma \vdash_S G_2?^{p_2} ; b_2 : B \rightsquigarrow C$ is derived by (Ct_Proj_S), we have

$$B = \star, \quad \Sigma \mid \Gamma \vdash_S b_2 : \Sigma(G_2) \rightsquigarrow C\ .$$

Since i_1 is not an identity coercion, $\Sigma \mid \Gamma \vdash i_1 : A \rightsquigarrow \star$ is derived by (Ct_Inj_S). Therefore,

$$i_1 = g_1 ; G_1!, \quad \Sigma \mid \Gamma \vdash_S g_1 : A \rightsquigarrow \Sigma(G_1) \quad (\exists g_1, G_1)\ .$$

By case analysis on whether $G_1 = G_2$ or not.

Case $G_1 = G_2$: By the IH, we have $(g_1 \fatsemi b_2) \fatsemi s_3 = g_1 \fatsemi (b_2 \fatsemi s_3)$. Therefore,

$$\begin{aligned}((g_1 ; G_1!) \fatsemi (G_1?^{p_2} ; b_2)) \fatsemi s_3 &= (g_1 \fatsemi b_2) \fatsemi s_3\\ &= g_1 \fatsemi (b_2 \fatsemi s_3),\\ (g_1 ; G_1!) \fatsemi ((G_1?^{p_2} ; b_2) \fatsemi s_3) &= (g_1 ; G_1!) \fatsemi (G_1?^{p_2} ; (b_2 \fatsemi s_3))\\ &= g_1 ; (b_2 \fatsemi s_3)\ .\end{aligned}$$

Case $G_1 \neq G_2$:

$$\begin{aligned}((g_1 \mathbin{;} G_1!) \fatsemi (G_2?^{p_2} \mathbin{;} b_2)) \fatsemi s_3 &= \bot^{p_2} \fatsemi s_3 \\ &= \bot^{p_2}, \\ (g_1 \mathbin{;} G_1!) \fatsemi ((G_2?^{p_2} \mathbin{;} b_2) \fatsemi s_3) &= (g_1 \mathbin{;} G_1!) \fatsemi (G_2?^{p_2} \mathbin{;} (b_2 \fatsemi s_3)) \\ &= \bot^{p_2} \ .\end{aligned}$$

Case $s_2 = \bot^{p_2}$ ($\exists p_2$):

$$\begin{aligned}(i_1 \fatsemi \bot^{p_2}) \fatsemi s_3 &= \bot^{p_2} \fatsemi s_3 = \bot^{p_2} \ , \\ i_1 \fatsemi (\bot^{p_2} \fatsemi s_3) &= i_1 \fatsemi \bot^{p_2} = \bot^{p_2} \ .\end{aligned}$$

Case $s_2 = i_2$ ($\exists i_2$): By case analysis on whether s_3 is an intermediate coercion or not.

Case $s_3 = G_3?^{p_3} \mathbin{;} b_3$ ($\exists G_3, p_3, b_3$): Because $\Sigma \mid \Gamma \vdash_S G_3?^{p_3} \mathbin{;} b_3 : C \rightsquigarrow D$ is derived by (Ct_Proj_S), we have

$$C = \star, \quad \Sigma \mid \Gamma \vdash_S b_3 : \Sigma(G_3) \rightsquigarrow D \ .$$

Since i_2 is not an identity coercion, $\Sigma \mid \Gamma \vdash i_2 : B \rightsquigarrow \star$ is derived by (Ct_Inj_S). Therefore,

$$i_2 = g_2 \mathbin{;} G_2!, \quad \Sigma \mid \Gamma \vdash_S g_2 : B \rightsquigarrow \Sigma(G_1) \quad (\exists g_2, G_2) \ .$$

By case analysis on whether $G_2 = G_3$ or not.

Case $G_2 = G_3$: By the IH, we have $(i_1 \fatsemi g_2) \fatsemi b_3 = i_1 \fatsemi (g_2 \fatsemi b_3)$. Therefore,

$$\begin{aligned}(i_1 \fatsemi (g_2 \mathbin{;} G_2!)) \fatsemi (G_2?^{p_3} \mathbin{;} b_3) &= ((i_1 \fatsemi g_2) \mathbin{;} G_2!) \fatsemi (G_2?^{p_3} \mathbin{;} b_3) \\ &= (i_1 \fatsemi g_2) \fatsemi b_3 \ , \\ i_1 \fatsemi ((g_2 \mathbin{;} G_2!) \fatsemi (G_2?^{p_3} \mathbin{;} b_3)) &= i_1 \fatsemi (g_2 \fatsemi b_3) \\ &= (i_1 \fatsemi g_2) \fatsemi b_3 \ .\end{aligned}$$

Case $G_2 \neq G_3$:

$$\begin{aligned}(i_1 \fatsemi (g_2 \mathbin{;} G_2!)) \fatsemi (G_3?^{p_3} \mathbin{;} b_3) &= ((i_1 \fatsemi g_2) \mathbin{;} G_2!) \fatsemi (G_3?^{p_3} \mathbin{;} b_3) \\ &= \bot^{p_3} \ , \\ i_1 \fatsemi ((g_2 \mathbin{;} G_2!) \fatsemi (G_3?^{p_3} \mathbin{;} b_3)) &= i_1 \fatsemi \bot^{p_3} \\ &= \bot^{p_3} \ .\end{aligned}$$

Case $s_3 = \bot^{p_3}$ ($\exists p_3$):

$$\begin{aligned}(i_1 \fatsemi i_2) \fatsemi \bot^{p_3} &= \bot^{p_3} \ , \\ i_1 \fatsemi (i_2 \fatsemi \bot^{p_3}) &= i_1 \fatsemi \bot^{p_3} = \bot^{p_3} \ .\end{aligned}$$

Case $s_3 = i_3$ ($\exists i_3$): (Proof follows in the next paragraph.)

Next, we consider the cases where s_1, s_2, or s_3 are intermediate coercions, but neither of them is not an identity coercion. By case analysis on whether i_1 is a ground coercion or not.

Case $i_1 = g_1 \mathbin{;} G_1!$ ($\exists g_1, G_1$): Since $\Sigma \mid \Gamma \vdash_S g_1 \mathbin{;} G_1! : A \rightsquigarrow B$ is derived by (Ct_Inj_S), we have

$$B = \star, \quad \Sigma \mid \Gamma \vdash_S g_1 : A \rightsquigarrow \Sigma(G_1) \ .$$

By case analysis on i_2.

Case $i_2 = g_2 \mathbin{;} G_2!$ ($\exists g_2, G_2$): Since $\Sigma \mid \Gamma \vdash_S g_2 \mathbin{;} G_2! : \star \rightsquigarrow C$ is derived by (Ct_Inj_S), we have

$$C = \star, \quad \Sigma \mid \Gamma \vdash_S g_2 : \star \rightsquigarrow \Sigma(G_2) \ .$$

By case analysis on i_3.

Case $i_3 = g_3 \mathbin{;} G_3!$ $(\exists g_3, G_3)$: Since $\Sigma \mid \Gamma \vdash_S g_3 \mathbin{;} G_3! : \star \rightsquigarrow D$ is derived by (Ct_Inj_S), we have

$$D = \star, \quad \Sigma \mid \Gamma \vdash_S g_3 : \star \rightsquigarrow \Sigma(G_3) \ .$$

By the IH, $(g_1 \mathbin{;} G_1!) \mathbin{⨟} ((g_2 \mathbin{;} G_2!) \mathbin{⨟} g_3) = ((g_1 \mathbin{;} G_1!) \mathbin{⨟} (g_2 \mathbin{;} G_2!)) \mathbin{⨟} g_3$. Therefore,

$$\begin{aligned}
((g_1 \mathbin{;} G_1!) \mathbin{⨟} (g_2 \mathbin{;} G_2!)) \mathbin{⨟} (g_3 \mathbin{;} G_3!) &= (((g_1 \mathbin{;} G_1!) \mathbin{⨟} g_2) \mathbin{;} G_2!) \mathbin{⨟} (g_3 \mathbin{;} G_3!) \\
&= ((((g_1 \mathbin{;} G_1!) \mathbin{⨟} g_2) \mathbin{;} G_2!) \mathbin{⨟} g_3) \mathbin{;} G_3! \ , \\
(g_1 \mathbin{;} G_1!) \mathbin{⨟} ((g_2 \mathbin{;} G_2!) \mathbin{⨟} (g_3 \mathbin{;} G_3!)) &= (g_1 \mathbin{;} G_1!) \mathbin{⨟} (((g_2 \mathbin{;} G_2!) \mathbin{⨟} g_3) \mathbin{;} G_3!) \\
&= ((g_1 \mathbin{;} G_1!) \mathbin{⨟} ((g_2 \mathbin{;} G_2!) \mathbin{⨟} g_3)) \mathbin{;} G_3! \\
&= (((g_1 \mathbin{;} G_1!) \mathbin{⨟} (g_2 \mathbin{;} G_2!)) \mathbin{⨟} g_3) \mathbin{;} G_3! \\
&= ((((g_1 \mathbin{;} G_1!) \mathbin{⨟} g_2) \mathbin{;} G_2!) \mathbin{⨟} g_3) \mathbin{;} G_3! \ .
\end{aligned}$$

Otherwise: Contradictory because C cannot be a dynamic type.

Otherwise: Contradictory because B cannot be a dynamic type.

Case $i_1 = g_1$ $(\exists g_1)$: By case analysis on whether i_2 is a ground coercion or not.

Case $i_2 = g_2 \mathbin{;} G_2!$ $(\exists g_2, G_2)$: Since $\Sigma \mid \Gamma \vdash_S g_2 \mathbin{;} G_2! : B \rightsquigarrow C$ is derived by (Ct_Inj_S), we have

$$C = \star, \quad \Sigma \mid \Gamma \vdash_S g_2 : B \rightsquigarrow \Sigma(G_2) \ .$$

By case analysis on i_3.

Case $i_3 = g_3 \mathbin{;} G_3!$ $(\exists g_3, G_3)$: Since $\Sigma \mid \Gamma \vdash_S g_3 \mathbin{;} G_3! : \star \rightsquigarrow D$ is derived by (Ct_Inj_S), we have

$$D = \star, \quad \Sigma \mid \Gamma \vdash_S g_3 : \star \rightsquigarrow \Sigma(G_3) \ .$$

Therefore,

$$\begin{aligned}
(g_1 \mathbin{⨟} (g_2 \mathbin{;} G_2!)) \mathbin{⨟} (g_3 \mathbin{;} G_3!) &= ((g_1 \mathbin{⨟} g_2) \mathbin{;} G_2!) \mathbin{⨟} (g_3 \mathbin{;} G_3!) \\
&= (((g_1 \mathbin{⨟} g_2) \mathbin{;} G_2!) \mathbin{⨟} g_3) \mathbin{;} G_3! \ , \\
g_1 \mathbin{⨟} ((g_2 \mathbin{;} G_2!) \mathbin{⨟} (g_3 \mathbin{;} G_3!)) &= g_1 \mathbin{⨟} (((g_2 \mathbin{;} G_2!) \mathbin{⨟} g_3) \mathbin{;} G_3!) \\
&= (g_1 \mathbin{⨟} ((g_2 \mathbin{;} G_2!) \mathbin{⨟} g_3)) \mathbin{;} G_3! \\
&= ((g_1 \mathbin{⨟} (g_2 \mathbin{;} G_2!)) \mathbin{⨟} g_3) \mathbin{;} G_3! \\
&= (((g_1 \mathbin{⨟} g_2) \mathbin{;} G_2!) \mathbin{⨟} g_3) \mathbin{;} G_3! \ .
\end{aligned}$$

Otherwise: Contradictory because C cannot be a dynamic type.

Case $i_2 = g_2$ $(\exists g_2)$: By case analysis on whether i_3 is a ground coercion or not.

Case $i_3 = g_3 \mathbin{;} G_3!$ $(\exists g_3, G_3)$: By the IH, $(g_1 \mathbin{⨟} g_2) \mathbin{⨟} g_3 = g_1 \mathbin{⨟} (g_2 \mathbin{⨟} g_3)$. Therefore,

$$\begin{aligned}
(g_1 \mathbin{⨟} g_2) \mathbin{⨟} (g_3 \mathbin{;} G_3!) &= ((g_1 \mathbin{⨟} g_2) \mathbin{⨟} g_3) \mathbin{;} G_3! \ , \\
g_1 \mathbin{⨟} (g_2 \mathbin{⨟} (g_3 \mathbin{;} G_3!)) & \\
&= g_1 \mathbin{⨟} ((g_2 \mathbin{⨟} g_3) \mathbin{;} G_3!) \\
&= (g_1 \mathbin{⨟} (g_2 \mathbin{⨟} g_3)) \mathbin{;} G_3! \\
&= ((g_1 \mathbin{⨟} g_2) \mathbin{⨟} g_3) \mathbin{;} G_3! \ .
\end{aligned}$$

Case $i_3 = g_3$: (Proof follows in the next paragraph.)

Finally, we consider the cases where s_1, s_2, and s_3 are ground coercions, but neither of them is not an identity coercion. By case analysis on g_1.

Case $g_1 = s'_1 \rightarrow t'_1\ (\exists s'_1, t'_1)$: Since $\Sigma \mid \Gamma \vdash_S s'_1 \rightarrow t'_1 : A \rightsquigarrow B$ is derived by (Ct_Arrow_S), we have

$$A = A_1 \rightarrow B_1, \quad B = A_2 \rightarrow B_2, \quad \Sigma \mid \Gamma \vdash_S s'_1 : A_2 \rightsquigarrow A_1, \quad \Sigma \mid \Gamma \vdash_S t'_1 : B_1 \rightsquigarrow B_2 \quad (\exists A_1, B_1, A_2, B_2)\ .$$

By case analysis on s_2.

Case $g_2 = s'_2 \rightarrow t'_2\ (\exists s'_2, t'_2)$: Since $\Sigma \mid \Gamma \vdash_S s'_2 \rightarrow t'_2 : B \rightsquigarrow C$ is derived by (Ct_Arrow_S), we have

$$C = A_3 \rightarrow B_3, \quad \Sigma \mid \Gamma \vdash_S s'_2 : A_3 \rightsquigarrow A_2, \quad \Sigma \mid \Gamma \vdash_S t'_2 : B_2 \rightsquigarrow B_3 \quad (\exists A_3, B_3)\ .$$

By case analysis on s_3.

Case $g_3 = s'_3 \rightarrow t'_3\ (\exists s'_3, t'_3)$: Since $\Sigma \mid \Gamma \vdash_S s'_3 \rightarrow t'_3 : C \rightsquigarrow D$ is derived by (Ct_Arrow_S), we have

$$D = A_4 \rightarrow B_4, \quad \Sigma \mid \Gamma \vdash_S s'_3 : A_4 \rightsquigarrow A_3, \quad \Sigma \mid \Gamma \vdash_S t'_3 : B_3 \rightsquigarrow B_4 \quad (\exists A_4, B_4)\ .$$

By the IHs, $(s'_3 \fatsemi s'_2) \fatsemi s'_1 = s'_3 \fatsemi (s'_2 \fatsemi s'_1)$ and $(t'_1 \fatsemi t'_2) \fatsemi t'_3 = t'_1 \fatsemi (t'_2 \fatsemi t'_3)$. Therefore,

$$\begin{aligned}
((s'_1 \rightarrow t'_1) \fatsemi (s'_2 \rightarrow t'_2)) \fatsemi (s'_3 \rightarrow t'_3) &= ((s'_2 \fatsemi s'_1) \rightarrow (t'_1 \fatsemi t'_2)) \fatsemi (s'_3 \rightarrow t'_3) \\
&= (s'_3 \fatsemi (s'_2 \fatsemi s'_1)) \rightarrow ((t'_1 \fatsemi t'_2) \fatsemi t'_3) \\
&= ((s'_3 \fatsemi s'_2) \fatsemi s'_1) \rightarrow ((t'_1 \fatsemi t'_2) \fatsemi t'_3)\ , \\
(s'_1 \rightarrow t'_1) \fatsemi ((s'_2 \rightarrow t'_2) \fatsemi (s'_3 \rightarrow t'_3)) &= (s'_1 \rightarrow t'_1) \fatsemi ((s'_3 \fatsemi s'_2) \rightarrow (t'_2 \fatsemi t'_3)) \\
&= ((s'_3 \fatsemi s'_2) \fatsemi s'_1) \rightarrow (t'_1 \fatsemi (t'_2 \fatsemi t'_3)) \\
&= ((s'_3 \fatsemi s'_2) \fatsemi s'_1) \rightarrow ((t'_1 \fatsemi t'_2) \fatsemi t'_3)\ .
\end{aligned}$$

Otherwise: Contradictory because C cannot be a function type.

Otherwise: Contradictory because B cannot be a funcion type.

Case $g_1 = \forall X. s'_1 \,,, t'_1\ (\exists X, s'_1, t'_1)$: Since $\Sigma \mid \Gamma \vdash_S \forall X. s'_1 \,,, t'_1 : A \rightsquigarrow B$ is derived by (Ct_All_S), we have

$$\begin{gathered}
A = \forall X. A_1, \quad B = \forall X. B_1, \quad \Sigma \mid \Gamma, X \vdash_S s'_1 : A_1 \rightsquigarrow B_1, \\
\Sigma \mid \Gamma \vdash_S t'_1 : A_1[X := \star] \rightsquigarrow B_1[X := \star] \quad (\exists A_1, B_1)\ .
\end{gathered}$$

By case analysis on s_2.

Case $g_2 = \forall Y. s'_2 \,,, t'_2\ (\exists Y, s'_2, t'_2)$: Since $\Sigma \mid \Gamma \vdash_S \forall X. s'_2 \,,, t'_2 : B \rightsquigarrow C$ is derived by (Ct_All_S), we have

$$\begin{gathered}
Y = X, \quad C = \forall X. C_1, \quad \Sigma \mid \Gamma, X \vdash_S s'_2 : B_1 \rightsquigarrow C_1, \\
\Sigma \mid \Gamma \vdash_S t'_2 : B_1[X := \star] \rightsquigarrow C_1[X := \star] \quad (\exists C_1)\ .
\end{gathered}$$

By case analysis on s_3.

Case $g_3 = \forall Z. s'_3 \,,, t'_3\ (\exists Z, s'_3, t'_3)$: Since $\Sigma \mid \Gamma \vdash_S \forall X. s'_3 \,,, t'_3 : B \rightsquigarrow C$ is derived by (Ct_All_S), we have

$$\begin{gathered}
Z = X, \quad D = \forall X. D_1, \quad \Sigma \mid \Gamma, X \vdash_S s'_3 : C_1 \rightsquigarrow D_1, \\
\Sigma \mid \Gamma \vdash_S t'_3 : C_1[X := \star] \rightsquigarrow D_1[X := \star] \quad (\exists D_1)\ .
\end{gathered}$$

Then, by the IH, $(s'_1 \fatsemi s'_2) \fatsemi s'_3 = s'_1 \fatsemi (s'_2 \fatsemi s'_3)$ and $(t'_1 \fatsemi t'_2) \fatsemi t'_3 = t'_1 \fatsemi (t'_2 \fatsemi t'_3)$. Therefore,

$$\begin{aligned}
((\forall X. s'_1 \,,, t'_1) \fatsemi (\forall X. s'_2 \,,, t'_2)) \fatsemi (\forall X. s'_3 \,,, t'_3) &= (\forall X. (s'_1 \fatsemi s'_2) \,,, (t'_1 \fatsemi t'_2)) \fatsemi (\forall X. s'_3 \,,, t'_3) \\
&= \forall X. ((s'_1 \fatsemi s'_2) \fatsemi s'_3) \,,, ((t'_1 \fatsemi t'_2) \fatsemi t'_3), \\
(\forall X. s'_1 \,,, t'_1) \fatsemi ((\forall X. s'_2 \,,, t'_2) \fatsemi (\forall X. s'_3 \,,, t'_3)) &= (\forall X. s'_1 \,,, t'_1) \fatsemi (\forall X. (s'_2 \,,, t'_2 \fatsemi s'_3 \,,, t'_3)) \\
&= \forall X. (s'_1 \fatsemi (s'_2 \fatsemi s'_3)) \,,, (t'_1 \fatsemi (t'_2 \fatsemi t'_3)) \\
&= \forall X. ((s'_1 \fatsemi s'_2) \fatsemi s'_3) \,,, ((t'_1 \fatsemi t'_2) \fatsemi t'_3)\ .
\end{aligned}$$

Otherwise: Contradictory because C cannot be a polymorphic type.

Otherwise: Contradictory because B cannot be a polymorphic type.

□

Lemma I.8 ($\mathit{coerce}^{\pm}_{\alpha}(A)$ Generates a No-op Coercion)**.** $\mathit{coerce}^{+}_{\alpha}(A)$ and $\mathit{coerce}^{-}_{\alpha}(A)$ are no-op coercions.

Proof. By straightforward induction on A. □

Lemma I.9 (Type Name Substitution Preserves Well-formedness)**.** Suppose $\vdash \Sigma, \alpha := \mathbb{C}$.

1. If $\Sigma \mid \Gamma[\alpha := \mathbb{C}] \vdash A$, then $\Sigma, \alpha := \mathbb{C} \mid \Gamma \vdash A$.
2. If $\Sigma \vdash \Gamma[\alpha := \mathbb{C}]$, then $\Sigma, \alpha := \mathbb{C} \vdash \Gamma$.
3. If $\Sigma \mid \Gamma[\alpha := \mathbb{C}] \vdash_C c : A \rightsquigarrow B$, then $\Sigma, \alpha := \mathbb{C} \mid \Gamma \vdash_C c : A \rightsquigarrow B$.

Proof.
1. By straightforward induction on $\Sigma \mid \Gamma[\alpha := \mathbb{C}] \vdash A$.
2. By straightforward induction on $\Sigma \vdash \Gamma[\alpha := \mathbb{C}]$.
3. By straightforward induction on $\Sigma \mid \Gamma[\alpha := \mathbb{C}] \vdash_C c : A \rightsquigarrow B$.

□

Lemma I.10 (Noop Coercion Exists for a Well-formed Type)**.** If $\vdash \Sigma$ and $\Sigma \mid \Gamma \vdash A$, there exist no-op coercions c^I and d^I such that $\Sigma \mid \Gamma \vdash_C c^I : A \rightsquigarrow \Sigma(A)$ and $\Sigma \mid \Gamma \vdash_C d^I : \Sigma(A) \rightsquigarrow A$.

Proof. By induction on Σ.

Case $\Sigma = \emptyset$: Since $\Sigma(A) = A$, it suffices to take $c^I = d^I = \mathsf{id}_A$.

Case $\Sigma = \Sigma_0, \alpha := \mathbb{B}$: By Lemma D.9, $\Sigma_0 \mid \Gamma[\alpha := \mathbb{B}] \vdash A[\alpha := \mathbb{B}]$. By the IH, there exist c^I_0 and d^I_0 such that

$$\Sigma_0 \mid \Gamma[\alpha := \mathbb{B}] \vdash_C c^I_0 : A[\alpha := \mathbb{B}] \rightsquigarrow \Sigma_0(A[\alpha := \mathbb{B}])$$
$$\Sigma_0 \mid \Gamma[\alpha := \mathbb{B}] \vdash_C d^I_0 : \Sigma_0(A[\alpha := \mathbb{B}]) \rightsquigarrow A[\alpha := \mathbb{B}]\ .$$

By Lemma I.9 and $\Sigma(A) = \Sigma_0(A[\alpha := \mathbb{B}])$,

$$\Sigma \mid \Gamma \vdash_C c^I_0 : A[\alpha := \mathbb{B}] \rightsquigarrow \Sigma(A) \quad \Sigma \mid \Gamma \vdash_C d^I_0 : \Sigma(A) \rightsquigarrow A[\alpha := \mathbb{B}]\ .$$

We have $\vdash \Sigma$ and $\alpha := \mathbb{B} \in \Sigma$ and $\Sigma \vdash \Gamma, X$ and $\Sigma \mid \Gamma, X \vdash A[\alpha := X]$ (by induction on A), and, by Lemma E.17,

$$\Sigma \mid \Gamma \vdash_C \mathit{coerce}^{+}_{\alpha}(A) : A \rightsquigarrow A[\alpha := \mathbb{B}] \quad \Sigma \mid \Gamma \vdash_C \mathit{coerce}^{-}_{\alpha}(A) : A[\alpha := \mathbb{B}] \rightsquigarrow A\ .$$

Let c^I be $\mathit{coerce}^{+}_{\alpha}(A); c^I_0$ and d^I be $d^I_0; \mathit{coerce}^{-}_{\alpha}(A)$, which are no-op coercions by Lemma I.8. By (CT_Seq_C),

$$\Sigma \mid \Gamma \vdash_C \mathit{coerce}^{+}_{\alpha}(A)\ ;\ c^I_0 : A \rightsquigarrow \Sigma(A) \quad \Sigma \mid \Gamma \vdash_C d^I_0\ ;\ \mathit{coerce}^{-}_{\alpha}(A) : \Sigma(A) \rightsquigarrow A\ .$$

□

Lemma I.11 (No-Op Coercion Translates to Unit)**.** Suppose $\Sigma \mid \Gamma \vdash_C c^I : A \rightsquigarrow B$ and $\Delta = \{X_1, \ldots, X_n\} \subseteq \Gamma$.

(1) If $\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S t : \Sigma(B[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow C$, then $|c^I|_{\Gamma\setminus\Delta} \fatsemi t = t$.

(2) If $\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S s : C \rightsquigarrow \Sigma(A[X_1 := \star, \cdots, X_n := \star])$, then $s \fatsemi |c^I|_{\Gamma\setminus\Delta} = s$.

Proof. We show both items simultaneously by induction on $\Sigma \mid \Gamma \vdash_C c^I : A \rightsquigarrow B$ with case analysis on the last typing rule used.

Case (CT_Id_C): We are given $c^I = \mathsf{id}_A$ (and $A = B$ and that A is neither a function nor universal type). If A is X_i for some i, then $|\mathsf{id}_A|_{\Gamma\setminus\Delta} = \mathsf{id}$ and, otherwise $|\mathsf{id}_A|_{\Gamma\setminus\Delta} = \mathsf{id}$. In either case, Lemma I.5 and Lemma I.6 finish the case.

Case (CT_Fail_C), (CT_Inj_C), (CT_Proj_C): Cannot happen because c^I is a no-op coercion.

Case (CT_Conceal_C): We are given $c^I = \alpha^-$ for some α. Since $|\alpha^-|_{\Gamma\setminus\Delta} = \mathsf{id}$, Lemma I.5 and Lemma I.6 finish the case.

Case (CT_Reveal_C): We are given $c^I = \alpha^+$ for some α. Since $|\alpha^+|_{\Gamma\setminus\Delta} = \mathsf{id}$, Lemma I.5 and Lemma I.6 finish the case.

Case (CT_Arrow_C): We are given:

$$c^I = c_1^I \to c_2^I, \quad A = A_1 \to A_2, \quad B = B_1 \to B_2,$$
$$\Sigma \mid \Gamma \vdash_C c_1^I : B_1 \rightsquigarrow A_1, \quad \Sigma \mid \Gamma \vdash_C c_2^I : A_2 \rightsquigarrow B_2 \quad (\exists c_1^I, c_2^I, A_1, A_2, B_1, B_2)$$

We have $|c^I|_{\Gamma\setminus\Delta} = |c_1^I|_{\Gamma\setminus\Delta} \to |c_2^I|_{\Gamma\setminus\Delta}$.

(1) Assume $\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S t : \Sigma(B_1[X_1 := \star, \cdots, X_n := \star]) \to \Sigma(B_2[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow C$. We will show that $|c^I|_{\Gamma\setminus\Delta} \mathbin{;} t = t$ by case analysis on the rule applied last to derive $\Sigma \mid \Sigma(\Gamma) \vdash_S t : \Sigma(B_1[X_1 := \star, \cdots, X_n := \star]) \to \Sigma(B_2[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow C$.

Case (Ct_Inj_S): We are given

$$t = g \mathbin{;} G!, \quad C = \star,$$
$$\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S g : \Sigma(B_1[X_1 := \star, \cdots, X_n := \star]) \to \Sigma(B_2[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow \Sigma(G) \quad (\exists G, g) .$$

By coercion typing rules, it must be the case that $\Sigma(G) = \star \to \star$ and $g = s' \to t'$ and $\Sigma \mid \Sigma(\Gamma\setminus\Delta) \vdash_S s' : \star \rightsquigarrow \Sigma(B_1[X_1 := \star, \cdots, X_n := \star])$ and $\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S t' : \Sigma(B_2[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow \star$ for some s' and t'. By the IH, $s' \mathbin{;} |c_1^I|_{\Gamma\setminus\Delta} = s'$ and $|c_2^I|_{\Gamma\setminus\Delta} \mathbin{;} t' = t'$. Thus, we have

$$\begin{aligned}(|c_1^I|_{\Gamma\setminus\Delta} \to |c_2^I|_{\Gamma\setminus\Delta}) \mathbin{;} ((s' \to t') \mathbin{;} G!) &= ((|c_1^I|_{\Gamma\setminus\Delta} \to |c_2^I|_{\Gamma\setminus\Delta}) \mathbin{;} (s' \to t')) \mathbin{;} G! \\ &= ((s' \mathbin{;} |c_1^I|_{\Gamma\setminus\Delta}) \to (|c_2^I|_{\Gamma\setminus\Delta} \mathbin{;} t')) \mathbin{;} G! \\ &= (s' \to t') \mathbin{;} G! .\end{aligned}$$

Case (Ct_Fail_S): We are given $t = \bot^p$ for some p. Then, $(|c_1^I|_{\Gamma\setminus\Delta} \to |c_2^I|_{\Gamma\setminus\Delta}) \mathbin{;} \bot^p = \bot^p$.

Case (Ct_Arrow_S): We are given

$$t = s' \to t', \quad C = C_1 \to C_2,$$
$$\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S s' : C_1 \rightsquigarrow \Sigma(B_1[X_1 := \star, \cdots, X_n := \star]),$$
$$\Sigma \mid \Gamma \setminus \Delta \vdash_S t' : \Sigma(B_2[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow C_2 \quad (\exists C_1, C_2, s', t') .$$

By the IH, $s' \mathbin{;} |c_1^I|_{\Gamma\setminus\Delta} = s'$ and $|c_2^I|_{\Gamma\setminus\Delta} \mathbin{;} t = t$. Therefore,

$$(|c_1^I|_{\Gamma\setminus\Delta} \to |c_2^I|_{\Gamma\setminus\Delta}) \mathbin{;} (s' \to t') = (s' \mathbin{;} |c_1^I|_{\Gamma\setminus\Delta}) \to (|c_2^I|_{\Gamma\setminus\Delta} \mathbin{;} t') = s' \to t' = t .$$

Otherwise: Cannot happen.

(2) Assume $\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S s : C \rightsquigarrow \Sigma(A_1[X_1 := \star, \cdots, X_n := \star]) \to \Sigma(A_2[X_1 := \star, \cdots, X_n := \star])$. We will show that $s \mathbin{;} |c^I|_{\Gamma\setminus\Delta} = s$ by case analysis on the rule applied last to derive $\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S s : C \rightsquigarrow \Sigma(A_1[X_1 := \star, \cdots, X_n := \star]) \to \Sigma(A_2[X_1 := \star, \cdots, X_n := \star])$.

Case (Ct_Proj_S): We are given

$$s = G?^p \mathbin{;} b, \quad C = \star,$$
$$\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S b : \Sigma(G) \rightsquigarrow \Sigma(A_1[X_1 := \star, \cdots, X_n := \star]) \to \Sigma(A_2[X_1 := \star, \cdots, X_n := \star]) \quad (\exists p, G, b) .$$

Then, b is either $\bot^q$ for some q or b is an intermediate coercion. The former case is easy because $s \mathbin{;} |c^I|_{\Gamma\setminus\Delta} = G?^p \mathbin{;} (\bot^q \mathbin{;} |c^I|_{\Gamma\setminus\Delta}) = G?^p \mathbin{;} \bot^q = s$. In the latter case, by coercion typing rules, it must be the case that $\Sigma(G) = \star \to \star$ and $b = s' \to t'$ and $\Sigma \mid \Sigma(\Gamma\setminus\Delta) \vdash_S s' : \Sigma(A_1[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow \star$ and $\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S t' : \star \rightsquigarrow \Sigma(A_2[X_1 := \star, \cdots, X_n := \star])$ for some s', t'. By the IH, $|c_1^I|_{\Gamma\setminus\Delta} \mathbin{;} s' = s'$ and $t' \mathbin{;} |c_2^I|_{\Gamma\setminus\Delta} = t'$. Thus, we have

$$\begin{aligned}(G?^p \mathbin{;} (s' \to t')) \mathbin{;} (|c_1^I|_{\Gamma\setminus\Delta} \to |c_2^I|_{\Gamma\setminus\Delta}) &= G?^p \mathbin{;} ((s' \to t') \mathbin{;} (|c_1^I|_{\Gamma\setminus\Delta} \to |c_2^I|_{\Gamma\setminus\Delta})) \\ &= G?^p \mathbin{;} ((|c_1^I|_{\Gamma\setminus\Delta} \mathbin{;} s') \to (t' \mathbin{;} |c_2^I|_{\Gamma\setminus\Delta})) \\ &= G?^p \mathbin{;} (s' \to t') = s .\end{aligned}$$

Case (CT_FAIL_S): We are given $s = \bot^p$ for some p. Then, $\bot^p \fatsemi |c^I|_{\Gamma\setminus\Delta} = \bot^p$.
Case (CT_ARROW_S): We are given

$$s = s' \to t', \quad C = C_1 \to C_2, \quad \Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S s' : \Sigma(A_1[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow C_1,$$
$$\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S t' : C_2 \rightsquigarrow \Sigma(A_2[X_1 := \star, \cdots, X_n := \star]) \quad (\exists C_1, C_2, s', t') \ .$$

By the IH, $|c_1^I|_{\Gamma\setminus\Delta} \fatsemi s' = s'$ and $t' \fatsemi |c_2^I|_{\Gamma\setminus\Delta} = t'$. Therefore,

$$(s' \to t') \fatsemi (|c_1^I|_{\Gamma\setminus\Delta} \to |c_2^I|_{\Gamma\setminus\Delta}) = (|c_1^I|_{\Gamma\setminus\Delta} \fatsemi s') \to (t' \fatsemi |c_2^I|_{\Gamma\setminus\Delta}) = s' \to t' = s \ .$$

Otherwise: Cannot happen.

Case (CT_ALL_C): We are given

$$c^I = \forall X.c_0^I, \quad A = \forall X.A_0, \quad B = \forall X.B_0, \quad \Sigma \mid \Gamma, X \vdash_C c_0^I : A_0 \rightsquigarrow B_0 \quad (\exists c_0^I, X, A_0, B_0) \ .$$

We have $|c^I|_{\Gamma\setminus\Delta} = \forall X.|c_0^I|_{(\Gamma\setminus\Delta),X} \,,, |c_0^I|_{\Gamma\setminus\Delta}$.

(1) Assume $\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S t : \forall X.\Sigma(B_0[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow C$. We will show that $|c^I| \fatsemi t = t$ by case analysis on the rule applied last to derive $\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S t : \forall X.\Sigma(B_0[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow C$.

Case (CT_INJ_S): We are given

$$t = g \,;\, G!, \quad C = \star, \quad \Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S g : \forall X.\Sigma(B_0[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow \Sigma(G) \quad (\exists G, g) \ .$$

By coercion typing rules, it must be the case that

$$\Sigma(G) = \forall X.\star, \quad g = \forall X.t' \,,, t'',$$
$$\Sigma \mid \Sigma(\Gamma \setminus \Delta), X \vdash_S t' : \Sigma(B_0[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow \star$$
$$\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S t'' : \Sigma(B_0[X_1 := \star, \cdots, X_n := \star][X := \star]) \rightsquigarrow \star$$

for some t' and t''. By the IH, $|c_0^I|_{(\Gamma,X)\setminus\Delta} \fatsemi t' = t'$ and $|c_0^I|_{(\Gamma,X)\setminus(\Delta,X)} \fatsemi t'' = t''$. Thus, we have

$$\begin{aligned}(|\forall X.c_0^I|_{\Gamma\setminus\Delta}) \fatsemi ((\forall X.t' \,,, t'') \,;\, G!) &= ((\forall X.|c_0^I|_{(\Gamma\setminus\Delta),X} \,,, |c_0^I|_{\Gamma\setminus\Delta}) \fatsemi (\forall X.t' \,,, t'')) \,;\, G! \\ &= ((\forall X.|c_0^I|_{(\Gamma,X)\setminus\Delta} \,,, |c_0^I|_{(\Gamma,X)\setminus(\Delta,X)}) \fatsemi (\forall X.t' \,,, t'')) \,;\, G! \\ &= (\forall X.((|c_0^I|_{(\Gamma,X)\setminus\Delta} \fatsemi t') \,,, (|c_0^I|_{(\Gamma,X)\setminus(\Delta,X)} \fatsemi t''))) \,;\, G! \\ &= (\forall X.t' \,,, t'') \,;\, G! = t \ .\end{aligned}$$

Case (CT_FAIL_S): We are given $t = \bot^p$ for some p. Then, $|\forall X.c_0^I|_{\Gamma\setminus\Delta} \fatsemi \bot^p = \bot^p$.
Case (CT_ALL_S): We are given

$$t = \forall X.t' \,,, t'', \quad C = \forall X.C_0, \quad \Sigma \mid \Gamma, X \vdash_S t' : \Sigma(B_0[X_1 := \star, \cdots, X_n := \star]) \rightsquigarrow C_0,$$
$$\Sigma \mid \Gamma \vdash_S t'' : \Sigma(B_0[X_1 := \star, \cdots, X_n := \star][X := \star]) \rightsquigarrow C_0[X := \star] \quad (\exists C_0, t', t'') \ .$$

By the IH, $|c_0^I|_{(\Gamma,X)\setminus\Delta} \fatsemi t' = t'$ and $|c_0^I|_{(\Gamma,X)\setminus(\Delta,X)} \fatsemi t'' = t''$. Therefore,

$$\begin{aligned}(|\forall X.c_0^I|_{\Gamma\setminus\Delta}) \fatsemi (\forall X.t' \,,, t'') &= (\forall X.|c_0^I|_{(\Gamma\setminus\Delta),X} \,,, |c_0^I|_{\Gamma\setminus\Delta}) \fatsemi (\forall X.t' \,,, t'') \\ &= (\forall X.|c_0^I|_{(\Gamma,X)\setminus\Delta} \,,, |c_0^I|_{(\Gamma,X)\setminus(\Delta,X)}) \fatsemi (\forall X.t' \,,, t'') \\ &= \forall X.((|c_0^I|_{(\Gamma,X)\setminus\Delta} \fatsemi t') \,,, (|c_0^I|_{(\Gamma,X)\setminus(\Delta,X)} \fatsemi t'')) \\ &= \forall X.t' \,,, t'' = t \ .\end{aligned}$$

Otherwise: Cannot happen.

(2) Assume $\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S s : C \rightsquigarrow \forall X.\Sigma(A_0[X_1 := \star, \cdots, X_n := \star])$. We will show that $s \fatsemi |c^I|_{\Gamma\setminus\Delta} = s$ by case analysis on the rule applied last to derive $\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S s : C \rightsquigarrow \forall X.\Sigma(A_0[X_1 := \star, \cdots, X_n := \star])$.

Case (Ct_Proj_S): We are given

$$s = G?^p \,;\, b, \quad C = \star, \quad \Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S b : \Sigma(G) \rightsquigarrow \forall X.\Sigma(A_0[X_1 := \star, \cdots, X_n := \star]) \quad (\exists p, G, b)\ .$$

Then, b is either $\bot^q$ for some q or an intermediate coercion. The former case is easy because $s \,\S\, |c_0^I|_{\Gamma\setminus\Delta} = G?^p \,;\, (\bot^q \,\S\, |c_0^I|_{\Gamma\setminus\Delta}) = G?^p \,;\, \bot^q$. In the latter case, by coercion typing rules, it must be the case that

$$\begin{gathered}\Sigma(G) = \forall X.\star, \quad b = \forall X.s' \,,,\, s'', \\ \Sigma \mid \Sigma(\Gamma \setminus \Delta), X \vdash_S s' : \star \rightsquigarrow \Sigma(A_0[X_1 := \star, \cdots, X_n := \star]), \\ \Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S s'' : \star \rightsquigarrow \Sigma(A_0[X_1 := \star, \cdots, X_n := \star][X := \star])\end{gathered}$$

for some s' and s''. By the IH, $s' \,\S\, |c_0^I|_{(\Gamma,X)\setminus\Delta} = s'$ and $s' \,\S\, |c_0^I|_{(\Gamma,X)\setminus(\Delta,X)} = s''$. Thus, we have

$$\begin{aligned}(G?^p \,;\, \forall X.s' \,,,\, s'') \,\S\, (|\forall X.c_0^I|_{\Gamma\setminus\Delta}) &= G?^p \,;\, ((\forall X.s' \,,,\, s'') \,\S\, (\forall X.|c_0^I|_{(\Gamma\setminus\Delta),X} \,,,\, |c_0^I|_{\Gamma\setminus\Delta})) \\ &= G?^p \,;\, ((\forall X.s' \,,,\, s'') \,\S\, (\forall X.|c_0^I|_{(\Gamma,X)\setminus\Delta} \,,,\, |c_0^I|_{(\Gamma,X)\setminus(\Delta,X)})) \\ &= G?^p \,;\, (\forall X.(s' \,\S\, |c_0^I|_{(\Gamma,X)\setminus\Delta}) \,\S\, (s'' \,,,\, |c_0^I|_{(\Gamma,X)\setminus(\Delta,X)})) \\ &= G?^p \,;\, (\forall X.s' \,,,\, s'')\ .\end{aligned}$$

Case (Ct_Fail_S): We are given $s = \bot^p$ for some p. Then, $\bot^p \,\S\, (|\forall X.c_0^I|_{\Gamma\setminus\Delta}) = \bot^p$.
Case (Ct_All_S): We are given

$$\begin{gathered}s = \forall X.s' \,,,\, s'', \quad C = \forall X.C_0, \quad \Sigma \mid \Gamma, X \vdash_S s' : C_0 \rightsquigarrow \Sigma(A_0[X_1 := \star, \cdots, X_n := \star]), \\ \Sigma \mid \Gamma \vdash_S s'' : C_0[X := \star] \rightsquigarrow \Sigma(A_0[X_1 := \star, \cdots, X_n := \star][X := \star]), \quad (\exists C_0, s', s'')\ .\end{gathered}$$

By the IH, $s' \,\S\, |c_0^I|_{(\Gamma,X)\setminus\Delta} = s'$ and $s'' \,\S\, |c_0^I|_{(\Gamma,X)\setminus(\Delta,X)} = s''$. Therefore,

$$\begin{aligned}(\forall X.s' \,,,\, s'') \,\S\, (|\forall X.c_0^I|_{\Gamma\setminus\Delta}) &= (\forall X.s' \,,,\, s'') \,\S\, (\forall X.|c_0^I|_{(\Gamma\setminus\Delta),X} \,,,\, |c_0^I|_{\Gamma\setminus\Delta}) \\ &= (\forall X.s' \,,,\, s'') \,\S\, (\forall X.|c_0^I|_{(\Gamma,X)\setminus\Delta} \,,,\, |c_0^I|_{(\Gamma,X)\setminus(\Delta,X)}) \\ &= \forall X.(s' \,\S\, |c_0^I|_{(\Gamma,X)\setminus\Delta}) \,,,\, (s'' \,\S\, |c_0^I|_{(\Gamma,X)\setminus(\Delta,X)}) \\ &= \forall X.s' \,,,\, s'' = s\ .\end{aligned}$$

Otherwise: Cannot happen.

Case (Ct_Seq_C): We are given

$$c^I = c_1^I \,;\, c_2^I, \quad \Sigma \mid \Gamma \vdash_C c_1^I : A \rightsquigarrow C, \quad \Sigma \mid \Gamma \vdash_C c_2^I : C \rightsquigarrow B \quad (\exists c_1^I, c_2^I, C)$$

(1) Assume $\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S t : B[X_1 := \star, \cdots, X_n := \star] \rightsquigarrow D$. We will show that $|c^I|_{\Gamma\setminus\Delta} \,\S\, t = t$. In fact,

$$\begin{aligned}|c^I|_{\Gamma\setminus\Delta} \,\S\, t &= (|c_1^I|_{\Gamma\setminus\Delta} \,\S\, |c_2^I|_{\Gamma\setminus\Delta}) \,\S\, t \\ &= |c_1^I|_{\Gamma\setminus\Delta} \,\S\, (|c_2^I|_{\Gamma\setminus\Delta} \,\S\, t) && \text{(Lemma I.7)} \\ &= |c_1^I|_{\Gamma\setminus\Delta} \,\S\, t && \text{(by the IH)} \\ &= t && \text{(by the IH).}\end{aligned}$$

(2) Assume $\Sigma \mid \Sigma(\Gamma \setminus \Delta) \vdash_S s : D \rightsquigarrow A[X_1 := \star, \cdots, X_n := \star]$. We will show that $s \,\S\, |c^I|_{\Gamma\setminus\Delta} = s$. In fact,

$$\begin{aligned}s \,\S\, |c^I|_{\Gamma\setminus\Delta} &= s \,\S\, (|c_1^I|_{\Gamma\setminus\Delta} \,\S\, |c_2^I|_{\Gamma\setminus\Delta}) \\ &= (s \,\S\, |c_1^I|_{\Gamma\setminus\Delta}) \,\S\, |c_2^I|_{\Gamma\setminus\Delta} && \text{(Lemma I.7)} \\ &= s \,\S\, |c_1^I|_{\Gamma\setminus\Delta} && \text{(by the IH)} \\ &= s && \text{(by the IH).}\end{aligned}$$

□

Lemma I.12 (Determinacy of Evaluation to Values)**.** The following holds.

(1) If $\Sigma \triangleright M \longrightarrow_C^* \Sigma_2 \triangleright V$ and $\Sigma \triangleright M \longrightarrow_C \Sigma_1 \triangleright M_1$, then $\Sigma_1 \triangleright M_1 \longrightarrow_C^* \Sigma_2 \triangleright V$.

(2) If $\Sigma \triangleright M \longrightarrow_S^* \Sigma_2 \triangleright V$ and $\Sigma \triangleright M \longrightarrow_S \Sigma_1 \triangleright M_1$, then $\Sigma_1 \triangleright M_1 \longrightarrow_S^* \Sigma_2 \triangleright V$.

Proof.

(1) By case analysis on the length of the evaluation sequence $\Sigma \triangleright M \longrightarrow_C^* \Sigma_2 \triangleright V$.

Case where the length is zero: We are given $M = V$, but it is contradictory with the assumption $\Sigma \triangleright M \longrightarrow_C \Sigma_1 \triangleright M_1$.

Case where the length is larger than zero: We are given

$$\Sigma \triangleright M \longrightarrow_C \Sigma_3 \triangleright M_3, \quad \Sigma_3 \triangleright M_3 \longrightarrow_C^* \Sigma_2 \triangleright V \quad (\exists \Sigma_3, M_3) .$$

By Theorem F.1, $\Sigma_3 = \Sigma_1$ and $M_3 = M_1$. Therefore, by the assumption, $\Sigma_1 \triangleright M_1 \longrightarrow_C^* \Sigma_2 \triangleright V$.

(2) Provable similarly to the case (1).

□

Lemma I.13 (Determinacy of Coercion Applications Evaluated to Values (Single))**.** If $\Sigma \mid \emptyset \vdash_S M : A$ and $\Sigma \mid \emptyset \vdash_S s : A \rightsquigarrow B$ and $\Sigma \triangleright M \longrightarrow_S \Sigma_1 \triangleright M_1$ and $\Sigma_1 \triangleright M_1\langle s\rangle \longrightarrow_S^* \Sigma_2 \triangleright V$, then $\Sigma \triangleright M\langle s\rangle \longrightarrow_S^* \Sigma_2 \triangleright V$.

Proof. By case analysis on the rule applied last to derive $\Sigma \triangleright M \longrightarrow_S \Sigma_1 \triangleright M_1$.

Case Reduction rules of $\xrightarrow{e}_S$: ((R_Delta_S), (R_Beta_S), (R_Wrap_S), (R_Tybeta_S), (R_TybetaC_S), (R_TybetaDyn_S), (R_TybetaDynC_S), (R_BlameE_S), (R_CtxE_S))

Because $\Sigma \triangleright M \xrightarrow{e}_S \Sigma_1 \triangleright M_1$, by (R_CtxC_S) we have $\Sigma \triangleright M\langle s\rangle \longrightarrow_S \Sigma_1 \triangleright M_1\langle s\rangle$. Therefore,

$$\Sigma \triangleright M\langle s\rangle \longrightarrow_S \Sigma_1 \triangleright M_1\langle s\rangle \longrightarrow_S^* \Sigma_2 \triangleright V .$$

Case (R_Id_S): We are given

$$M = U\langle \mathsf{id}\rangle, \quad M_1 = U, \quad \Sigma_1 = \Sigma \quad (\exists U) .$$

Lemma I.5 implies $\mathsf{id} \fatsemi s = s$. Therefore, by (R_Merge_S),

$$\begin{aligned} \Sigma \triangleright U\langle \mathsf{id}\rangle\langle s\rangle &\longrightarrow_S \Sigma \triangleright U\langle \mathsf{id} \fatsemi s\rangle \\ &= \Sigma \triangleright U\langle s\rangle \\ &\longrightarrow_S^* \Sigma_2 \triangleright V . \end{aligned}$$

Case (R_Fail_S): We are given

$$M = U\langle \bot^p\rangle, \quad M_1 = \mathsf{blame}\, p, \quad \Sigma_1 = \Sigma \quad (\exists p, U) .$$

By (R_BlameC_S), $\Sigma \triangleright (\mathsf{blame}\, p)\langle s\rangle \longrightarrow_S \Sigma \triangleright \mathsf{blame}\, p$. Therefore, Lemma I.12 implies $\Sigma \triangleright \mathsf{blame}\, p \longrightarrow_S^* \Sigma_2 \triangleright V$, but it does not hold. Hence, there is a contradiction.

Case (R_Merge_S): We are given

$$M = M'\langle s'\rangle\langle t'\rangle, \quad M_1 = M'\langle s' \fatsemi t'\rangle, \quad \Sigma_1 = \Sigma \quad (\exists s', t', M') .$$

The inversion of the derivation of $\Sigma \mid \emptyset \vdash_S M'\langle s'\rangle\langle t'\rangle : A$ implies

$$\Sigma \mid \emptyset \vdash_S s' : A' \rightsquigarrow B', \quad \Sigma \mid \emptyset \vdash_S t' : B' \rightsquigarrow A \quad (\exists A', B') .$$

By (R_Merge_S), $\Sigma \triangleright M'\langle s' \fatsemi t'\rangle\langle s\rangle \longrightarrow_S \Sigma \triangleright M'\langle (s' \fatsemi t') \fatsemi s\rangle$. Therefore, Lemma I.12 implies $\Sigma \triangleright M'\langle (s' \fatsemi t') \fatsemi s\rangle \longrightarrow_S^* \Sigma_2 \triangleright V$. Furthermore, by (R_Merge_S),

$$\Sigma \triangleright M'\langle s'\rangle\langle t'\rangle\langle s\rangle \longrightarrow_S \Sigma \triangleright M'\langle s'\rangle\langle t' \fatsemi s\rangle \longrightarrow_S \Sigma \triangleright M'\langle s' \fatsemi (t' \fatsemi s)\rangle .$$

Also, Lemma I.7 implies $(s' \fatsemi t') \fatsemi s = s' \fatsemi (t' \fatsemi s)$. Therefore,

$$\Sigma \triangleright M'\langle s'\rangle\langle t'\rangle\langle s\rangle \longrightarrow_S^* \Sigma \triangleright M'\langle s' \fatsemi (t' \fatsemi s)\rangle = \Sigma \triangleright M'\langle (s' \fatsemi t') \fatsemi s\rangle \longrightarrow_S^* \Sigma_2 \triangleright V .$$

Case (R_BlameC_S): We are given

$$M = (\mathsf{blame}\, p)\langle s' \rangle, \quad M_1 = \mathsf{blame}\, p, \quad \Sigma_1 = \Sigma \quad (\exists p, s') .$$

By (R_BlameC_S), $\Sigma \triangleright (\mathsf{blame}\, p)\langle s \rangle \longrightarrow_S \Sigma \triangleright \mathsf{blame}\, p$. Therefore, Lemma I.12 implies $\Sigma \triangleright \mathsf{blame}\, p \longrightarrow_S^* \Sigma_2 \triangleright V$, but it does not hold. Hence, there is a contradiction.

Case (R_CtxC_S): We are given

$$M = M'\langle s' \rangle, \quad M_1 = M_1'\langle s' \rangle, \quad \Sigma \triangleright M' \xrightarrow{e}_S \Sigma_1 \triangleright M_1' \quad (\exists s', M', M_1') .$$

By (R_Merge_S), $\Sigma_1 \triangleright M_1'\langle s' \rangle\langle s \rangle \longrightarrow_S \Sigma_1 \triangleright M_1'\langle s' \mathbin{\text{⨾}} s \rangle$. Hence, Lemma I.12 implies $\Sigma_1 \triangleright M_1'\langle s' \mathbin{\text{⨾}} s \rangle \longrightarrow_S^* \Sigma_2 \triangleright V$. Furthermore, by (R_Merge_S), $\Sigma \triangleright M'\langle s' \rangle\langle s \rangle \longrightarrow_S \Sigma \triangleright M'\langle s' \mathbin{\text{⨾}} s \rangle$. Also, by (R_CtxC_S), $\Sigma \triangleright M'\langle s' \mathbin{\text{⨾}} s \rangle \longrightarrow_S \Sigma_1 \triangleright M_1'\langle s' \mathbin{\text{⨾}} s \rangle$. Therefore,

$$\Sigma \triangleright M'\langle s' \rangle\langle s \rangle \longrightarrow_S \Sigma \triangleright M'\langle s' \mathbin{\text{⨾}} s \rangle \longrightarrow_S \Sigma_1 \triangleright M_1'\langle s' \mathbin{\text{⨾}} s \rangle \longrightarrow_S^* \Sigma_2 \triangleright V .$$

□

Lemma I.14 (Determinacy of Coercion Applications Evaluated to Values (Multi))**.** If $\Sigma \mid \emptyset \vdash_S M : A$ and $\Sigma \mid \emptyset \vdash_S s : A \rightsquigarrow B$ and $\Sigma \triangleright M \longrightarrow_S^* \Sigma_1 \triangleright M_1$ and $\Sigma_1 \triangleright M_1\langle s \rangle \longrightarrow_S^* \Sigma_2 \triangleright V$, then $\Sigma \triangleright M\langle s \rangle \longrightarrow_S^* \Sigma_2 \triangleright V$.

Proof. By induction on the length of the evaluation sequence $\Sigma \triangleright M \longrightarrow_S^* \Sigma_1 \triangleright M_1$. □

Lemma I.15 (Determinacy of Blame)**.** The following holds.

(1) If $\Sigma \triangleright M \longrightarrow_C^* \Sigma_2 \triangleright \mathsf{blame}\, p$ and $\Sigma \triangleright M \longrightarrow_C \Sigma_1 \triangleright M_1$, then $\Sigma_1 \triangleright M_1 \longrightarrow_C^* \Sigma_2 \triangleright \mathsf{blame}\, p$.

(2) If $\Sigma \triangleright M \longrightarrow_S^* \Sigma_2 \triangleright \mathsf{blame}\, p$ and $\Sigma \triangleright M \longrightarrow_S \Sigma_1 \triangleright M_1$, then $\Sigma_1 \triangleright M_1 \longrightarrow_S^* \Sigma_2 \triangleright \mathsf{blame}\, p$.

Proof. Provable similarly to Lemma I.12. □

Lemma I.16 (Determinacy of Coercion Applications Evaluated to Blame (Single))**.** If $\Sigma \mid \emptyset \vdash_S M : A$ and $\Sigma \mid \emptyset \vdash_S s : A \rightsquigarrow B$ and $\Sigma \triangleright M \longrightarrow_S \Sigma_1 \triangleright M_1$ and $\Sigma_1 \triangleright M_1\langle s \rangle \longrightarrow_S^* \Sigma_2 \triangleright \mathsf{blame}\, p$, then $\Sigma \triangleright M\langle s \rangle \longrightarrow_S^* \Sigma_2 \triangleright \mathsf{blame}\, p$.

Proof. By case analysis on the rule applied last to derive $\Sigma \triangleright M \longrightarrow_S \Sigma_1 \triangleright M_1$.

Case (R_Fail_S): We are given

$$M = U\langle \bot^{p'} \rangle, \quad M_1 = \mathsf{blame}\, p', \quad \Sigma_1 = \Sigma \quad (\exists p', U) .$$

By (R_BlameC_S), $\Sigma \triangleright (\mathsf{blame}\, p')\langle s \rangle \longrightarrow_S \Sigma \triangleright \mathsf{blame}\, p'$. Therefore, Lemma I.15 implies $\Sigma \triangleright \mathsf{blame}\, p' \longrightarrow_S^* \Sigma_2 \triangleright \mathsf{blame}\, p$. Hence, $\Sigma = \Sigma_2$ and $p' = p$. Also, $\bot^p \mathbin{\text{⨾}} s = \bot^p$. Therefore, by (R_Merge_S) and (R_Fail_S),

$$\Sigma \triangleright U\langle \bot^p \rangle\langle s \rangle \longrightarrow_S \Sigma \triangleright U\langle \bot^p \mathbin{\text{⨾}} s \rangle = \Sigma \triangleright U\langle \bot^p \rangle \longrightarrow_S \Sigma \triangleright \mathsf{blame}\, p .$$

Case (R_BlameC_S): We are given

$$M = (\mathsf{blame}\, p')\langle s' \rangle, \quad M_1 = \mathsf{blame}\, p', \quad \Sigma_1 = \Sigma \quad (\exists p', s') .$$

By (R_BlameC_S), $\Sigma \triangleright (\mathsf{blame}\, p')\langle s \rangle \longrightarrow_S \Sigma \triangleright \mathsf{blame}\, p'$. Therefore, Lemma I.15 implies $\Sigma \triangleright \mathsf{blame}\, p' \longrightarrow_S^* \Sigma_2 \triangleright \mathsf{blame}\, p$. Hence, $\Sigma = \Sigma_2$ and $p' = p$. Therefore, by (R_Merge_S) and (R_BlameC_S),

$$\Sigma \triangleright (\mathsf{blame}\, p)\langle s' \rangle\langle s \rangle \longrightarrow_S \Sigma \triangleright (\mathsf{blame}\, p)\langle s' \mathbin{\text{⨾}} s \rangle \longrightarrow_S \Sigma \triangleright \mathsf{blame}\, p .$$

Otherwise: Provable similarly to Lemma I.13. The proof uses Lemma I.15.

□

Lemma I.17 (Determinacy of Coercion Applications Evaluated to Blame (Multi))**.** If $\Sigma \mid \emptyset \vdash_S M : A$ and $\Sigma \mid \emptyset \vdash_S s : A \rightsquigarrow B$ and $\Sigma \triangleright M \longrightarrow_S^* \Sigma_1 \triangleright M_1$ and $\Sigma_1 \triangleright M_1\langle s \rangle \longrightarrow_S^* \Sigma_2 \triangleright \mathsf{blame}\, p$, then $\Sigma \triangleright M\langle s \rangle \longrightarrow_S^* \Sigma_2 \triangleright \mathsf{blame}\, p$.

Proof. Provable similarly to Lemma I.14. The proof uses Lemma I.16. □

Lemma I.18 (Determinacy of Consecutive Coercion Applications Evaluated to Values)**.** If $\Sigma \mid \emptyset \vdash_S M\langle s \rangle : B$ and $\Sigma \mid \emptyset \vdash_S t : B \rightsquigarrow C$ and $\Sigma \triangleright M\langle s \rangle \longrightarrow_S^* \Sigma_1 \triangleright M_1$ and $\Sigma_1 \triangleright M_1\langle t \rangle \longrightarrow_S^* \Sigma_2 \triangleright V$, then $\Sigma \triangleright M\langle s \fatsemi t \rangle \longrightarrow_S^* \Sigma_2 \triangleright V$.

Proof. Lemma I.14 implies $\Sigma \triangleright M\langle s \rangle\langle t \rangle \longrightarrow_S^* \Sigma_2 \triangleright V$. By (R_Merge_S), $\Sigma \triangleright M\langle s \rangle\langle t \rangle \longrightarrow_S \Sigma \triangleright M\langle s \fatsemi t \rangle$. Therefore, Lemma I.12 implies $\Sigma \triangleright M\langle s \fatsemi t \rangle \longrightarrow_S^* \Sigma_2 \triangleright V$. □

Lemma I.19 (Determinacy of Consecutive Coercion Applications Evaluated to Blame)**.** If $\Sigma \mid \emptyset \vdash_S M\langle s \rangle : B$ and $\Sigma \mid \emptyset \vdash_S t : B \rightsquigarrow C$ and $\Sigma \triangleright M\langle s \rangle \longrightarrow_S^* \Sigma_1 \triangleright M_1$ and $\Sigma_1 \triangleright M_1\langle t \rangle \longrightarrow_S^* \Sigma_2 \triangleright \mathsf{blame}\, p$, then $\Sigma \triangleright M\langle s \fatsemi t \rangle \longrightarrow_S^* \Sigma_2 \triangleright \mathsf{blame}\, p$.

Proof. Provable similarly to Lemma I.18. The proof uses Lemma I.17. □

Lemma I.20 (Evaluation of Coercion Applications)**.** If $\Sigma \mid \emptyset \vdash_S M : A$ and $\Sigma \mid \emptyset \vdash_S s : A \rightsquigarrow B$ and $\Sigma \triangleright M \longrightarrow_S^* \Sigma_1 \triangleright M_1$, then one of the following holds:

(1) $\Sigma \triangleright M\langle s \rangle \longrightarrow_S^* \Sigma_1 \triangleright M_1\langle s \rangle$;

(2) there exist some s_2 and M_2 such that $\Sigma \triangleright M\langle s \rangle \longrightarrow_S^* \Sigma_1 \triangleright M_2\langle s_2 \fatsemi s \rangle$ and $M_1 = M_2\langle s_2 \rangle$; or

(3) there exists some p such that $\Sigma \triangleright M\langle s \rangle \longrightarrow_S^* \Sigma_1 \triangleright M_1$ and $M_1 = \mathsf{blame}\, p$.

Proof. By induction on the length of the evaluation sequence $\Sigma \triangleright M \longrightarrow_S^* \Sigma_1 \triangleright M_1$. We perform case analysis on the length.

Case the length is zero: Because $\Sigma = \Sigma_1$ and $M = M_1$, we have $\Sigma \triangleright M\langle s \rangle \longrightarrow_S^* \Sigma \triangleright M\langle s \rangle$.

Case the length is larger than zero: We are given

$$\Sigma \triangleright M \longrightarrow_S^* \Sigma' \triangleright M', \quad \Sigma' \triangleright M' \longrightarrow_S \Sigma_1 \triangleright M_1 \quad (\exists \Sigma', M') .$$

Corollary E.46 implies $\Sigma' \mid \emptyset \vdash_S M' : A$. By Lemma E.44 and Lemma E.30, we have $\Sigma' \mid \emptyset \vdash_S s : A \rightsquigarrow B$. By case analysis on the result of the IH.

Case (1): We are given

$$\Sigma \triangleright M\langle s \rangle \longrightarrow_S^* \Sigma' \triangleright M'\langle s \rangle .$$

By case analysis on the rule applied last to derive $\Sigma' \triangleright M' \longrightarrow_S \Sigma_1 \triangleright M_1$.

Case (R_Delta_S), (R_Beta_S), (R_Wrap_S), (R_Tybeta_S), (R_TybetaC_S), (R_BlameE_S), (R_CtxE_S): We show the case (1). Because $\Sigma' \triangleright M' \xrightarrow{e}_S \Sigma_1 \triangleright M_1$, the assumption and (R_CtxC_S) imply

$$\Sigma \triangleright M\langle s \rangle \longrightarrow_S^* \Sigma' \triangleright M'\langle s \rangle \longrightarrow_S \Sigma_1 \triangleright M_1\langle s \rangle .$$

Case (R_Id_S): We are given

$$M' = U\langle \mathsf{id} \rangle, \quad M_1 = U, \quad \Sigma_1 = \Sigma' \quad (\exists U) .$$

We show the case (1). Lemma I.5 implies $\mathsf{id} \fatsemi s = s$. Therefore, the assumption and (R_Merge_S) imply

$$\begin{aligned} \Sigma \triangleright M\langle s \rangle &\longrightarrow_S^* \Sigma' \triangleright U\langle \mathsf{id} \rangle\langle s \rangle \\ &\longrightarrow_S \Sigma' \triangleright U\langle \mathsf{id} \fatsemi s \rangle \\ &= \Sigma' \triangleright U\langle s \rangle . \end{aligned}$$

Case (R_Fail_S): We are given

$$M' = U\langle \bot^p \rangle, \quad M_1 = \mathsf{blame}\, p, \quad \Sigma_1 = \Sigma' \quad (\exists p, U) .$$

We show the case (3). The assumption, (R_Merge_S), and (R_Fail_S) imply

$$\Sigma \rhd M\langle s \rangle \longrightarrow_S^* \Sigma' \rhd U\langle \bot^p \rangle\langle s \rangle \longrightarrow_S \Sigma' \rhd U\langle \bot^p \mathbin{;} s \rangle = \Sigma' \rhd U\langle \bot^p \rangle \longrightarrow_S \Sigma' \rhd \mathsf{blame}\, p .$$

Case (R_Merge_S): We are given

$$M' = M''\langle s' \rangle\langle t' \rangle, \quad M_1 = M''\langle s' \mathbin{;} t' \rangle, \quad \Sigma_1 = \Sigma' \quad (\exists s', t', M'') .$$

We show the case (2). By inversion of the derivation of $\Sigma' \mid \emptyset \vdash_S M''\langle s' \rangle\langle t' \rangle : A$, we have

$$\Sigma' \mid \emptyset \vdash_S s' : A' \rightsquigarrow B', \quad \Sigma' \mid \emptyset \vdash_S t' : B' \rightsquigarrow A \quad (\exists A', B') .$$

Therefore, Lemma I.7 implies $s' \mathbin{;} (t' \mathbin{;} s) = (s' \mathbin{;} t') \mathbin{;} s$. Hence, the assumption and (R_Merge_S) imply

$$\begin{aligned} \Sigma \rhd M\langle s \rangle &\longrightarrow_S^* \Sigma' \rhd M''\langle s' \rangle\langle t' \rangle\langle s \rangle \\ &\longrightarrow_S \Sigma' \rhd M''\langle s' \rangle\langle t' \mathbin{;} s \rangle \\ &\longrightarrow_S \Sigma' \rhd M''\langle s' \mathbin{;} (t' \mathbin{;} s) \rangle \\ &= \Sigma' \rhd M''\langle (s' \mathbin{;} t') \mathbin{;} s \rangle . \end{aligned}$$

Case (R_BlameC_S): We are given

$$M' = (\mathsf{blame}\, p)\langle s' \rangle, \quad M_1 = \mathsf{blame}\, p, \quad \Sigma_1 = \Sigma \quad (\exists p, s') .$$

We show the case (3). The assumption, (R_Merge_S), and (R_BlameC_S) imply

$$\Sigma \rhd M\langle s \rangle \longrightarrow_S^* \Sigma' \rhd (\mathsf{blame}\, p)\langle s' \rangle\langle s \rangle \longrightarrow_S \Sigma' \rhd (\mathsf{blame}\, p)\langle s' \mathbin{;} s \rangle \longrightarrow_S \Sigma' \rhd \mathsf{blame}\, p .$$

Case (R_CtxC_S): We are given

$$M' = M''\langle s' \rangle, \quad M_1 = M'''\langle s' \rangle, \quad \Sigma' \rhd M'' \xrightarrow{e}_S \Sigma_1 \rhd M''' \quad (\exists s', M'', M''') .$$

We show the case (2). The assumption, (R_Merge_S), and (R_CtxC_S) imply

$$\Sigma \rhd M\langle s \rangle \longrightarrow_S^* \Sigma' \rhd M''\langle s' \rangle\langle s \rangle \longrightarrow_S \Sigma' \rhd M''\langle s' \mathbin{;} s \rangle \longrightarrow_S \Sigma_1 \rhd M'''\langle s' \mathbin{;} s \rangle .$$

Case (2): We are given

$$\Sigma \rhd M\langle s \rangle \longrightarrow_S^* \Sigma' \rhd M_2\langle s_2 \mathbin{;} s \rangle, \quad M' = M_2\langle s_2 \rangle \quad (\exists s_2, M_2) .$$

By case analysis on the rule applied last to derive $\Sigma' \rhd M_2\langle s_2 \rangle \longrightarrow_S \Sigma_1 \rhd M_1$. It suffices to consider only the following cases.

Case (R_Id_S): We are given

$$M_2 = U, \quad s_2 = \mathsf{id}, \quad M_1 = U, \quad \Sigma_1 = \Sigma' \quad (\exists U) .$$

We show the case (1). Lemma I.5 implies $\mathsf{id} \mathbin{;} s = s$. Therefore, by the assumption,

$$\begin{aligned} \Sigma \rhd M\langle s \rangle &\longrightarrow_S^* \Sigma' \rhd U\langle \mathsf{id} \mathbin{;} s \rangle \\ &= \Sigma' \rhd U\langle s \rangle . \end{aligned}$$

Case (R_Fail_S): We are given

$$M_2 = U, \quad s_2 = \bot^p, \quad M_1 = \mathsf{blame}\, p, \quad \Sigma_1 = \Sigma' \quad (\exists p, U) .$$

We show the case (3). We have $\bot^p \mathbin{;} s = \bot^p$. Therefore, the assumption and (R_Fail_S) imply

$$\Sigma \rhd M\langle s \rangle \longrightarrow_S^* \Sigma' \rhd U\langle \bot^p \rangle \longrightarrow_S \Sigma' \rhd \mathsf{blame}\, p .$$

Case (R_MERGE_S): We are given

$$M_2 = M''\langle s'\rangle, \quad M_1 = M''\langle s' \mathbin{;} s_2'\rangle, \quad \Sigma_1 = \Sigma' \quad (\exists s', M'') .$$

We show the case (2). By inversion of the derivation of $\Sigma' \mid \emptyset \vdash_S M''\langle s'\rangle\langle s_2\rangle : A$, we have

$$\Sigma' \mid \emptyset \vdash_S s' : A' \rightsquigarrow C \quad \Sigma' \mid \emptyset \vdash_S s_2 : C \rightsquigarrow A \quad (\exists A', C) .$$

Therefore, Lemma I.7 implies $s' \mathbin{;} (s_2 \mathbin{;} s) = (s' \mathbin{;} s_2) \mathbin{;} s$. Hence, the assumption and (R_MERGE_S) imply

$$\Sigma \triangleright M\langle s\rangle \longrightarrow_S^* \Sigma' \triangleright M''\langle s'\rangle\langle s_2 \mathbin{;} s\rangle \longrightarrow_S \Sigma' \triangleright M''\langle s' \mathbin{;} (s_2 \mathbin{;} s)\rangle = \Sigma' \triangleright M''\langle (s' \mathbin{;} s_2) \mathbin{;} s\rangle .$$

Case (R_BLAMEC_S): We are given

$$M_2 = \mathsf{blame}\, p, \quad M_1 = \mathsf{blame}\, p, \quad \Sigma_1 = \Sigma' \quad (\exists p) .$$

We show the case (3). By the assumption and (R_BLAMEC_S),

$$\Sigma \triangleright M\langle s\rangle \longrightarrow_S^* \Sigma' \triangleright (\mathsf{blame}\, p)\langle s_2 \mathbin{;} s\rangle \longrightarrow_S \Sigma' \triangleright \mathsf{blame}\, p .$$

Case (R_CTXC_S): We are given

$$M_1 = M_3\langle s_2\rangle, \quad \Sigma' \triangleright M_2 \xrightarrow{e}_S \Sigma_1 \triangleright M_3 \quad (\exists M_3) .$$

We show the case (2). By the assumption and (R_CTXC_S),

$$\Sigma \triangleright M\langle s\rangle \longrightarrow_S^* \Sigma' \triangleright M_2\langle s_2 \mathbin{;} s\rangle \longrightarrow_S \Sigma_1 \triangleright M_3\langle s_2 \mathbin{;} s\rangle .$$

Case (3): We are given

$$\Sigma \triangleright M\langle s\rangle \longrightarrow_S^* \Sigma' \triangleright M', \quad M' = \mathsf{blame}\, p \quad (\exists p) .$$

Contradictory because $\Sigma' \triangleright \mathsf{blame}\, p \longrightarrow_S \Sigma_1 \triangleright M_1$ does not hold.

□

Lemma I.21 (Evaluation of Application of Composed Coercions)**.** If $\Sigma \mid \emptyset \vdash_S M\langle s\rangle : B$ and $\Sigma \mid \emptyset \vdash_S t : B \rightsquigarrow C$ and $\Sigma \triangleright M\langle s\rangle \longrightarrow_S \Sigma_1 \triangleright M_1 \longrightarrow_S^* \Sigma_2 \triangleright M_2$, then one of the following holds:

(1) $\Sigma \triangleright M\langle s \mathbin{;} t\rangle \longrightarrow_S^* \Sigma_2 \triangleright M_2\langle t\rangle$;

(2) there exist some D, s_3, and M_3 such that $\Sigma \triangleright M\langle s \mathbin{;} t\rangle \longrightarrow_S^* \Sigma_2 \triangleright M_3\langle (s_3 \mathbin{;} s) \mathbin{;} t\rangle$ and $M_2 = M_3\langle s_3 \mathbin{;} s\rangle$ and $\Sigma_2 \mid \emptyset \vdash_S s_3 : D \rightsquigarrow A$.

(3) there exists some p such that $\Sigma \triangleright M\langle s \mathbin{;} t\rangle \longrightarrow_S^* \Sigma_2 \triangleright \mathsf{blame}\, p$ and $M_2 = \mathsf{blame}\, p$.

Proof. By induction on the length of the evaluation sequence $\Sigma_1 \triangleright M_1 \longrightarrow_S^* \Sigma_2 \triangleright M_2$. We perform case analysis on the length.

Case The length is zero: We have $\Sigma_1 = \Sigma_2$ and $M_1 = M_2$. By case analysis on the rule applied last to derive $\Sigma \triangleright M\langle s\rangle \longrightarrow_S \Sigma_1 \triangleright M_1$. It suffices to consider only the following cases.

Case (R_ID_S): We are given

$$M = U, \quad s = \mathsf{id}, \quad A = B, \quad M_1 = U, \quad \Sigma_1 = \Sigma \quad (\exists U) .$$

We show the case (1). Lemma I.5 implies $\mathsf{id} \mathbin{;} t = t$. Furthermore, $\Sigma \triangleright U\langle t\rangle \longrightarrow_S^* \Sigma \triangleright U\langle t\rangle$ holds trivially.

Case (R_FAIL_S): We are given

$$M = U, \quad s = \bot^p, \quad M_1 = \mathsf{blame}\, p, \quad \Sigma_1 = \Sigma \quad (\exists p, U) .$$

We show the case (3). We have $\bot^p \mathbin{;} t = \bot^p$. Furthermore, by (R_FAIL), $\Sigma \triangleright U\langle \bot^p\rangle \longrightarrow_S \Sigma \triangleright \mathsf{blame}\, p$.

Case (R_MERGE_S): We are given

$$M = M'\langle s'\rangle, \quad M_1 = M'\langle s' \mathbin{;} s\rangle, \quad \Sigma_1 = \Sigma \quad (\exists s', M') .$$

We show the case (2). By inversion of the derivation of $\Sigma \mid \emptyset \vdash_S M'\langle s'\rangle\langle s\rangle : B$, we have $\Sigma \mid \emptyset \vdash_S s' : A' \rightsquigarrow A$ and $\Sigma \mid \emptyset \vdash_S s : A \rightsquigarrow B$. Therefore, Lemma I.7 implies $s' \mathbin{;} (s \mathbin{;} t) = (s' \mathbin{;} s) \mathbin{;} t$. Hence, by (R_MERGE_S),

$$\Sigma \triangleright M'\langle s'\rangle\langle s \mathbin{;} t\rangle \longrightarrow_S \Sigma \triangleright M'\langle s' \mathbin{;} (s \mathbin{;} t)\rangle = \Sigma \triangleright M'\langle (s' \mathbin{;} s) \mathbin{;} t\rangle .$$

Case (R_BLAMEC_S): We are given

$$M = \mathsf{blame}\, p, \quad M_1 = \mathsf{blame}\, p, \quad \Sigma_1 = \Sigma \quad (\exists p) .$$

We show the case (3). By (R_BLAMEC_S),

$$\Sigma \triangleright (\mathsf{blame}\, p)\langle s \mathbin{;} t\rangle \longrightarrow_S \Sigma \triangleright \mathsf{blame}\, p .$$

Case (R_CTXC_S): We are given

$$M_1 = M'\langle s\rangle, \quad \Sigma \triangleright M \xrightarrow{e}_S \Sigma_1 \triangleright M' \quad (\exists M') .$$

We show the case (2). Theorem E.45 implies $\Sigma_1 \mid \emptyset \vdash_S M'\langle s\rangle : B$. Therefore, Lemma E.32 implies $\vdash \Sigma_1$ and $\emptyset \vdash \emptyset$. Hence, because $\Sigma_1 \mid \emptyset \vdash_S M'\langle s\rangle : B$ is derived by (T_CRC_S), we have $\Sigma_1 \mid \emptyset \vdash_S s : A \rightsquigarrow B$ $(\exists A)$. Furthermore, since Lemma I.5 implies $\mathsf{id} \mathbin{;} s = s$, by (R_CTXC_S),

$$\Sigma \triangleright M\langle s \mathbin{;} t\rangle \longrightarrow_S \Sigma_1 \triangleright M'\langle s \mathbin{;} t\rangle = \Sigma_1 \triangleright M'\langle (\mathsf{id} \mathbin{;} s) \mathbin{;} t\rangle .$$

Furthermore, $M_1 = M'\langle \mathsf{id} \mathbin{;} s\rangle$. Therefore, it suffices to show that $\Sigma_1 \mid \emptyset \vdash_S \mathsf{id} : A \rightsquigarrow A$. Lemma E.22 implies $\emptyset \mid \emptyset \vdash A$. Hence, by Lemma D.2 (1), $\Sigma_1 \mid \emptyset \vdash A$. Therefore, by (CT_ID_S), $\Sigma_1 \mid \emptyset \vdash_S \mathsf{id} : \Sigma(A) \rightsquigarrow \Sigma(A)$. Since A does not contain any type names, $\Sigma(A) = A$. Hence, we have $\Sigma_1 \mid \emptyset \vdash_S \mathsf{id} : A \rightsquigarrow A$.

Case The length is larger than zero: We are given

$$\Sigma_1 \triangleright M_1 \longrightarrow_S \Sigma' \triangleright M', \quad \Sigma' \triangleright M' \longrightarrow_S^* \Sigma_2 \triangleright M_2 \quad (\exists \Sigma', M') .$$

By case analysis on the rule applied last to derive $\Sigma \triangleright M\langle s\rangle \longrightarrow_S \Sigma_1 \triangleright M_1$. It suffices to consider only the following cases.

Case (R_ID_S): We are given

$$M = U, \quad s = \mathsf{id}, \quad A = B, \quad M_1 = U, \quad \Sigma_1 = \Sigma \quad (\exists U) .$$

Contradictory because $\Sigma \triangleright U \longrightarrow_S \Sigma' \triangleright M'$ does not hold.

Case (R_FAIL_S): We are given

$$M = U, \quad s = \bot^p, \quad M_1 = \mathsf{blame}\, p, \quad \Sigma_1 = \Sigma \quad (\exists p, U) .$$

Contradictory because $\Sigma \triangleright \mathsf{blame}\, p \longrightarrow_S \Sigma' \triangleright M'$ does not hold.

Case (R_MERGE_S): We are given

$$M = M''\langle s'\rangle, \quad M_1 = M''\langle s' \mathbin{;} s\rangle, \quad \Sigma_1 = \Sigma \quad (\exists s', M'') .$$

By inversion of the derivation of $\Sigma \mid \emptyset \vdash_S M''\langle s'\rangle\langle s\rangle : B$, we have

$$\Sigma \mid \emptyset \vdash_S M'' : A', \quad \Sigma \mid \emptyset \vdash_S s' : A' \rightsquigarrow A, \quad \Sigma \mid \emptyset \vdash_S s : A \rightsquigarrow B \quad (\exists A') .$$

Therefore, Lemma E.24 and (T_CRC_S) imply $\Sigma \mid \emptyset \vdash_S M''\langle s' \mathbin{;} s\rangle : B$. Furthermore, Lemma I.7 implies $s' \mathbin{;} (s \mathbin{;} t) = (s' \mathbin{;} s) \mathbin{;} t$. Therefore, by (R_MERGE_S),

$$\Sigma \triangleright M''\langle s'\rangle\langle s \mathbin{;} t\rangle \longrightarrow_S \Sigma \triangleright M''\langle s' \mathbin{;} (s \mathbin{;} t)\rangle = \Sigma \triangleright M''\langle (s' \mathbin{;} s) \mathbin{;} t\rangle \quad \cdots (*) .$$

By case analysis on the result of the IH.

Case (1): We are given

$$\Sigma \triangleright M''\langle (s' \mathbin{;} s) \mathbin{;} t\rangle \longrightarrow_S^* \Sigma_2 \triangleright M_2\langle t\rangle \ .$$

We show the case (1). By (∗), we have

$$\Sigma \triangleright M''\langle s'\rangle\langle s \mathbin{;} t\rangle \longrightarrow_S \Sigma \triangleright M''\langle (s' \mathbin{;} s) \mathbin{;} t\rangle \longrightarrow_S^* \Sigma_2 \triangleright M_2\langle t\rangle \ .$$

Case (2): We are given

$$\Sigma \triangleright M''\langle (s' \mathbin{;} s) \mathbin{;} t\rangle \longrightarrow_S^* \Sigma_2 \triangleright M_3\langle (s_3 \mathbin{;} (s' \mathbin{;} s)) \mathbin{;} t\rangle, \quad M_2 = M_3\langle s_3 \mathbin{;} (s' \mathbin{;} s)\rangle, \quad \Sigma_2 \mid \emptyset \vdash_S s_3 : D \rightsquigarrow A' \quad (\exists D, s_3, M_3) \ .$$

We show the case (2). By Lemma E.44 and Lemma E.30, we have $\Sigma_2 \mid \emptyset \vdash_S s' : A' \rightsquigarrow A$ and $\Sigma_2 \mid \emptyset \vdash_S s : A \rightsquigarrow B$. Therefore, Lemma I.7 implies $s_3 \mathbin{;} (s' \mathbin{;} s) = (s_3 \mathbin{;} s') \mathbin{;} s$. Hence, by (∗), we have

$$\begin{aligned}\Sigma \triangleright M''\langle s'\rangle\langle s \mathbin{;} t\rangle &\longrightarrow_S \Sigma \triangleright M''\langle (s' \mathbin{;} s) \mathbin{;} t\rangle \\ &\longrightarrow_S^* \Sigma_2 \triangleright M_3\langle (s_3 \mathbin{;} (s' \mathbin{;} s)) \mathbin{;} t\rangle \\ &= \Sigma_2 \triangleright M_3\langle ((s_3 \mathbin{;} s') \mathbin{;} s) \mathbin{;} t\rangle \ .\end{aligned}$$

Furthermore, $M_2 = M_3\langle (s_3 \mathbin{;} s') \mathbin{;} s\rangle$. Moreover, Lemma E.24 implies $\Sigma_2 \mid \emptyset \vdash_S s_3 \mathbin{;} s' : D \rightsquigarrow A$.

Case (3): We are given

$$\Sigma \triangleright M''\langle (s' \mathbin{;} s) \mathbin{;} t\rangle \longrightarrow_S^* \Sigma_2 \triangleright \mathsf{blame}\, p, \quad M_2 = \mathsf{blame}\, p \quad (\exists p) \ .$$

We show the case (3). By (∗),

$$\Sigma \triangleright M''\langle s'\rangle\langle s \mathbin{;} t\rangle \longrightarrow_S \Sigma \triangleright M''\langle (s' \mathbin{;} s) \mathbin{;} t\rangle \longrightarrow_S^* \Sigma_2 \triangleright \mathsf{blame}\, p \ .$$

Case (R_BlameC_S): We are given

$$M = \mathsf{blame}\, p, \quad M_1 = \mathsf{blame}\, p, \quad \Sigma_1 = \Sigma \quad (\exists p) \ .$$

Contradictory because $\Sigma \triangleright \mathsf{blame}\, p \longrightarrow_S \Sigma' \triangleright M'$ does not hold.

Case (R_CtxC_S): We are given

$$M_1 = M''\langle s\rangle, \quad \Sigma \triangleright M \xrightarrow{e}_S \Sigma_1 \triangleright M'' \quad (\exists M'') \ .$$

Theorem E.45 implies $\Sigma_1 \mid \emptyset \vdash_S M''\langle s\rangle : B$. Furthermore, By Lemma E.44 and Lemma E.30, we have $\Sigma_1 \mid \emptyset \vdash_S t : B \rightsquigarrow C$. Therefore, by the IH, one of the following holds:

(1) $\Sigma_1 \triangleright M''\langle s \mathbin{;} t\rangle \longrightarrow_S^* \Sigma_2 \triangleright M_2\langle t\rangle$;

(2) there exist some D, s_3, and M_3 such that $\Sigma_1 \triangleright M''\langle s \mathbin{;} t\rangle \longrightarrow_S^* \Sigma_2 \triangleright M_3\langle (s_3 \mathbin{;} s) \mathbin{;} t\rangle$ and $M_2 = M_3\langle s_3 \mathbin{;} s\rangle$ and $\Sigma_2 \mid \emptyset \vdash_S s_3 : D \rightsquigarrow A$; or

(3) there exists some p such that $\Sigma_1 \triangleright M''\langle s \mathbin{;} t\rangle \longrightarrow_S^* \Sigma_2 \triangleright \mathsf{blame}\, p$ and $M_2 = \mathsf{blame}\, p$.

Furthermore, by (R_CtxC_S), $\Sigma \triangleright M\langle s \mathbin{;} t\rangle \longrightarrow_S \Sigma_1 \triangleright M''\langle s \mathbin{;} t\rangle$. Hence, we have the conclusion.

□

Lemma I.22 (Typability of Bisimulation)**.** If $\Sigma \mid \Gamma \vdash M \approx M' : A$, then $\Sigma \mid \Gamma \vdash_C M : A$ and $\Sigma \mid \Sigma(\Gamma) \vdash_S M' : \Sigma(A)$.

Proof. By induction on the derivation of $\Sigma \mid \Gamma \vdash M \approx M' : A$. We perform case analysis on the rule applied last to derive $\Sigma \mid \Gamma \vdash M \approx M' : A$.

Case (Bs_Const): We are given

$$M = M' = k, \quad \vdash \Sigma, \quad \Sigma \vdash \Gamma, \quad ty(k) = A \quad (\exists k) \ .$$

Therefore, by (T_Const_C), $\Sigma \mid \Gamma \vdash_C k : A$. Furthermore, by Lemma D.10 (5), $\emptyset \vdash \Sigma(\Gamma)$. Moreover, $ty(k) = A = \Sigma(A)$. Therefore, by (T_Const_S), $\Sigma \mid \Sigma(\Gamma) \vdash_S k : \Sigma(A)$.

Case (Bs_Var): We are given

$$M = M' = x, \quad \vdash \Sigma, \quad \Sigma \vdash \Gamma, \quad x : A \in \Gamma \quad (\exists x) .$$

Therefore, by (T_Var_C), $\Sigma \mid \Gamma \vdash_C x : A$. Furthermore, by Lemma D.10 (5), $\emptyset \vdash \Sigma(\Gamma)$. Moreover, $x : \Sigma(A) \in \Sigma(\Gamma)$. Therefore, by (T_Var_S), $\Sigma \mid \Sigma(\Gamma) \vdash_S x : \Sigma(A)$.

Case (Bs_Abs): We are given

$$A = A' \to B, \quad M = \lambda x{:}A'.M_1, \quad M' = \lambda x{:}A'.M_1', \quad \Sigma \mid \Gamma, x : A' \vdash M_1 \approx M_1' : B \quad (\exists A', B, x, M_1, M_1') .$$

By the IH,

$$\Sigma \mid \Gamma, x : A' \vdash_C M_1 : B, \quad \Sigma \mid \Sigma(\Gamma), x : \Sigma(A') \vdash_S M_1' : \Sigma(B) .$$

Therefore, by (T_Abs_C), $\Sigma \mid \Gamma \vdash_C \lambda x{:}A'.M_1 : A' \to B$. Furthermore, by (T_Abs_S), $\Sigma \mid \Sigma(\Gamma) \vdash_S \lambda x{:}\Sigma(A').M_1' : \Sigma(A') \to \Sigma(B)$. Since $\Sigma(A') \to \Sigma(B) = \Sigma(A' \to B)$, we have the conclusion.

Case (Bs_App): We are given

$$M = M_1\, M_2, \quad M' = M_1'\, M_2', \quad \Sigma \mid \Gamma \vdash M_1 \approx M_1' : B \to A, \quad \Sigma \mid \Gamma \vdash M_2 \approx M_2' : B \quad (\exists B, M_1, M_2, M_1', M_2') .$$

By the IHs,

$$\Sigma \mid \Gamma \vdash_C M_1 : B \to A, \quad \Sigma \mid \Sigma(\Gamma) \vdash_S M_1' : \Sigma(B) \to \Sigma(A), \quad \Sigma \mid \Gamma \vdash_C M_2 : B, \quad \Sigma \mid \Sigma(\Gamma) \vdash_S M_2' : \Sigma(B) .$$

Therefore, by (T_App_C), $\Sigma \mid \Gamma \vdash_C M_1\, M_2 : A$. Furthermore, by (T_App_S), $\Sigma \mid \Sigma(\Gamma) \vdash_S M_1'\, M_2' : \Sigma(A)$.

Case (Bs_Tyabs): We are given

$$A = \forall X.A', \quad M = \Lambda X.(M_1 : A'), \quad M' = \Lambda X.M_1', \quad \Sigma \mid \Gamma, X \vdash M_1 \approx M_1' : A' \quad (\exists X, A', M_1, M_1') .$$

By the IH,

$$\Sigma \mid \Gamma, X \vdash_C M_1 : A', \quad \Sigma \mid \Sigma(\Gamma), X \vdash_S M_1' : \Sigma(A') .$$

Therefore, by (T_Tyabs_C), $\Sigma \mid \Gamma \vdash_C \Lambda X.(M_1 : A') : \forall X.A'$. Furthermore, by (T_Tyabs_S), $\Sigma \mid \Sigma(\Gamma) \vdash_S \Lambda X.M_1' : \forall X.\Sigma(A')$. Since $\forall X.\Sigma(A') = \Sigma(\forall X.A')$, we have the conclusion.

Case (Bs_Tyapp): We are given

$$A = B[X := A'], \quad M = M_1\, A', \quad M' = M_1'\, A', \quad \Sigma \mid \Gamma \vdash M_1 \approx M_1' : \forall X.B, \quad \Sigma \mid \Gamma \vdash A' \quad (\exists X, A', B, M_1, M_1') .$$

By the IH,

$$\Sigma \mid \Gamma \vdash_C M_1 : \forall X.B, \quad \Sigma \mid \Sigma(\Gamma) \vdash_S M_1' : \forall X.\Sigma(B) .$$

Therefore, by (T_Tyapp_C), $\Sigma \mid \Gamma \vdash_C M_1\, A' : B[X := A']$. Furthermore, by (T_Tyapp_S), $\Sigma \mid \Sigma(\Gamma) \vdash_S M_1'\, A' : \Sigma(B)[X := \Sigma(A')]$. Since $\Sigma(B)[X := \Sigma(A')] = \Sigma(B[X := A'])$, we have the conclusion.

Case (Bs_Blame): We are given

$$M = \mathsf{blame}\, p, \quad M' = \mathsf{blame}\, p, \quad \vdash \Sigma, \quad \Sigma \vdash \Gamma, \quad \Sigma \mid \Gamma \vdash A \quad (\exists p) .$$

Therefore, by (T_Blame_C), $\Sigma \mid \Gamma \vdash_C \mathsf{blame}\, p : A$. Furthermore, Lemma D.10 (5) and Lemma D.10 (3) imply $\emptyset \vdash \Sigma(\Gamma)$ and $\emptyset \mid \Sigma(\Gamma) \vdash \Sigma(A)$. Hence, by (T_Blame_S), $\Sigma \mid \Sigma(\Gamma) \vdash_S \mathsf{blame}\, p : \Sigma(A)$.

Case (Bs_Crc): We are given

$$M = M_1\langle c \rangle, \quad M' = M_1'\langle |c|_\Gamma \rangle, \quad \Sigma \mid \Gamma \vdash M_1 \approx M_1' : B, \quad \Sigma \mid \Gamma \vdash_C c : B \rightsquigarrow A \quad (\exists M_1, c, M_1', B) .$$

By the IH, $\Sigma \mid \Gamma \vdash_C M_1 : B$ and $\Sigma \mid \Sigma(\Gamma) \vdash_S M_1' : \Sigma(B)$. By (T_Crc_C), $\Sigma \mid \Gamma \vdash_C M_1\langle c \rangle : A$. By Lemma I.3, $\Sigma \mid \Sigma(\Gamma) \vdash_S |c|_\Gamma : \Sigma(B) \rightsquigarrow \Sigma(A)$. Finally, (T_Crc_S) finihes the case.

Case (Bs_CrcId): We are given

$$M' = M_1'\langle|\mathsf{id}_A|_\emptyset\rangle, \quad \Sigma \mid \Gamma \vdash M \approx M_1' : A, \quad \Sigma \mid \emptyset \vdash_C \mathsf{id}_A : A \rightsquigarrow A \quad (\exists M_1') .$$

By the IH,

$$\Sigma \mid \Gamma \vdash_C M : A, \quad \Sigma \mid \Sigma(\Gamma) \vdash_S M_1' : \Sigma(A) .$$

By Lemma I.3, $\Sigma \mid \emptyset \vdash_S |\mathsf{id}_A|_\emptyset : \Sigma(A) \rightsquigarrow \Sigma(A)$. Hence, by (T_Crc_S), $\Sigma \mid \Sigma(\Gamma) \vdash_S M_1'\langle|\mathsf{id}_A|_\emptyset\rangle : \Sigma(A)$.

Case (Bs_CrcMore): We are given

$$M = M_1\langle c\rangle, \quad M' = M_1'\langle s \,;\, |c|_\emptyset\rangle,$$
$$\Sigma \mid \Gamma \vdash M_1 \approx M_1'\langle s\rangle : B, \quad \Sigma \mid \emptyset \vdash_C c : B \rightsquigarrow A \quad (\exists B, c, s, M_1, M_1') .$$

By the IH,

$$\Sigma \mid \Gamma \vdash_C M_1 : B, \quad \Sigma \mid \Sigma(\Gamma) \vdash_S M_1'\langle s\rangle : \Sigma(B) .$$

By (C_Crc_C), we have $\Sigma \mid \Gamma \vdash_C M : A$ is derived. The judgment $\Sigma \mid \Sigma(\Gamma) \vdash_S M_1'\langle s\rangle : \Sigma(B)$ must be derived by (T_Crc_S) and we have $\Sigma \mid \Sigma(\Gamma) \vdash_S M_1' : C$ and $\Sigma \mid \Sigma(\Gamma) \vdash_S s : C \rightsquigarrow \Sigma(B)$ for some C. Since $\mathrm{ftv}(s) = \emptyset$, we have $\Sigma \mid \emptyset \vdash_S s : C \rightsquigarrow \Sigma(B)$ by Lemma E.11 and Lemma I.2. By Lemma I.3, $\Sigma \mid \emptyset \vdash_S |c|_\emptyset : \Sigma(B) \rightsquigarrow \Sigma(A)$. By Lemma E.24, $\Sigma \mid \emptyset \vdash_S s \,;\, |c|_\emptyset : C \rightsquigarrow \Sigma(B)$. Finally, by (T_Crc_S), we have $\Sigma \mid \Sigma(\Gamma) \vdash_S M' : \Sigma(A)$.

Case (Bs_CrcIdL): We are given

$$M = M_1\langle c^I\rangle, \quad \Sigma \mid \Gamma \vdash M_1 \approx M' : B, \quad \Sigma \mid \Gamma \vdash_C c^I : B \rightsquigarrow A \quad (\exists c^I, M_1, B) .$$

By the IH,

$$\Sigma \mid \Gamma \vdash_C M_1 : B, \quad \Sigma \mid \Sigma(\Gamma) \vdash_S M' : \Sigma(B) .$$

Therefore, by (T_Crc_C), $\Sigma \mid \Gamma \vdash_C M_1\langle c^I\rangle : A$. Furthermore, by Lemma I.48, $\Sigma \mid \Sigma(\Gamma) \vdash_S M' : \Sigma(A)$

□

Lemma I.23 (Well-Formedness of Bisimulation)**.** If $\Sigma \mid \Gamma \vdash M \approx M' : A$, then $\vdash \Sigma$, $\Sigma \vdash \Gamma$, and $\Sigma \mid \Gamma \vdash A$.

Proof. By Lemma I.22, $\Sigma \mid \Gamma \vdash_C M : A$. Then, by Lemma E.10, $\vdash \Sigma$ and $\Sigma \vdash \Gamma$ and $\Sigma \mid \Gamma \vdash A$. □

Lemma I.24 (Variable Weakening of Bisimulation)**.** If $\Sigma \mid \Gamma_1, \Gamma_2 \vdash M \approx M' : A$ and $\Sigma \mid \Gamma_1 \vdash B$ and $x \notin \mathrm{dom}(\Gamma_1, \Gamma_2)$, then $\Sigma \mid \Gamma_1, x : B, \Gamma_2 \vdash M \approx M' : A$.

Proof. By straightforward induction on the derivation of $\Sigma \mid \Gamma_1, \Gamma_2 \vdash M \approx M' : A$. □

Lemma I.25 (Type Variable Weakening of Bisimulation)**.** If $\Sigma \mid \Gamma \vdash M \approx M' : A$ and $X \notin \mathrm{dom}(\Gamma)$ is fresh, then $\Sigma \mid X, \Gamma \vdash M \approx M' : A$.

Proof. By straightforward induction on the derivation of $\Sigma \mid \Gamma \vdash M \approx M' : A$. □

Lemma I.26 (Type Binding Weakening of Bisimulation)**.** If $\Sigma \mid \Gamma \vdash M \approx M' : A$ and $\Sigma \mid \emptyset \vdash A'$ and α is fresh, then $\Sigma, \alpha := A' \mid \Gamma \vdash M \approx M' : A$.

Proof. By straightforward induction on the derivation of $\Sigma \mid \Gamma \vdash M \approx M' : A$. □

Lemma I.27 (Value Substitution for Bisimulation)**.** If $\Sigma \mid \Gamma_1, x : B, \Gamma_2 \vdash M \approx M' : A$ and $\Sigma \mid \Gamma_1, \Gamma_2 \vdash V \approx V' : B$, then $\Sigma \mid \Gamma_1, \Gamma_2 \vdash M[x := V] \approx M'[x := V'] : A$.

Proof. By induction on the derivation of $\Sigma \mid \Gamma_1, x : B, \Gamma_2 \vdash M \approx M' : A$. The proof is similar to Lemma E.12 except that Lemmas I.24 and I.25 are used, instead of Lemmas E.7.) □

Lemma I.28 (Distribution of Substitution on Composition)**.** If $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_S s : A \rightsquigarrow B$ and $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_S t : B \rightsquigarrow C$ and α is fresh, then $(s \fatsemi t)[X := \alpha] = s[X := \alpha] \fatsemi t[X := \alpha]$.

Proof. By induction on the sum of the sizes of s and t. We perform case analysis on s.

Case $s = \mathsf{id}$: Lemma I.5 implies $\mathsf{id} \fatsemi t = t$ and $\mathsf{id} \fatsemi t[X := \alpha] = t[X := \alpha]$. Therefore,

$$\begin{aligned}(\mathsf{id} \fatsemi t)[X := \alpha] &= t[X := \alpha] \\ &= \mathsf{id} \fatsemi t[X := \alpha] \\ &= \mathsf{id}[X := \alpha] \fatsemi t[X := \alpha] \ .\end{aligned}$$

Case $s = s_1 \to t_1$ $(\exists s_1, t_1)$: Because $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_S s_1 \to t_1 : A \rightsquigarrow B$ is derived by (Ct_Arrow_S), we have

$$\begin{gathered}A = A_1 \to B_1, \quad B = A_2 \to B_2, \\ \Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_S s_1 : A_2 \rightsquigarrow A_1, \quad \Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_S t_1 : B_1 \rightsquigarrow B_2 \quad (\exists A_1, A_2, B_1, B_2) \ .\end{gathered}$$

We perform case analysis on t.

Case $t = s_2 \to t_2$ $(\exists s_2, t_2)$: Because $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_S s_2 \to t_2 : (A_2 \to B_2) \rightsquigarrow C$ is derived by (Ct_Arrow_S), we have

$$C = A_3 \to B_3, \quad \Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_S s_2 : A_3 \rightsquigarrow A_2, \quad \Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_S t_2 : B_2 \rightsquigarrow B_3 \quad (\exists A_3, B_3) \ .$$

Therefore, by the IHs, $(s_2 \fatsemi s_1)[X := \alpha] = s_2[X := \alpha] \fatsemi s_1[X := \alpha]$ and $(t_1 \fatsemi t_2)[X := \alpha] = t_1[X := \alpha] \fatsemi t_2[X := \alpha]$. Hence,

$$\begin{aligned}((s_1 \to t_1) \fatsemi (s_2 \to t_2))[X := \alpha] &= ((s_2 \fatsemi s_1) \to (t_1 \fatsemi t_2))[X := \alpha] \\ &= (s_2 \fatsemi s_1)[X := \alpha] \to (t_1 \fatsemi t_2)[X := \alpha] \\ &= (s_2[X := \alpha] \fatsemi s_1[X := \alpha]) \to (t_1[X := \alpha] \fatsemi t_2[X := \alpha]) \\ &= (s_1[X := \alpha] \to t_1[X := \alpha]) \fatsemi (s_2[X := \alpha] \to t_2[X := \alpha]) \\ &= (s_1 \to t_1)[X := \alpha] \fatsemi (s_2 \to t_2)[X := \alpha] \ .\end{aligned}$$

Case $t = g_2 \,;\, G_2!(\exists g_2, G_2)$: Because $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_S g_2 \,;\, G_2! : (A_2 \to B_2) \rightsquigarrow C$ is derived by (Ct_Inj_S), we have $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_S g_2 : (A_2 \to B_2) \rightsquigarrow \Sigma(G_2)$. Therefore, by the IH, $((s_1 \to t_1) \fatsemi g_2)[X := \alpha] = (s_1 \to t_1)[X := \alpha] \fatsemi g_2[X := \alpha]$. Hence,

$$\begin{aligned}((s_1 \to t_1) \fatsemi (g_2 \,;\, G_2!))[X := \alpha] &= (((s_1 \to t_1) \fatsemi g_2) \,;\, G_2!)[X := \alpha] \\ &= ((s_1 \to t_1) \fatsemi g_2)[X := \alpha] \,;\, G_2![X := \alpha] \\ &= ((s_1 \to t_1)[X := \alpha] \fatsemi g_2[X := \alpha]) \,;\, G_2![X := \alpha] \\ &= (s_1 \to t_1)[X := \alpha] \fatsemi (g_2[X := \alpha] \,;\, G_2![X := \alpha]) \\ &= (s_1 \to t_1)[X := \alpha] \fatsemi (g_2 \,;\, G_2!)[X := \alpha] \ .\end{aligned}$$

Case $t = \bot^{p_2}$ $(\exists p_2)$:

$$\begin{aligned}((s_1 \to t_1) \fatsemi \bot^{p_2})[X := \alpha] &= \bot^{p_2}[X := \alpha] \\ &= \bot^{p_2} \\ &= (s_1 \to t_1) \fatsemi \bot^{p_2}[X := \alpha] \\ &= (s_1 \to t_1)[X := \alpha] \fatsemi \bot^{p_2}[X := \alpha] \ .\end{aligned}$$

Otherwise: Contradictory because $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_S t : (A_2 \to B_2) \rightsquigarrow C$ does not hold.

Case $s = \forall Y.s_1$ $(\exists Y, s_1)$: Without loss of generality, we can assume $Y \neq X$. Because $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_S \forall Y.s_1 : A \rightsquigarrow B$ is derived by (Ct_All_S), we have

$$A = \forall Y.A', \quad B = \forall Y.B', \quad \Sigma \mid \Gamma_1, X, \Gamma_2, Y \vdash_S s_1 : A' \rightsquigarrow B' \quad (\exists A', B') \ .$$

By case analysis on t.

Case $t = \forall Z.t_1$ $(\exists Z, t_1)$: Because $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_S \forall Z.t_1 : \forall Y.B' \rightsquigarrow C$ is derived by (CT_ALL_S), we have

$$Z = Y, \quad C = \forall Y.C', \quad \Sigma \mid \Gamma_1, X, \Gamma_2, Y \vdash_S t_1 : B' \rightsquigarrow C', \quad (\exists C') .$$

Therefore, by the IH, $(s_1 \fatsemi t_1)[X := \alpha] = s_1[X := \alpha] \fatsemi t_1[X := \alpha]$. Hence,

$$\begin{aligned}((\forall Y.s_1) \fatsemi (\forall Y.t_1))[X := \alpha] &= (\forall Y.(s_1 \fatsemi t_1))[X := \alpha] \\ &= \forall Y.(s_1 \fatsemi t_1)[X := \alpha] \\ &= \forall Y.(s_1[X := \alpha] \fatsemi t_1[X := \alpha]) \\ &= (\forall Y.s_1[X := \alpha]) \fatsemi (\forall Y.t_1[X := \alpha]) \\ &= (\forall Y.s_1)[X := \alpha] \fatsemi (\forall Y.t_1)[X := \alpha] .\end{aligned}$$

Case $t = g \,;\, G!$ $(\exists g, G)$: Provable similarly to the case of $s = s_1 \to t_1$.

Case $t = \bot^{p_2}$ $(\exists p_2)$: Provable similarly to the case of $s = s_1 \to t_1$.

Otherwise: Contradictory because $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_S t : \forall Y.B' \rightsquigarrow C$ does not hold.

Case $s = g_1 \,;\, G_1!$ $(\exists G_1, g_1)$: Because $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_S g_1 \,;\, G_1! : A \rightsquigarrow B$ is derived by (CT_INJ_S), we have

$$B = \star, \quad \Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_S g_1 : A \rightsquigarrow \Sigma(G_1) .$$

By case analysis on t.

Case $t = \mathsf{id}$: Because $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_S \mathsf{id} : \star \rightsquigarrow C$ is derived by (CT_ID_S), we have $C = \star$. By Lemma I.6, $(g_1 \,;\, G_1!) \fatsemi \mathsf{id} = g_1 \,;\, G_1!$ and $(g_1[X := \alpha] \,;\, G_1[X := \alpha]!) \fatsemi \mathsf{id} = g_1[X := \alpha] \,;\, G_1[X := \alpha]!$. Therefore,

$$\begin{aligned}((g_1 \,;\, G_1!) \fatsemi \mathsf{id})[X := \alpha] &= (g_1 \,;\, G_1!)[X := \alpha] \\ &= g_1[X := \alpha] \,;\, G_1[X := \alpha]! \\ &= (g_1[X := \alpha] \,;\, G_1[X := \alpha]!) \fatsemi \mathsf{id} \\ &= (g_1[X := \alpha] \,;\, G_1[X := \alpha]!) \fatsemi \mathsf{id}[X := \alpha] .\end{aligned}$$

Case $t = G_1?^{p_2} \,;\, t_2$ $(\exists p_2, t_2)$: Because $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_S G_1?^{p_2} \,;\, t_2 : \star \rightsquigarrow C$ is derived by (CT_PROJ_S), we have $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_S t_2 : \Sigma(G_1) \rightsquigarrow C$. Therefore, by the IH, $(g_1 \fatsemi t_2)[X := \alpha] = g_1[X := \alpha] \fatsemi t_2[X := \alpha]$. Hence,

$$\begin{aligned}((g_1 \,;\, G_1!) \fatsemi (G_1?^{p_2} \,;\, t_2))[X := \alpha] &= (g_1 \fatsemi t_2)[X := \alpha] \\ &= g_1[X := \alpha] \fatsemi t_2[X := \alpha] \\ &= (g_1[X := \alpha] \,;\, G_1[X := \alpha]!) \fatsemi (G_1[X := \alpha]?^{p_2} \,;\, t_2[X := \alpha]) \\ &= (g_1 \,;\, G_1!)[X := \alpha] \fatsemi (G_1?^{p_2} \,;\, t_2)[X := \alpha] .\end{aligned}$$

Case $t = H_2?^{p_2} \,;\, t_2 (H_2 \neq G_1)(\exists p_2, H_2, t_2)$: Because α is fresh, we have $G_1 \neq \alpha$ and $H_2 \neq \alpha$. Therefore, $G_1[X := \alpha] \neq H_2[X := \alpha]$. Hence,

$$\begin{aligned}((g_1 \,;\, G_1!) \fatsemi (H_2?^{p_2} \,;\, t_2))[X := \alpha] &= \bot^{p_2}[X := \alpha] \\ &= (g_1[X := \alpha] \,;\, G_1[X := \alpha]!) \fatsemi (H_2[X := \alpha]?^{p_2} \,;\, t_2[X := \alpha]) \\ &= (g_1 \,;\, G_1!)[X := \alpha] \fatsemi (H_2?^{p_2} \,;\, t_2)[X := \alpha] .\end{aligned}$$

Case $t = \bot^{p_2}$ $(\exists p_2)$: Provable similarly to the case of $s = s_1 \to t_1$.

Case $s = G_1?^{p_1} \,;\, b_1$ $(\exists G_1, p_1, b_1)$: Because $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_S G_1?^{p_1} \,;\, b_1 : A \rightsquigarrow B$ is derived by (CT_PROJ_S), we have $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_S b_1 : \Sigma(G_1) \rightsquigarrow B$. Therefore, by the IH, $(b_1 \fatsemi t)[X := \alpha] = b_1[X := \alpha] \fatsemi t[X := \alpha]$. Hence,

$$\begin{aligned}((G_1?^{p_1} \,;\, b_1) \fatsemi t)[X := \alpha] &= (G_1?^{p_1} \,;\, (b_1 \fatsemi t))[X := \alpha] \\ &= G_1?^{p_1}[X := \alpha] \,;\, (b_1 \fatsemi t)[X := \alpha] \\ &= G_1?^{p_1}[X := \alpha] \,;\, (b_1[X := \alpha] \fatsemi t[X := \alpha]) \\ &= (G_1?^{p_1}[X := \alpha] \,;\, b_1[X := \alpha]) \fatsemi t[X := \alpha] \\ &= (G_1?^{p_1} \,;\, b_1)[X := \alpha] \fatsemi t[X := \alpha] .\end{aligned}$$

Case $s = \bot^{p_1}$ $(\exists p_1)$:

$$\begin{aligned}(\bot^{p_1} \mathbin{;} t)[X := \alpha] &= \bot^{p_1}[X := \alpha] \\ &= \bot^{p_1} \\ &= \bot^{p_1} \mathbin{;} t[X := \alpha] \\ &= \bot^{p_1}[X := \alpha] \mathbin{;} t[X := \alpha] \ .\end{aligned}$$

□

Lemma I.29 (Identity Coercion Translation Simulates Dynamic Type Substitution)**.** If $\vdash \Sigma$ and $\Sigma \mid \Gamma \vdash_C \mathsf{id}_A : A \rightsquigarrow A$ and $X \notin \mathrm{dom}(\Gamma)$, then $|\mathsf{id}_A[X := \star]|_\Gamma = |\mathsf{id}_A|_\Gamma$.

Proof. By induction on A.

Case $A = \iota$ $(\exists \iota)$: $|\mathsf{id}_\iota[X := \star]|_\Gamma = |\mathsf{id}_{\iota[X:=\star]}|_\Gamma = |\mathsf{id}_\iota|_\Gamma$.

Case $A = \star, A = \beta$ $(\exists \beta)$: Provable similarly to the case of $A = \iota$.

Case $A = A' \to B'$ $(\exists A', B')$: By the IHs, $|\mathsf{id}_{A'}[X := \star]|_\Gamma = |\mathsf{id}_{A'}|_\Gamma$ and $|\mathsf{id}_{B'}[X := \star]|_\Gamma = |\mathsf{id}_{B'}|_\Gamma$. Therefore,

$$\begin{aligned}|\mathsf{id}_{A' \to B'}[X := \star]|_\Gamma &= |\mathsf{id}_{(A' \to B')[X:=\star]}|_\Gamma \\ &= |\mathsf{id}_{A'[X:=\star] \to B'[X:=\star]}|_\Gamma \\ &= |\mathsf{id}_{A'[X:=\star]}|_\Gamma \to |\mathsf{id}_{B'[X:=\star]}|_\Gamma \\ &= |\mathsf{id}_{A'}[X := \star]|_\Gamma \to |\mathsf{id}_{B'}[X := \star]|_\Gamma \\ &= |\mathsf{id}_{A'}|_\Gamma \to |\mathsf{id}_{B'}|_\Gamma \\ &= |\mathsf{id}_{A' \to B'}|_\Gamma \ .\end{aligned}$$

Case $A = \forall Y.A'$ $(\exists Y, A')$: Without loss of generality, we can assume $Y \neq X$. Since $X \notin \mathrm{dom}(\Gamma)$ and $Y \neq X$, $X \notin \mathrm{dom}(\Gamma, Y)$ By the IHs, $|\mathsf{id}_{A'}[X := \star]|_\Gamma = |\mathsf{id}_{A'}|_\Gamma$ and $|\mathsf{id}_{A'}[X := \star]|_{\Gamma, Y} = |\mathsf{id}_{A'}|_{\Gamma, Y}$. Therefore,

$$\begin{aligned}|\mathsf{id}_{\forall Y.A'}[X := \star]|_\Gamma &= |\mathsf{id}_{(\forall Y.A')[X:=\star]}|_\Gamma \\ &= |\mathsf{id}_{\forall Y.A'[X:=\star]}|_\Gamma \\ &= \forall Y.|\mathsf{id}_{A'[X:=\star]}|_{\Gamma, Y} \ ,, |\mathsf{id}_{A'[X:=\star]}|_\Gamma \\ &= \forall Y.|\mathsf{id}_{A'}|_{\Gamma, Y} \ ,, |\mathsf{id}_{A'}|_\Gamma \\ &= |\mathsf{id}_{\forall Y.A'}|_\Gamma \ .\end{aligned}$$

Case $A = Y$ $(\exists Y)$: We perform case analysis on whether $Y = X$ or not.

Case $Y = X$:

$$\begin{aligned}|\mathsf{id}_X[X := \star]|_\Gamma &= |\mathsf{id}_{X[X:=\star]}|_\Gamma \\ &= |\mathsf{id}_\star|_\Gamma \\ &= \mathsf{id} \\ &= |\mathsf{id}_X|_\Gamma \ .\end{aligned}$$

Case $Y \neq X$:

$$\begin{aligned}|\mathsf{id}_Y[X := \star]|_\Gamma &= |\mathsf{id}_{Y[X:=\star]}|_\Gamma \\ &= |\mathsf{id}_Y|_\Gamma \ .\end{aligned}$$

□

Lemma I.30 (Coercion Translation Simulates Dynamic Type Substitution)**.** If $\vdash \Sigma$ and $\Sigma \mid \Gamma \vdash_C c : A \rightsquigarrow B$ and $X \notin \mathrm{dom}(\Gamma)$, then $|c[X := \star]|_\Gamma = |c|_\Gamma$.

Proof. By induction on c with case analysis on c.

Case $c = \mathsf{id}_{A'}$ $(\exists A')$: By Lemma I.29, $|\mathsf{id}_{A'}[X := \star]|_\Gamma = |\mathsf{id}_{A'}|_\Gamma$.

Case $c = G!$ $(\exists G)$:

Case $G = X$: We are given

$$|X!|_\Gamma = \mathsf{id}, \quad X![X := \star] = \mathsf{id}_\star \ .$$

Therefore,

$$\begin{aligned} |X![X := \star]|_\Gamma &= |\mathsf{id}_\star|_\Gamma \\ &= \mathsf{id} \\ &= |X!|_\Gamma \ . \end{aligned}$$

Otherwise: We are given $G![X := \star] = G!$. Therefore,

$$|G![X := \star]|_\Gamma = |G!|_\Gamma \ .$$

Case $c = G?^p$ $(\exists p, G)$: Provable similarly to the case of $c = G!$.

Case $c = \alpha^-$ $(\exists \alpha)$: $|\alpha^-[X := \star]|_\Gamma = |\alpha^-|_\Gamma$.

Case $c = \alpha^+$ $(\exists \alpha)$: Provable similarly to the case of $c = \alpha^-$.

Case $c = c' \to d'$ $(\exists c', d')$: Since $\Sigma \mid \Gamma \vdash_C c' \to d' : A \rightsquigarrow B$ is derived by (Ct_Arrow_S), we have

$$\begin{gathered} A = A_1 \to B_1, \quad B = A_2 \to B_2, \quad \Sigma \mid \Gamma \vdash_C c' : A_2 \rightsquigarrow A_1, \\ \Sigma \mid \Gamma \vdash_C d' : B_1 \rightsquigarrow B_2 \quad (\exists A_1, A_2, B_1, B_2) \ . \end{gathered}$$

By the IHs, $|c'[X := \star]|_\Gamma = |c'|_\Gamma$ and $|d'[X := \star]|_\Gamma = |d'|_\Gamma$. Therefore,

$$\begin{aligned} |(c' \to d')[X := \star]|_\Gamma &= |c'[X := \star] \to d'[X := \star]|_\Gamma \\ &= |c'[X := \star]|_\Gamma \to |d'[X := \star]|_\Gamma \\ &= |c'|_\Gamma \to |d'|_\Gamma \\ &= |c' \to d'|_\Gamma \ . \end{aligned}$$

Case $c = \forall Y.c'$ $(\exists Y, c')$: Without loss of generality, we can assume $Y \neq X$. Since $\Sigma \mid \Gamma \vdash_C \forall Y.c' : A \rightsquigarrow B$ is derived by (Ct_All_S), we have

$$A = \forall Y.A_1, \quad B = \forall Y.B_1, \quad \Sigma \mid \Gamma, Y \vdash_C c' : A_1 \rightsquigarrow B_1 \quad (\exists A_1, B_1) \ .$$

Since $X \notin \mathrm{dom}(\Gamma)$ and $Y \neq X$, we have $X \notin \mathrm{dom}(\Gamma, Y)$. By the IHs, $|c'[X := \star]|_\Gamma = |c'|_\Gamma$ and $|c'[X := \star]|_{\Gamma,Y} = |c'|_{\Gamma,Y}$. Hence,

$$\begin{aligned} |(\forall Y.c')[X := \star]|_\Gamma &= |\forall Y.c'[X := \star]|_\Gamma \\ &= \forall Y.|c'[X := \star]|_{\Gamma,Y} \,,, |c'[X := \star]|_\Gamma \\ &= \forall Y.|c'|_{\Gamma,Y} \,,, |c'|_\Gamma \\ &= |\forall Y.c'|_\Gamma \ . \end{aligned}$$

Case $c = c' \,;\, d'$ $(\exists c', d')$: Since $\Sigma \mid \Gamma \vdash_C c' \,;\, d' : A \rightsquigarrow B$ is derived by (Ct_Seq_S), we have

$$\Sigma \mid \Gamma \vdash_C c' : A \rightsquigarrow C, \quad \Sigma \mid \Gamma \vdash_C d' : C \rightsquigarrow B \quad (\exists C) \ .$$

By the IHs, $|c'[X := \star]|_\Gamma = |c'|_\Gamma$ and $|d'[X := \star]|_\Gamma = |d'|_\Gamma$. Therefore,

$$\begin{aligned} |(c' \,;\, d')[X := \star]|_\Gamma &= |c'[X := \star] \,;\, d'[X := \star]|_\Gamma \\ &= |c'[X := \star]|_\Gamma \mathbin{⨾} |d'[X := \star]|_\Gamma \\ &= |c'|_\Gamma \mathbin{⨾} |d'|_\Gamma \\ &= |c' \,;\, d'|_\Gamma \ . \end{aligned}$$

Case $c = \bot^p_{A' \rightsquigarrow B'}$ $(\exists p, A', B')$:

$$\begin{aligned} |\bot^p_{A' \rightsquigarrow B'}[X := \star]|_\Gamma &= |\bot^p_{A'[X:=\star] \rightsquigarrow B'[X:=\star]}|_\Gamma \\ &= \bot^p \\ &= |\bot^p_{A' \rightsquigarrow B'}|_\Gamma \ . \end{aligned}$$

□

Lemma I.31 (Identity Coercion Translation and Substitution)**.** Suppose $\vdash \Sigma$ and $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_C \mathsf{id}_A : A \rightsquigarrow A$.

(1) If α is fresh, $|\mathsf{id}_A|_\Gamma[X := \alpha] = |\mathsf{id}_A[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])}$.

(2) $|\mathsf{id}_A|_{\Gamma_1,X,\Gamma_2}[X := \star] = |\mathsf{id}_A[X := \star]|_{\Gamma_1,\Gamma_2[X:=\star]}$.

Proof. (1) By induction on A.

Case $A = \iota$ $(\exists \iota)$:

$$\begin{aligned} |\mathsf{id}_\iota|_{\Gamma_1,X,\Gamma_2}[X := \alpha] &= \mathsf{id}[X := \alpha] \\ &= \mathsf{id} \\ &= |\mathsf{id}_\iota|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |\mathsf{id}_{\iota[X:=\alpha]}|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |\mathsf{id}_\iota[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \ . \end{aligned}$$

Case $A = \star, A = \beta$ $(\exists \beta)$: Provable similarly to the case of $A = \iota$.

Case $A = A' \to B'$ $(\exists A', B')$: By the IHs, $|\mathsf{id}_{A'}|_{\Gamma_1,X,\Gamma_2}[X := \alpha] = |\mathsf{id}_{A'}[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])}$ and $|\mathsf{id}_{B'}|_{\Gamma_1,X,\Gamma_2}[X := \alpha] = |\mathsf{id}_{B'}[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])}$. Therefore,

$$\begin{aligned} |\mathsf{id}_{A' \to B'}|_{\Gamma_1,X,\Gamma_2}[X := \alpha] &= (|\mathsf{id}_{A'}|_{\Gamma_1,X,\Gamma_2} \to |\mathsf{id}_{B'}|_{\Gamma_1,X,\Gamma_2})[X := \alpha] \\ &= |\mathsf{id}_{A'}|_{\Gamma_1,X,\Gamma_2}[X := \alpha] \to |\mathsf{id}_{B'}|_{\Gamma_1,X,\Gamma_2}[X := \alpha] \\ &= |\mathsf{id}_{A'}[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \to |\mathsf{id}_{B'}[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |\mathsf{id}_{A'[X:=\alpha]}|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \to |\mathsf{id}_{B'[X:=\alpha]}|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |\mathsf{id}_{A'[X:=\alpha] \to B'[X:=\alpha]}|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |\mathsf{id}_{(A' \to B')[X:=\alpha]}|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |\mathsf{id}_{A' \to B'}[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \ . \end{aligned}$$

Case $A = \forall Y.A'$ $(\exists Y, A')$: Without loss of generality, we can assume $Y \neq X$. Since $X \in \mathrm{dom}(\Gamma_1, X, \Gamma_2)$, $X \in \mathrm{dom}(\Gamma_1, X, \Gamma_2, Y)$. By the IHs, $|\mathsf{id}_{A'}|_{\Gamma_1,X,\Gamma_2}[X := \alpha] = |\mathsf{id}_{A'}[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])}$ and $|\mathsf{id}_{A'}|_{\Gamma_1,X,\Gamma_2,Y}[X := \alpha] = |\mathsf{id}_{A'}[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha]),Y}$. Therefore,

$$\begin{aligned} |\mathsf{id}_{\forall Y.A'}|_{\Gamma_1,X,\Gamma_2}[X := \alpha] &= (\forall Y.|\mathsf{id}_{A'}|_{\Gamma_1,X,\Gamma_2,Y} \,,, |\mathsf{id}_{A'}|_{\Gamma_1,X,\Gamma_2})[X := \alpha] \\ &= \forall Y.|\mathsf{id}_{A'}|_{\Gamma_1,X,\Gamma_2,Y}[X := \alpha] \,,, |\mathsf{id}_{A'}|_{\Gamma_1,X,\Gamma_2}[X := \alpha] \\ &= \forall Y.|\mathsf{id}_{A'}[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha]),Y} \,,, |\mathsf{id}_{A'}[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= \forall Y.|\mathsf{id}_{A'[X:=\alpha]}|_{\Gamma_1,(\Gamma_2[X:=\alpha]),Y} \,,, |\mathsf{id}_{A'[X:=\alpha]}|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |\mathsf{id}_{\forall Y.(A'[X:=\alpha])}|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |\mathsf{id}_{(\forall Y.A')[X:=\alpha]}|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |\mathsf{id}_{\forall Y.A'}[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \ . \end{aligned}$$

Case $A = Y$ $(\exists Y)$: We consider the following cases.

Case $Y \in \mathrm{dom}(\Gamma_1, X, \Gamma_2)$: We have

$$\begin{aligned}|\mathsf{id}_Y|_{\Gamma_1,X,\Gamma_2}[X := \alpha] &= \mathsf{id}[X := \alpha] \\ &= \mathsf{id}\ .\end{aligned}$$

We perform case analysis on whether $Y = X$ or not.
Case $Y = X$:

$$\begin{aligned}\mathsf{id} &= |\mathsf{id}_\alpha|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |\mathsf{id}_{X[X:=\alpha]}|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |\mathsf{id}_X[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])}\ .\end{aligned}$$

Case $Y \neq X$: We are given

$$Y \in \Gamma_1, (\Gamma_2[X := \alpha]), \quad |\mathsf{id}_Y|_{\Gamma_1,(\Gamma_2[X:=\alpha])} = \mathsf{id}\ .$$

Therefore,

$$\begin{aligned}\mathsf{id} &= |\mathsf{id}_Y|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |\mathsf{id}_{Y[X:=\alpha]}|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |\mathsf{id}_Y[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])}\ .\end{aligned}$$

Case $Y \notin \mathrm{dom}(\Gamma_1, X, \Gamma_2)$: Contradictory.

(2) Provable similarly to the case (1).

□

Lemma I.32 (Coercion Translation and Type Substitution)**.** Let $\vdash \Sigma$ and $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_C c : A \rightsquigarrow B$.

(1) If α is fresh, then $|c|_{\Gamma_1,X,\Gamma_2}[X := \alpha] = |c[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])}$.

(2) $|c|_{\Gamma_1,X,\Gamma_2}[X := \star] = |c[X := \star]|_{\Gamma_1,(\Gamma_2[X:=\star])}$.

Proof. (1) By induction on c with case analysis on c.

Case $c = \mathsf{id}_{A'}$ $(\exists A')$: Straightforward by Lemma I.31.

Case $c = G!$ $(\exists G)$:

Case $G = Y$ and $Y \notin (\Gamma_1, X, \Gamma_2)$ $(\exists Y)$: Contradictory.

Otherwise: Since $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_C G! : A \rightsquigarrow B$ is derived by (Ct_Inj_C), we have

$$\vdash \Sigma, \quad \Sigma \vdash \Gamma_1, X, \Gamma_2, \quad \Sigma \mid \Gamma_1, X, \Gamma_2 \vdash G\ .$$

By (Ct_Id_C), we have $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_C \mathsf{id}_G : G \rightsquigarrow G$. Therefore, by Lemma I.31,

$$|\mathsf{id}_G|_{\Gamma_1,X,\Gamma_2}[X := \alpha] = |\mathsf{id}_G[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])}\ .$$

Hence,

$$\begin{aligned}|G!|_{\Gamma_1,X,\Gamma_2}[X := \alpha] &= (|\mathsf{id}_G|_{\Gamma_1,X,\Gamma_2}\ ;\ G!)[X := \alpha] \\ &= |\mathsf{id}_G|_{\Gamma_1,X,\Gamma_2}[X := \alpha]\ ;\ (G![X := \alpha]) \\ &= |\mathsf{id}_G[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])}\ ;\ (G![X := \alpha]) \\ &= |\mathsf{id}_{G[X:=\alpha]}|_{\Gamma_1,(\Gamma_2[X:=\alpha])}\ ;\ (G![X := \alpha]) \\ &= |\mathsf{id}_{G[X:=\alpha]}|_{\Gamma_1,(\Gamma_2[X:=\alpha])}\ ;\ G[X := \alpha]! \\ &= |G[X := \alpha]!|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |G![X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])}\ .\end{aligned}$$

Case $c = G?^p$ $(\exists p, G)$: Provable similarly to the case of $c = G!$.

Case $c = \beta^-$ $(\exists \beta)$:

$$\begin{aligned} |\beta^-|_{\Gamma_1,X,\Gamma_2}[X := \alpha] &= \mathsf{id}[X := \alpha] \\ &= \mathsf{id} \\ &= |\beta^-|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |\beta^-[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \ . \end{aligned}$$

Case $c = \beta^+$ $(\exists \beta)$: Provable similarly to the case of $c = \beta^-$.

Case $c = c' \to d'$ $(\exists c', d')$: Because $\Sigma \mid \Gamma \vdash_C c' \to d' : A \rightsquigarrow B$ is derived by (Ct_Arrow_C), we have

$$A = A_1 \to B_1, \quad B = A_2 \to B_2, \quad \Sigma \mid \Gamma \vdash_C c' : A_2 \rightsquigarrow A_1, \quad \Sigma \mid \Gamma \vdash_C d' : B_1 \rightsquigarrow B_2 \quad (\exists A_1, A_2, B_1, B_2) \ .$$

Therefore, by the IHs, we have

$$\begin{aligned} |c'|_{\Gamma_1,X,\Gamma_2}[X := \alpha] &= |c'[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ |d'|_{\Gamma_1,X,\Gamma_2}[X := \alpha] &= |d'[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \ . \end{aligned}$$

Hence,

$$\begin{aligned} |c' \to d'|_{\Gamma_1,X,\Gamma_2}[X := \alpha] &= (|c'|_{\Gamma_1,X,\Gamma_2} \to |d'|_{\Gamma_1,X,\Gamma_2})[X := \alpha] \\ &= |c'|_{\Gamma_1,X,\Gamma_2}[X := \alpha] \to |d'|_{\Gamma_1,X,\Gamma_2}[X := \alpha] \\ &= |c'[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \to |d'[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |c'[X := \alpha] \to d'[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |(c' \to d')[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \ . \end{aligned}$$

Case $c = \forall Y.c'$ $(\exists Y, c')$: Without loss of generality, we can assume $Y \neq X$. Because $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_C \forall Y.c' : A \rightsquigarrow B$ is derived by (Ct_All_C), we have

$$A = \forall Y.A', \quad B = \forall Y.B', \quad \Sigma \mid \Gamma_1, X, \Gamma_2, Y \vdash_C c' : A' \rightsquigarrow B' \quad (\exists A', B') \ .$$

Since $Y \neq X$, we have $(\Gamma_2, Y)[X := \alpha] = (\Gamma_2[X := \alpha]), Y$. Therefore, by the IHs, we have

$$\begin{aligned} |c'|_{\Gamma_1,X,\Gamma_2}[X := \alpha] &= |c'[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ |c'|_{\Gamma_1,X,\Gamma_2,Y}[X := \alpha] &= |c'[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha]),Y} \ . \end{aligned}$$

Hence,

$$\begin{aligned} |\forall Y.c'|_{\Gamma_1,X,\Gamma_2}[X := \alpha] &= (\forall Y.|c'|_{\Gamma_1,X,\Gamma_2,Y} \,,, |c'|_{\Gamma_1,X,\Gamma_2})[X := \alpha] \\ &= \forall Y.|c'|_{\Gamma_1,X,\Gamma_2,Y}[X := \alpha] \,,, |c'|_{\Gamma_1,X,\Gamma_2}[X := \alpha] \\ &= \forall Y.|c'[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha]),Y} \,,, |c'[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |\forall Y.c'[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |(\forall Y.c')[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \ . \end{aligned}$$

Case $c = c' \,; d'$ $(\exists c', d')$: Because $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_C c' \,; d' : A \rightsquigarrow B$ is derived by (Ct_Seq_C), we have

$$\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_C c' : A \rightsquigarrow A', \quad \Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_C d' : A' \rightsquigarrow B \quad (\exists A') \ .$$

By the IHs,

$$\begin{aligned} |c'|_{\Gamma_1,X,\Gamma_2}[X := \alpha] &= |c'[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ |d'|_{\Gamma_1,X,\Gamma_2}[X := \alpha] &= |d'[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \ . \end{aligned}$$

Furthermore, by Lemma I.3,

$$\Sigma \mid \Sigma(\Gamma_1, X, \Gamma_2) \vdash_S |c'|_{\Gamma_1,X,\Gamma_2} : \Sigma(A) \rightsquigarrow \Sigma(A'), \quad \Sigma \mid \Sigma(\Gamma_1, X, \Gamma_2) \vdash_S |d'|_{\Gamma_1,X,\Gamma_2} : \Sigma(A') \rightsquigarrow \Sigma(B) \ .$$

Hence, by Lemma I.28,

$$(|c'|_{\Gamma_1,X,\Gamma_2} \mathbin{⨟} |d'|_{\Gamma_1,X,\Gamma_2})[X := \alpha] = |c'|_{\Gamma_1,X,\Gamma_2}[X := \alpha] \mathbin{⨟} |d'|_{\Gamma_1,X,\Gamma_2}[X := \alpha] \ .$$

Therefore,

$$\begin{aligned} |c' \,;\, d'|_{\Gamma_1,X,\Gamma_2}[X := \alpha] &= (|c'|_{\Gamma_1,X,\Gamma_2} \mathbin{⨟} |d'|_{\Gamma_1,X,\Gamma_2})[X := \alpha] \\ &= |c'|_{\Gamma_1,X,\Gamma_2}[X := \alpha] \mathbin{⨟} |d'|_{\Gamma_1,X,\Gamma_2}[X := \alpha] \\ &= |c'[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \mathbin{⨟} |d'[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |c'[X := \alpha] \,;\, d'[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |(c' \,;\, d')[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \ . \end{aligned}$$

Case $c = \bot^p_{A' \rightsquigarrow B'}$ $(\exists p, A', B')$:

$$\begin{aligned} |\bot^p_{A' \rightsquigarrow B'}|_{\Gamma_1,X,\Gamma_2}[X := \alpha] &= \bot^p[X := \alpha] \\ &= \bot^p \\ &= |\bot^p_{A'[X:=\alpha] \rightsquigarrow B'[X:=\alpha]}|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \\ &= |\bot^p_{A' \rightsquigarrow B'}[X := \alpha]|_{\Gamma_1,(\Gamma_2[X:=\alpha])} \ . \end{aligned}$$

(2) We only show the interesting cases because the other cases are proved similarly as the case (1). By induction on c with case analysis on c.

Case $c = G!$ $(\exists G)$:

Case $G = Y$ $(\exists Y)$:

Case $Y \notin (\Gamma_1, X, \Gamma_2)$: Contradictory.

Case $Y \in (\Gamma_1, X, \Gamma_2)$: Since $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_C Y! : A \rightsquigarrow B$ is derive by (Ct_Inj_C),

$$A = G, \quad B = \star, \quad \vdash \Sigma, \quad \Sigma \vdash \Gamma_1, X, \Gamma_2, \quad \Sigma \mid \Gamma_1, X, \Gamma_2 \vdash Y \ .$$

By (Ct_Id_C), $\Sigma \mid \Gamma_1, X, \Gamma_2 \vdash_C \mathsf{id}_Y : Y \rightsquigarrow Y$. By case analysis on whether $Y = X$ or not.

Case $Y = X$: By Lemma I.31, $|\mathsf{id}_X|_{\Gamma_1,X,\Gamma_2}[X := \star] = |\mathsf{id}_X[X := \star]|_{\Gamma_1,(\Gamma_2[X:=\star])}$. Therefore,

$$\begin{aligned} |X!|_{\Gamma_1,X,\Gamma_2}[X := \star] &= (|\mathsf{id}_X|_{\Gamma_1,X,\Gamma_2} \,;\, X!)[X := \star] \\ &= |\mathsf{id}_X|_{\Gamma_1,X,\Gamma_2}[X := \star] \\ &= |\mathsf{id}_X[X := \star]|_{\Gamma_1,(\Gamma_2[X:=\star])} \\ &= |\mathsf{id}_{X[X:=\star]}|_{\Gamma_1,(\Gamma_2[X:=\star])} \\ &= |\mathsf{id}_\star|_{\Gamma_1,(\Gamma_2[X:=\star])} \\ &= |X![X := \star]|_{\Gamma_1,(\Gamma_2[X:=\star])} \ . \end{aligned}$$

Case $Y \neq X$: By Lemma I.31, $|\mathsf{id}_Y|_{\Gamma_1,X,\Gamma_2}[X := \star] = |\mathsf{id}_Y[X := \star]|_{\Gamma_1,(\Gamma_2[X:=\star])}$. Furthermore, since $Y \in (\Gamma_1, X, \Gamma_2)$, we have $Y \in (\Gamma_1, (\Gamma_2[X := \star]))$. Therefore,

$$|Y!|_{\Gamma_1,(\Gamma_2[X:=\star])} = |\mathsf{id}_Y|_{\Gamma_1,(\Gamma_2[X:=\star])} \,;\, Y! \ .$$

Hence,

$$\begin{aligned} |Y!|_{\Gamma_1,X,\Gamma_2}[X := \star] &= (|\mathsf{id}_Y|_{\Gamma_1,X,\Gamma_2} \,;\, Y!)[X := \star] \\ &= |\mathsf{id}_Y|_{\Gamma_1,X,\Gamma_2}[X := \star] \,;\, Y! \\ &= |\mathsf{id}_Y[X := \star]|_{\Gamma_1,(\Gamma_2[X:=\star])} \,;\, Y! \\ &= |\mathsf{id}_{Y[X:=\star]}|_{\Gamma_1,(\Gamma_2[X:=\star])} \,;\, Y! \\ &= |\mathsf{id}_Y|_{\Gamma_1,(\Gamma_2[X:=\star])} \,;\, Y! \\ &= |Y!|_{\Gamma_1,(\Gamma_2[X:=\star])} \\ &= |Y![X := \star]|_{\Gamma_1,(\Gamma_2[X:=\star])} \ . \end{aligned}$$

Otherwise: Since G does not contain type variables, we have $G[X := \star] = G$ and $G![X := \star] = G!$. By Lemma I.31, $|\mathsf{id}_G|_{\Gamma_1,X,\Gamma_2}[X := \star] = |\mathsf{id}_G[X := \star]|_{\Gamma_1,(\Gamma_2[X:=\star])}$. Therefore,

$$\begin{aligned}
|G!|_{\Gamma_1,X,\Gamma_2}[X := \star] &= (|\mathsf{id}_G|_{\Gamma_1,X,\Gamma_2} \mathbin{;} G!)[X := \star] \\
&= |\mathsf{id}_G|_{\Gamma_1,X,\Gamma_2}[X := \star] \mathbin{;} G! \\
&= |\mathsf{id}_G[X := \star]|_{\Gamma_1,(\Gamma_2[X:=\star])} \mathbin{;} G! \\
&= |\mathsf{id}_{G[X:=\star]}|_{\Gamma_1,(\Gamma_2[X:=\star])} \mathbin{;} G! \\
&= |\mathsf{id}_G|_{\Gamma_1,(\Gamma_2[X:=\star])} \mathbin{;} G! \\
&= |G!|_{\Gamma_1,(\Gamma_2[X:=\star])} \\
&= |G![X := \star]|_{\Gamma_1,(\Gamma_2[X:=\star])} \ .
\end{aligned}$$

Case $c = G?^p$ $(\exists p, G)$: Provable similarly to the case of $c = G!$.

Otherwise: Provable similarly to the case (1).

□

Lemma I.33 (Type Substitution for Bisimulation)**.** Let $\Sigma \mid X, \Gamma \vdash M \approx M' : A$.

(1) If α is fresh and $\Sigma \mid \emptyset \vdash B$, then $\Sigma, \alpha := B \mid \Gamma[X := \alpha] \vdash M[X := \alpha] \approx M'[X := \alpha] : A[X := \alpha]$.

(2) $\Sigma \mid \Gamma[X := \star] \vdash M[X := \star] \approx M'[X := \star] : A[X := \star]$.

Proof. By induction on the derivation of $\Sigma \mid X, \Gamma \vdash M \approx M' : A$. We perform case analysis on the rule applied last to derive $\Sigma \mid X, \Gamma \vdash M \approx M' : A$. We show only $\Sigma, \alpha := B \mid \Gamma[X := \alpha] \vdash M[X := \alpha] \approx M'[X := \alpha] : A[X := \alpha]$. $\Sigma \mid \Gamma[X := \star] \vdash M[X := \star] \approx M'[X := \star] : A[X := \star]$ can be similarly shown.

Case (Bs_Crc): We are given

$$M = M_1\langle c\rangle, \quad M' = M_1'\langle |c|_{X,\Gamma}\rangle, \quad \Sigma \mid X, \Gamma \vdash M_1 \approx M_1' : B, \quad \Sigma \mid X, \Gamma \vdash_C c : B \rightsquigarrow A \quad (\exists M_1, M_1', c, B) \ .$$

By the IH, we have $\Sigma \mid \Gamma[X := \alpha] \vdash M_1[X := \alpha] \approx M_1'[X := \alpha] : B[X := \alpha]$. By Lemma E.13, we have $\Sigma \mid \Gamma[X := \alpha] \vdash_C c[X := \alpha] : B[X := \alpha] \rightsquigarrow A[X := \alpha]$. By Lemma I.32, we have $|c[X := \alpha]|_{\Gamma[X:=\alpha]} = (|c|_{X,\Gamma})[X := \alpha]$. Rule (Bs_Crc) finishes the case.

Case (Bs_CrcId): We are given

$$M' = M_1'\langle |\mathsf{id}_A|_\emptyset\rangle, \quad \Sigma \mid X, \Gamma \vdash M \approx M_1' : A, \quad \Sigma \mid \emptyset \vdash_C \mathsf{id}_A : A \rightsquigarrow A \quad (\exists M_1') \ .$$

We have

$$(M_1'\langle |\mathsf{id}_A|_\emptyset\rangle)[X := \alpha] = M_1'[X := \alpha]\langle (|\mathsf{id}_A|_\emptyset)[X := \alpha]\rangle \ .$$

By Lemma I.31, we have $|\mathsf{id}_A|_\emptyset[X := \alpha] = |\mathsf{id}_A[X := \alpha]|_\emptyset = |\mathsf{id}_{A[X:=\alpha]}|_\emptyset$. By the IH,

$$\Sigma, \alpha := B \mid \Gamma[X := \alpha] \vdash M[X := \alpha] \approx M_1'[X := \alpha] : A[X := \alpha] \ .$$

Rule (Bs_CrcId) finishes the case.

Case (Bs_CrcMore): We are given

$$\begin{gathered}
M = M_1\langle c\rangle, \quad M' = M_1'\langle s \mathbin{⨟} |c|_\emptyset\rangle, \\
\Sigma \mid X, \Gamma \vdash M_1 \approx M_1'\langle s\rangle : C, \quad \Sigma \mid \emptyset \vdash_C c : C \rightsquigarrow A \quad (\exists C, c, s, M_1, M_1') \ .
\end{gathered}$$

We have

$$\begin{gathered}
(M_1\langle c\rangle)[X := \alpha] = M_1[X := \alpha]\langle c[X := \alpha]\rangle, \\
(M_1'\langle s \mathbin{⨟} |c|_\emptyset\rangle)[X := \alpha] = M_1'[X := \alpha]\langle (s \mathbin{⨟} |c|_\emptyset)[X := \alpha]\rangle \ .
\end{gathered}$$

By the IH,

$$\Sigma, \alpha := B \mid \Gamma[X := \alpha] \vdash M_1[X := \alpha] \approx M_1'[X := \alpha]\langle s[X := \alpha]\rangle : C[X := \alpha] \ .$$

By Lemma E.13, $\Sigma \mid \emptyset \vdash_C c[X := \alpha] : C[X := \alpha] \rightsquigarrow A[X := \alpha]$. Since $\mathrm{ftv}(s) = \mathrm{ftv}(c) = \emptyset$, we have $\mathrm{ftv}(s[X := \alpha]) = \emptyset$. By Lemma I.32, $|c[X := \alpha]|_\emptyset = |c|_{\emptyset,X}[X := \alpha]$. By Lemma I.28, $(s \,⨟\, |c|_\emptyset)[X := \alpha] = s[X := \alpha] \,⨟\, |c|_\emptyset[X := \alpha]$. Therefore,

$$\begin{aligned} s[X := \alpha] \,⨟\, |c[X := \alpha]|_\emptyset &= s[X := \alpha] \,⨟\, |c|_\emptyset[X := \alpha] \\ &= (s \,⨟\, |c|_\emptyset)[X := \alpha] \ . \end{aligned}$$

Rule (Bs_CrcMore) finishes the case.

Case (Bs_CrcIdL): We are given

$$M = M_1\langle c^I\rangle, \quad \Sigma \mid X, \Gamma \vdash M_1 \approx M' : D, \quad \Sigma \mid \emptyset \vdash_C c^I : D \rightsquigarrow A \quad (\exists M_1, c^I, D) \ .$$

By the IH,

$$\Sigma, \alpha := B \mid \Gamma[X := \alpha] \vdash M_1[X := \alpha] \approx M'[X := \alpha] : D[X := \alpha] \ .$$

By Lemma E.6 and Lemma E.13,

$$\Sigma, \alpha := B \mid \emptyset \vdash_C c^I[X := \alpha] : D[X := \alpha] \rightsquigarrow A[X := \alpha] \ .$$

Since no-op coercions are closed under type name substitution, we have, by (Bs_CrcIdL),

$$\Sigma, \alpha := B \mid \Gamma[X := \alpha] \vdash (M_1[X := \alpha])\langle c^I[X := \alpha]\rangle \approx M'[X := \alpha] : D[X := \alpha] \ ,$$

finishing the case.

Otherwise: Similarly to Lemma E.14. The proof uses Lemmas I.24 and I.25 instead of Lemma E.7.

□

Lemma I.34 (Coercion Composition Preserves Bisimilarity)**.** If $\Sigma \mid \emptyset \vdash M \approx M'\langle s\rangle\langle t\rangle : A$, then $\Sigma \mid \emptyset \vdash M \approx M'\langle s \,⨟\, t\rangle : A$.

Proof. By induction on the derivation of $\Sigma \mid \emptyset \vdash M \approx M'\langle s\rangle\langle t\rangle : A$. We perform case analysis on the rule applied last to derive $\Sigma \mid \emptyset \vdash M \approx M'\langle s\rangle\langle t\rangle : A$, which is either (Bs_Crc), (Bs_CrcId), (Bs_CrcMore), or (Bs_CrcIdL).

Case (Bs_Crc): We are given

$$M = M_1\langle c\rangle, \quad t = |c|_\emptyset, \quad \Sigma \mid \emptyset \vdash M_1 \approx M'\langle s\rangle : B, \quad \Sigma \mid \emptyset \vdash_C c : B \rightsquigarrow A \quad (\exists M_1, M_1', c)$$

By Lemma I.22, we have $\Sigma \mid \emptyset \vdash_S M'\langle s\rangle : \Sigma(B)$. By inversion, $\Sigma \mid \emptyset \vdash_S s : C \rightsquigarrow \Sigma(B)$ for some C. Thus, $\mathrm{ftv}(s) = \emptyset$. By (Bs_CrcMore),

$$\Sigma \mid \emptyset \vdash M\langle c\rangle \approx M'\langle s \,⨟\, |c|_\emptyset\rangle : A \ .$$

Case (Bs_CrcId): We are given

$$t = |\mathsf{id}_A|_\emptyset, \quad \Sigma \mid \emptyset \vdash M \approx M'\langle s\rangle : A, \quad \Sigma \mid \emptyset \vdash_C \mathsf{id}_A : A \rightsquigarrow A \ .$$

By Lemma I.22, we have $\Sigma \mid \emptyset \vdash_S M'\langle s\rangle : \Sigma(A)$. By inversion, $\Sigma \mid \emptyset \vdash_S s : C \rightsquigarrow \Sigma(A)$ for some C. Thus, $\mathrm{ftv}(s) = \emptyset$. By (Bs_CrcMore),

$$\Sigma \mid \emptyset \vdash M\langle \mathsf{id}_A\rangle \approx M'\langle s \,⨟\, |\mathsf{id}_A|_\emptyset\rangle : A \ .$$

Case (Bs_CrcMore): We are given

$$M = M_1\langle c\rangle, \quad t = t' \fatsemi |c|_\emptyset, \quad \mathrm{ftv}(t') = \emptyset,$$
$$\Sigma \mid \emptyset \vdash M_1 \approx M'\langle s\rangle\langle t'\rangle : B, \quad \Sigma \mid \emptyset \vdash_C c : B \rightsquigarrow A \quad (\exists B, c, t', M_1)\ .$$

By the IH, $\Sigma \mid \emptyset \vdash M_1 \approx M'\langle s \fatsemi t'\rangle : B$. By Lemma I.22, we have $\Sigma \mid \emptyset \vdash_S M'\langle s \fatsemi t'\rangle : \Sigma(B)$. By inversion, $\Sigma \mid \emptyset \vdash_S s \fatsemi t' : C \rightsquigarrow \Sigma(B)$ for some C. Thus, $\mathrm{ftv}(s \fatsemi t') = \emptyset$. By (Bs_CrcMore),

$$\Sigma \mid \emptyset \vdash M\langle \mathsf{id}_A\rangle \approx M'\langle (s \fatsemi t') \fatsemi |c|_\emptyset\rangle : A\ .$$

Lemma I.7 finishes the case.

Case (Bs_CrcIdL): We are given

$$M = M_1\langle c^I\rangle, \quad \Sigma \mid \Gamma \vdash_C c^I : B \rightsquigarrow A, \quad \Sigma \mid \Gamma \vdash M_1 \approx M'\langle s\rangle\langle t\rangle : B \quad (\exists M_1, c^I, B)\ .$$

By the IH, $\Sigma \mid \Gamma \vdash M_1 \approx M'\langle s \fatsemi t\rangle : B$. Applying (Bs_CrcIdL) finishes the case.

□

Lemma I.35 (Typing Coercions in Coercion Sequences)**.** Let $n > 0$. If $\Sigma \vdash \emptyset, \langle c_1\rangle, \ldots, \langle c_n\rangle : A \rightsquigarrow B$, then there exist some $A_0, \ldots, A_n$ such that $A_0 = A$ and $A_n = B$ and, for any i such that $n \geq i > 0$, $\Sigma \mid \emptyset \vdash_C c_i : A_{i-1} \rightsquigarrow A_i$.

Proof. By induction on n.

Case $n = 1$: Because $\Sigma \vdash \emptyset, \langle c_1\rangle : A \rightsquigarrow B$ is derived by (Ct_Cons_C), we have

$$\Sigma \vdash \emptyset : A \rightsquigarrow A', \quad \Sigma \mid \emptyset \vdash_C c_1 : A' \rightsquigarrow B \quad (\exists A')\ .$$

Because $\Sigma \vdash \emptyset : A \rightsquigarrow A'$ is derived by (Ct_Nil_C), we have $A = A'$. Therefore, $\Sigma \mid \emptyset \vdash_C c_1 : A \rightsquigarrow B$.

Case $n = k + 1 (k \geq 1)$: Because $\Sigma \vdash \emptyset, \langle c_1\rangle, \ldots, \langle c_k\rangle, \langle c_{k+1}\rangle : A \rightsquigarrow B$ is derived by (Ct_Cons_C), we have

$$\Sigma \vdash \emptyset, \langle c_1\rangle, \ldots, \langle c_k\rangle : A \rightsquigarrow A', \quad \Sigma \mid \emptyset \vdash_C c_{k+1} : A' \rightsquigarrow B \quad (\exists A')\ .$$

Therefore, by the IH, there exist some $A_0, \ldots, A_k$ such that $A_0 = A$ and $A_k = A'$ and, for any i such that $k \geq i > 0$, $\Sigma \mid \emptyset \vdash_C c_i : A_{i-1} \rightsquigarrow A_i$. Therefore, we have the conclusion by letting $A_{k+1} = B$.

□

Lemma I.36 (Canonical Forms of Coercion Applications)**.** For any M and c, there exist some M', $n > 0$, and $c_1, \ldots, c_n$ such that M' is not a coercion application and $M\langle c\rangle = M'\langle c_1\rangle \cdots \langle c_n\rangle$ and $c_n = c$.

Proof. By induction on M. We perform case analysis on M.

Case $M = M''\langle c''\rangle$ $(\exists c'', M'')$: By the IH, there exist some M''' that is not a coercion application, $m > 0$, and $c'_1, \ldots, c'_m$ such that $M''\langle c''\rangle = M'''\langle c'_1\rangle \cdots \langle c'_m\rangle$ and $c'_m = c''$. Therefore, because $M\langle c\rangle = M'''\langle c'_1\rangle \cdots \langle c'_m\rangle\langle c\rangle$, we have the conclusion by letting $M' = M'''$, $n = m + 1$, and $c_{m+1} = c$.

Otherwise: Because M is not a coercion application, we have the conclusion by letting $M' = M$, $n = 1$, and $c_1 = c$.

□

Lemma I.37 (Canonical Forms of Coercion Applications with Function Coercions)**.** If $\Sigma \mid \emptyset \vdash_C V\langle c \rightarrow d\rangle : A \rightarrow B$, then there exist some V' that is not a coercion application, $n > 0$, and $c_1, \ldots, c_n, d_1, \ldots, d_n$ such that $V\langle c \rightarrow d\rangle = V'\langle c_1 \rightarrow d_1\rangle \cdots \langle c_n \rightarrow d_n\rangle$ and $c_n = c$ and $d_n = d$.

Proof. By induction on V. By inversion of the derivation of $\Sigma \mid \emptyset \vdash_C V\langle c \rightarrow d\rangle : A \rightarrow B$, we have

$$\Sigma \mid \emptyset \vdash_C V : A' \rightarrow B', \quad \Sigma \mid \emptyset \vdash_C c : A \rightsquigarrow A', \quad \Sigma \mid \emptyset \vdash_C d : B' \rightsquigarrow B \quad (\exists A', B')\ .$$

By case analysis on V.

Case $V = k$ $(\exists k)$, $V = \lambda x : A''.M$ $(\exists A'', x, M)$: Obvious.

Case $V = V'\langle c' \to d' \rangle$ $(\exists c', d', V')$: By the IH, there exist some V'' that is not a coercion application, $m > 0$, and $c_1, \dots, c_m, d_1, \dots, d_m$ such that $V'\langle c' \to d' \rangle = V''\langle c_1 \to d_1 \rangle \cdots \langle c_m \to d_m \rangle$ and $c_m = c'$ and $d_m = d'$. Therefore, because $V\langle c \to d \rangle = V''\langle c_1 \to d_1 \rangle \cdots \langle c_m \to d_m \rangle \langle c \to d \rangle$, we have the conclusion by letting $V' = V''$, $n = m + 1$, $c_{m+1} = c$, and $d_{m+1} = d$.

Otherwise: Contradictory because $\Sigma \mid \emptyset \vdash_C V : A' \to B'$ does not hold.

□

Lemma I.38 (Typing Coercions in Function Coercion Sequences)**.** Let $n > 0$. If $\Sigma \mid \emptyset \vdash_C M\langle c_1 \to d_1 \rangle \cdots \langle c_n \to d_n \rangle : A \to B$, then there exist some $A_0, \dots, A_n, B_0, \dots, B_n$ such that $A_n = A$ and $B_n = B$ and, for any i such that $n \geq i > 0$, $\Sigma \mid \emptyset \vdash_C c_i : A_i \rightsquigarrow A_{i-1}$ and $\Sigma \mid \emptyset \vdash_C d_i : B_{i-1} \rightsquigarrow B_i$.

Proof. By induction on n

Case $n = 1$: By inversion of the derivation of $\Sigma \mid \emptyset \vdash_C M\langle c_1 \to d_1 \rangle : A \to B$, there exist some A_0, B_0 such that $\Sigma \mid \emptyset \vdash_C c_1 : A \rightsquigarrow A_0$ and $\Sigma \mid \emptyset \vdash_C d_1 : B_0 \rightsquigarrow B$. Therefore, we have the conclusion by letting $A_1 = A$ and $B_1 = B$.

Case $n = k + 1 (k \geq 1)$: By inversion of the derivation of $\Sigma \mid \emptyset \vdash_C M\langle c_1 \to d_1 \rangle \cdots \langle c_k \to d_k \rangle \langle c_{k+1} \to d_{k+1} \rangle : A \to B$, we have

$$\Sigma \mid \emptyset \vdash_C M\langle c_1 \to d_1 \rangle \cdots \langle c_k \to d_k \rangle : A' \to B', \quad \Sigma \mid \emptyset \vdash_C c_{k+1} : A \rightsquigarrow A' \quad \Sigma \mid \emptyset \vdash_C d_{k+1} : B' \rightsquigarrow B \quad (\exists A', B') .$$

Therefore, by the IH, there exist some $A_0, \dots, A_k, B_0, \dots, B_k$ such that $A_k = A'$ and $B_k = B'$ and, for any i such that $k \geq i > 0$, $\Sigma \mid \emptyset \vdash_C c_i : A_i \rightsquigarrow A_{i-1}$ and $\Sigma \mid \emptyset \vdash_C d_i : B_{i-1} \rightsquigarrow B_i$. Hence, we have the conclusion by letting $A_{k+1} = A$ and $B_{k+1} = B$.

□

Lemma I.39 (Bisimulation and Composition)**.** If neither M nor M' is a coercion application and $\Sigma \mid \emptyset \vdash M\langle c_1 \rangle \cdots \langle c_n \rangle \approx M'\langle s \rangle : A_{n+1}$ (for $n > 0$), then there exist a nonnegative integer i and $A_1, \dots A_n$ such that:

- $\Sigma \mid \emptyset \vdash M \approx M' : A_1$, whose derivation is a subderivation of $\Sigma \mid \emptyset \vdash M\langle c_1 \rangle \cdots \langle c_n \rangle \approx M'\langle s \rangle : A_{n+1}$;
- $\Sigma \mid \emptyset \vdash_C c_1 : A_1 \rightsquigarrow A_2, \dots, \Sigma \mid \emptyset \vdash_C c_n : A_n \rightsquigarrow A_{n+1}$;
- $i \leq n$
- the first i coercions $c_1, \dots, c_i$ are no-op; and
- $s = |\mathsf{id}_{A_{i+1}}|_\emptyset \mathbin{;} |c_{i+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n|_\emptyset$.

Proof. By induction on n with case analysis on the rule last applied to derive $\Sigma \mid \emptyset \vdash M\langle c_1 \rangle \cdots \langle c_n \rangle \approx M'\langle s \rangle : A_{n+1}$, which is either (Bs_CrcId), (Bs_CrcMore), or (Bs_CrcIdL).

Case (Bs_Crc): We are given

$$\Sigma \mid \emptyset \vdash M\langle c_1 \rangle \cdots \langle c_{n-1} \rangle \approx M' : A_n, \quad s = |c_n|_\emptyset .$$

Since M' is not a coercion application, it must be the case that $\Sigma \mid \emptyset \vdash M\langle c_1 \rangle \cdots \langle c_{n-1} \rangle \approx M' : A_n$ is derived by a sequence of applications of (Bs_CrcIdL). Thus, taking $i = n - 1$, we have

- $\Sigma \mid \emptyset \vdash M \approx M' : A_1$, whose derivation is a subderivation of $\Sigma \mid \emptyset \vdash M\langle c_1 \rangle \cdots \langle c_n \rangle \approx M'\langle s \rangle : A_{n+1}$;
- $\Sigma \mid \emptyset \vdash_C c_1 : A_1 \rightsquigarrow A_2, \dots, \Sigma \mid \emptyset \vdash_C c_n : A_n \rightsquigarrow A_{n+1}$;
- $c_1, \dots, c_{n-1}$ are no-op; and
- $s = |\mathsf{id}_{A_{i+1}}|_\emptyset \mathbin{;} |c_n|_\emptyset$.

Case (Bs_CrcId): We are given

$$s = |\mathsf{id}_{A_{n+1}}|_\emptyset, \quad \Sigma \mid \emptyset \vdash M\langle c_1\rangle \cdots \langle c_n\rangle \approx M' : A_{n+1}, \quad \Sigma \mid \emptyset \vdash_C \mathsf{id}_{A_{n+1}} : A_{n+1} \rightsquigarrow A_{n+1} \ .$$

Since M' is not a coercion application, it must be the case that $\Sigma \mid \emptyset \vdash M\langle c_1\rangle \cdots \langle c_n\rangle \approx M' : A_{n+1}$ is derived by applying (Bs_CrcIdL) n times, we have for all $j \in [1 \dots n]$

$$c_j = c_j^I, \quad \Sigma \mid \emptyset \vdash M \approx M' : A_1, \quad \Sigma \mid \emptyset \vdash_C c_j^I : A_j \rightsquigarrow A_{j+1} \quad (\exists A_1, \dots, A_n, c_1^I, \dots, c_n^I) \ .$$

Taking $i = n$ finishes the case. (Here, $|\mathsf{id}_{A_{i+1}}|_\emptyset \mathbin{;} |c_{i+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n|_\emptyset$ means $|\mathsf{id}_{A_{i+1}}|_\emptyset$.)

Case (Bs_CrcMore): We are given

$$s = s' \mathbin{;} |c_n|_\emptyset, \quad \Sigma \mid \emptyset \vdash M\langle c_1\rangle \cdots \langle c_{n-1}\rangle \approx M'\langle s'\rangle : A_n, \quad \Sigma \mid \emptyset \vdash_C c_n : A_n \rightsquigarrow A_{n+1} \quad (\exists A_n, s') \ .$$

By the IH, for some nonnegative i, A_1, ..., A_{n-1}, we have

- $\Sigma \mid \emptyset \vdash M \approx M' : A_1$, whose derivation is a subderivation of $\Sigma \mid \emptyset \vdash M\langle c_1\rangle \cdots \langle c_{n-1}\rangle \approx M'\langle s'\rangle : A_n$;
- $\Sigma \mid \emptyset \vdash c_1 : A_1 \rightsquigarrow A_2$, ..., $\Sigma \mid \emptyset \vdash c_{n-1} : A_{n-1} \rightsquigarrow A_n$;
- $i \leq n-1$
- the first i coercions c_1, ..., c_i are no-op; and
- $s' = |\mathsf{id}_{A_{i+1}}|_\emptyset \mathbin{;} |c_{i+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_{n-1}|_\emptyset$.

It is immediate that $s = s' \mathbin{;} |c_n|_\emptyset = |\mathsf{id}_{A_{i+1}}|_\emptyset \mathbin{;} |c_{i+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n|_\emptyset$, finishing the case.

Case (Bs_CrcIdL): We are given

$$c_n = c^I, \quad \Sigma \mid \emptyset \vdash M\langle c_1\rangle \cdots \langle c_{n-1}\rangle \approx M'\langle s\rangle : A_n, \quad \Sigma \mid \emptyset \vdash_C c_n : A_n \rightsquigarrow A_{n+1} \quad (\exists A_n, c^I) \ .$$

By the IH, for some nonnegative i, A_1, ..., A_{n-1}, we have

- $\Sigma \mid \emptyset \vdash M \approx M' : A_1$, whose derivation is a subderivation of $\Sigma \mid \emptyset \vdash M\langle c_1\rangle \cdots \langle c_{n-1}\rangle \approx M'\langle s\rangle : A_n$;
- $\Sigma \mid \emptyset \vdash c_1 : A_1 \rightsquigarrow A_2$, ..., $\Sigma \mid \emptyset \vdash c_{n-1} : A_{n-1} \rightsquigarrow A_n$;
- $i \leq n-1$
- the first i coercions c_1, ..., c_i are no-op; and
- $s = |\mathsf{id}_{A_{i+1}}|_\emptyset \mathbin{;} |c_{i+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_{n-1}|_\emptyset$.

By Lemma I.11, $(|\mathsf{id}_{A_{i+1}}|_\emptyset \mathbin{;} |c_{i+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_{n-1}|_\emptyset) \mathbin{;} |c_n|_\emptyset = |\mathsf{id}_{A_{i+1}}|_\emptyset \mathbin{;} |c_{i+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_{n-1}|_\emptyset = s$, finishing the case.

□

Lemma I.40 (Uncoerced Values are Bisimilar to Values)**.** If $\Sigma \mid \Gamma \vdash M \approx U : A$, then there exists some V that is not a coercion application, nonnegative integer n, $A_0, \dots, A_n$ and $c_1^I, \dots, c_n^I$ such that

$$M = V\langle c_1^I\rangle \cdots \langle c_n^I\rangle, \quad \Sigma \mid \Gamma \vdash_C c_i^I : A_{i-1} \rightsquigarrow A_i \ (1 \leq i \leq n), \quad A_n = A, \quad \Sigma \mid \Gamma \vdash V \approx U : A_0 \ .$$

Furthermore,

1. If $U = k$ for some k, then $V = k$ and $A_0 = ty(k)$.
2. If $U = \lambda x\!:\!B_0.M_1'$ for some x, B_0 and M_1', then

$$V = \lambda x\!:\!B_0.M_1, \quad A_0 = B_0 \to C_0, \quad \Sigma \mid \Gamma \vdash M_1 \approx M_1' : C_0 \quad (\exists M_1) \ .$$

3. If $U = \Lambda X.M_1'$ for some X and M_1', then

$$V = \Lambda X.(M_1 : B_0), \quad A_0 = \forall X.B_0, \quad \Sigma \mid \Gamma, X \vdash M_1 \approx M_1' : B_0 \quad (\exists M_1, B_0) \ .$$

Proof. Straightforward if M is not a coercion application. If M is a coercion application, by Lemma I.36, there exists some term M_1 such that M_1 is not a coercion application and

$$M = M_1\langle c_1^I\rangle \cdots \langle c_n^I\rangle \quad (\exists c_1^I, \ldots, c_n^I)\ .$$

Since $\Sigma \mid \Gamma \vdash M_1\langle c_1^I\rangle \cdots \langle c_n^I\rangle \approx U : A$ is derived by (Bs_CrcIdL), we heve

$$A_n = A, \quad \Sigma \mid \Gamma \vdash M_1 \approx U : A_0, \quad \Sigma \mid \Gamma \vdash_C c_i^I : A_{i-1} \rightsquigarrow A_i \ (1 \le i \le n) \quad (\exists A_0, \ldots A_n)\ .$$

Furthermore,

Case $U = k$ $(\exists k)$: Since M_1 is not a cercion application, $\Sigma \mid \Gamma \vdash M_1 \approx k : A_0$ is derived by (Bs_Const). Therefore, we have $M_1 = k$ and $A_0 = ty(k)$.

Case $U = \lambda x\!:\!B_0.M_2'$ $(\exists B_0, x, M_2')$: Since M_1 is not a coercion application, $\Sigma \mid \Gamma \vdash M_1 \approx \lambda x\!:\!B_0.M_2' : A_0$ is derived by (Bs_Abs). Therefore, we have

$$M_1 = \lambda x\!:\!B_0.M_2, \quad A_0 = B_0 \to C_0, \quad \Sigma \mid \Gamma \vdash M_2 \approx M_2' : C_0 \quad (\exists M_2, C_0)\ .$$

Case $U = \Lambda X.M_2'$ $(\exists X, M_2')$: Since M_1 is not a coercion application, $\Sigma \mid \Gamma \vdash M_1 \approx \Lambda X.M_2' : A_0$ is derived by (Bs_Tyabs). Therefore, we have

$$M_1 = \Lambda X.(M_2 : B_0), \quad A_0 = \forall X.B_0, \quad \Sigma \mid \Gamma, X \vdash M_2 \approx M_2' : B_0 \quad (\exists M_2, B_0)\ .$$

□

Definition I.1 (Value Coercions)**.** We define value coercions vc, which are a subset of coercions in $\lambda C^{\forall}_{mp}$, and space-efficient value coercion vs, which are a subset of space-efficient coercions in $\lambda S^{\forall}_{mp}$, as follows.

$$\begin{aligned} vc &::= G! \mid \alpha^- \mid c \to d \mid \forall X.c \\ vs &::= i \end{aligned}$$

Definition I.2 (No-op Value Coercions)**.** We define No-op value coercions vc^I, which are a subset of value coercions in $\lambda C^{\forall}_{mp}$ as follows.

$$vc^I ::= G! \mid \alpha^- \mid c^I \to d^I \mid \forall X.c^I$$

Lemma I.41 (Ground Coercion Composition)**.** If $\Sigma \mid \emptyset \vdash_S g_1 : A \rightsquigarrow B$ and $\Sigma \mid \emptyset \vdash_S g_2 : B \rightsquigarrow C$, then there exists some g_3 such that $g_3 = g_1 \fatsemi g_2$ and $\Sigma \mid \emptyset \vdash_S g_3 : A \rightsquigarrow C$.

Proof. First, we consider the cases where either g_1 or g_2 is an identity coercion.

Case $g_1 = \mathsf{id}$: By Lemma I.5, $\mathsf{id} \fatsemi g_2 = g_2$.

Case $g_2 = \mathsf{id}$: By Lemma I.6, $g_1 \fatsemi \mathsf{id} = g_1$.

Next, we consider the cases where neither g_1 nor g_2 is an identity coercion. By case analysis on g_2.

Case $g_2 = s_2 \to t_2$ $(\exists s_2, t_2)$: Because $\Sigma \mid \emptyset \vdash_S s_2 \to t_2 : B \rightsquigarrow C$ is derived by (Ct_Arrow_S), we have

$$\begin{aligned} B = B_1 \to B_2, \quad C = C_1 \to C_2, \quad &\Sigma \mid \emptyset \vdash_S s_2 : C_1 \rightsquigarrow B_1, \\ &\Sigma \mid \emptyset \vdash_S t_2 : B_2 \rightsquigarrow C_2 \quad (\exists B_1, B_2, C_1, C_2)\ . \end{aligned}$$

By case analysis on g_1.

Case $g_1 = s_1 \to t_1$ $(\exists g_1, G_1)$: Because $\Sigma \mid \emptyset \vdash_S s_1 \to t_1 : A \rightsquigarrow B$ is derived by (Ct_Arrow_S), we have

$$A = A_1 \to A_2, \quad \Sigma \mid \emptyset \vdash_S s_1 : B_1 \rightsquigarrow A_1, \quad \Sigma \mid \emptyset \vdash_S t_1 : A_2 \rightsquigarrow B_2 \quad (\exists A_1, A_2)$$

Therefore, by Lemma E.24, there exists some s such that $s = s_2 \fatsemi s_1$ and $\Sigma \mid \emptyset \vdash_S s : C_1 \rightsquigarrow A_1$. Similarly, by Lemma E.24, there exists some t such that $t = t_1 \fatsemi t_2$ and $\Sigma \mid \emptyset \vdash_S t : A_2 \rightsquigarrow C_2$. Hence,

$$\begin{aligned} (s_1 \to t_1) \fatsemi (s_2 \to t_2) &= (s_2 \fatsemi s_1) \to (t_1 \fatsemi t_2) \\ &= s \to t\ . \end{aligned}$$

Furthermore, by (Ct_Arrow_S), $\Sigma \mid \emptyset \vdash_S s \to t : (A_1 \to A_2) \rightsquigarrow (C_1 \to C_2)$.

Otherwise: Contradictory because B cannot be a function type for all space-efficient value coercions except id.

Case $g_2 = \forall X.s_2 \,,,\, t_2$ $(\exists X, s_2, t_2)$: Because $\Sigma \mid \emptyset \vdash_S \forall X.s_2 : B \rightsquigarrow C$ is derived by (Ct_All_S), we have

$$B = \forall X.B_1, \quad C = \forall X.C_1, \quad \Sigma \mid \emptyset, X \vdash_S s_2 : B_1 \rightsquigarrow C_1,$$
$$\Sigma \mid \emptyset, X \vdash_S t_2 : B_1[X := \star] \rightsquigarrow C_1[X := \star] \quad (\exists B_1, C_1) \; .$$

By case analysis on g_1.

Case $g_1 = \forall Y.s_1 \,,,\, t_1$ $(\exists Y_1, s_1, t_1)$: Because $\Sigma \mid \emptyset \vdash_S \forall Y.s_1 \,,,\, t_1 : A \rightsquigarrow (\forall X.B_1)$ is derived by (Ct_All_S), we have

$$Y = X, \quad A = \forall X.A_1, \quad \Sigma \mid \emptyset \vdash_S s_1 : A_1 \rightsquigarrow B_1,$$
$$\Sigma \mid \emptyset \vdash_S t_1 : A_1[X := \star] \rightsquigarrow B_1[X := \star] \quad (\exists A_1)$$

Therefore, by Lemma E.24, there exists some s such that $s = s_1 \fatsemi s_2$ and $\Sigma \mid \emptyset \vdash_S s : A_1 \rightsquigarrow C_1$. Similarly, by Lemma E.24, there exists some t such that $t = t_1 \fatsemi t_2$ and $\Sigma \mid \emptyset \vdash_S t : A_1[X := \star] \rightsquigarrow C_1[X := \star]$. Hence,

$$\begin{aligned}(\forall X.s_1 \,,,\, t_1) \fatsemi (\forall X.s_2 \,,,\, t_2) &= \forall X.(s_1 \fatsemi s_2) \,,,\, (t_1 \fatsemi t_2) \\ &= \forall X.s \,,,\, t \; .\end{aligned}$$

Furthermore, by (Ct_Arrow_S), $\Sigma \mid \emptyset \vdash_S \forall X.s \,,,\, t : \forall X.A_1 \rightsquigarrow \forall X.C_1$.

Otherwise: Contradictory because B cannot be a polymorphic type for all space-efficient value coercions except id.

□

Lemma I.42 (Space-efficient Value Coercion Composition)**.** If $\Sigma \mid \emptyset \vdash_S vs_1 : A \rightsquigarrow B$ and $\Sigma \mid \emptyset \vdash_S vs_2 : B \rightsquigarrow C$, then there exists some vs_3 such that $vs_3 = vs_1 \fatsemi vs_2$ and $\Sigma \mid \emptyset \vdash_S vs_3 : A \rightsquigarrow C$.

Proof. If both vs_1 and vs_2 is a ground coercion, straightforward by Lemma I.41. We consider the cases where neither vs_1 or vs_2 is a ground coercion. By case analysis on vs_2.

Case $vs_2 = g_2 \,;\, G_2!$ $(\exists g_2, G_2)$: Because $\Sigma \mid \emptyset \vdash_S g_2 \,;\, G_2! : B \rightsquigarrow C$ is derived by (Ct_Inj_S), we have

$$C = \star, \quad \Sigma \mid \emptyset \vdash_S g_2 : B \rightsquigarrow \Sigma(G_2) \; .$$

By case analysis on vs_1.

Case $vs_1 = g_1 \,;\, G_1!$ $(\exists g_1, G_1)$: Because $\Sigma \mid \emptyset \vdash_S g_1 \,;\, G_1! : A \rightsquigarrow B$ is derived by (Ct_Inj_S), we have

$$B = \star, \quad \Sigma \mid \emptyset \vdash_S g_1 : A \rightsquigarrow \Sigma(G_1) \; .$$

Since $B = \star$, $g_2 = \mathsf{id}$ (otherwise B cannot be a dynamic type, which is contradiction). Since $\Sigma \mid \emptyset \vdash_S \mathsf{id} : \star \rightsquigarrow \Sigma(G_2)$ is derived by (Ct_Id_S), we have $\star = \Sigma(G_2)$. There is contradiction because $G_2 \neq \star$ and for any $\mathbb{A}$ in $\alpha := \mathbb{A} \in \Sigma$ is not a dynamic type.

Case $vs_1 = g_1$ $(\exists g_1)$: By Lemma I.41, there exists a ground coercion g_3 such that $g_3 = g_1 \fatsemi g_2$ and $\Sigma \mid \emptyset \vdash_C g_3 : A \rightsquigarrow \Sigma(G_2)$. Therefore,

$$\begin{aligned}g_1 \fatsemi (g_2 \,;\, G_2!) &= (g_1 \fatsemi g_2) \,;\, G_2! \\ &= g_3 \,;\, G_2! \; .\end{aligned}$$

By (Ct_Inj_S), $\Sigma \mid \emptyset \vdash_C g_3 \,;\, G_2! : A \rightsquigarrow \star$.

Case $vs_2 = s_2 \to t_2$ $(\exists s_2, t_2)$: Since vs_2 is a ground coercion, $vs_1 = g_1 \mathbin{;} G_1!$ $(\exists g_1, G_1)$. Because $\Sigma \mid \emptyset \vdash_S s_2 \to t_2 : B \rightsquigarrow C$ is derived by (CT_ARROW_S), we have

$$B = B_1 \to B_2, \quad C = C_1 \to C_2, \quad \Sigma \mid \emptyset \vdash_S s_2 : C_1 \rightsquigarrow B_1, \\ \Sigma \mid \emptyset \vdash_S t_2 : B_2 \rightsquigarrow C_2 \quad (\exists B_1, B_2, C_1, C_2) \ .$$

Since $\Sigma \mid \emptyset \vdash_S g_1 \mathbin{;} G_1! : A \rightsquigarrow B$ is derived by (CT_INJ_S), however, contradictory because B cannot be a dynamic type.

Case $vs_2 = \forall X.s_2 \mathbin{,,} t_2$ $(\exists X, s_2, t_2)$: Since vs_2 is a ground coercion, $vs_1 = g_1 \mathbin{;} G_1!$ $(\exists g_1, G_1)$. Because $\Sigma \mid \emptyset \vdash_S \forall X.s_2 : B \rightsquigarrow C$ is derived by (CT_ALL_S), we have

$$B = \forall X.B_1, \quad C = \forall X.C_1, \quad \Sigma \mid \emptyset, X \vdash_S s_2 : B_1 \rightsquigarrow C_1, \\ \Sigma \mid \emptyset, X \vdash_S t_2 : B_1[X := \star] \rightsquigarrow C_1[X := \star] \quad (\exists B_1, C_1) \ .$$

Since $\Sigma \mid \emptyset \vdash_S g_1 \mathbin{;} G_1! : A \rightsquigarrow B$ is derived by (CT_INJ_S), however, contradictory because B cannot be a dynamic type.

□

Lemma I.43 (Value Coercions are Translated to Space-Efficient Value Coercions)**.** If $\Sigma \mid \emptyset \vdash_C vc : A \rightsquigarrow B$, then $|vc|_\emptyset$ is a space-efficient value coercion and $\Sigma \mid \emptyset \vdash_S |vc|_\emptyset : \Sigma(A) \rightsquigarrow \Sigma(B)$.

Proof. By Lemma I.3, there exists some $|vc|_\emptyset$ such that $\Sigma \mid \emptyset \vdash_S |vc|_\emptyset : \Sigma(A) \rightsquigarrow \Sigma(B)$. Therefore, it suffices to show that $|vc|_\emptyset$ is a space-efficient value coercion. By case analysis on vc.

Case $vc = G!$ $(\exists G)$:

Case $G = X$ $(\exists X)$: $|G!|_\emptyset = \mathsf{id}$. id is a space-efficient value coercion.

Case G is not type variable: $|G!|_\emptyset = |\mathsf{id}_G|_\emptyset \mathbin{;} G!$.

Case $G = \iota$ $(\exists \iota)$: $|\mathsf{id}_\iota|_\emptyset \mathbin{;} \iota! = \mathsf{id} \mathbin{;} \iota!$. $\mathsf{id} \mathbin{;} \iota!$ is a space-efficient value coercion.

Case $G = X$ $(\exists X)$: $|\mathsf{id}_X|_\emptyset \mathbin{;} X! = \mathsf{id} \mathbin{;} X!$. $\mathsf{id} \mathbin{;} X!$ is a space-efficient value coercion.

Case $G = \alpha$ $(\exists \alpha)$: $|\mathsf{id}_\alpha|_\emptyset \mathbin{;} \alpha! = \mathsf{id} \mathbin{;} \alpha!$. $\mathsf{id} \mathbin{;} \alpha!$ is a space-efficient value coercion.

Case $G = \star \to \star$: $|\mathsf{id}_{\star\to\star}|_\emptyset \mathbin{;} (\star \to \star)! = (|\mathsf{id}_\star|_\emptyset \to |\mathsf{id}_\star|_\emptyset) \mathbin{;} (\star \to \star)! = (\mathsf{id} \to \mathsf{id}) \mathbin{;} (\star \to \star)!$. $(\mathsf{id} \to \mathsf{id}) \mathbin{;} (\star \to \star)!$ is a space-efficient value coercion.

Case $G = \forall X.\star$ $(\exists X)$: $|\mathsf{id}_{\forall X.\star}|_\emptyset \mathbin{;} (\forall X.\star)! = (\forall X.|\mathsf{id}_\star|_\emptyset) \mathbin{;} (\forall X.\star)! = (\forall X.\mathsf{id}) \mathbin{;} (\forall X.\star)!$. $(\forall X.\mathsf{id}) \mathbin{;} (\forall X.\star)!$ is a space-efficient value coercion.

Case $vc = \alpha^-$ $(\exists \alpha)$: $|\alpha^-|_\emptyset = \mathsf{id}$, and id is a space-efficient value coercion.

Case $vc = c \to d$ $(\exists c, d)$: $|c \to d|_\emptyset = |c|_\emptyset \to |d|_\emptyset$, and $|c|_\emptyset \to |d|_\emptyset$ is a space-efficient value coercion.

Case $vc = \forall X.c$ $(\exists X, c)$: $|\forall X.c|_\emptyset = \forall X.|c|_X \mathbin{,,} |c|_\emptyset$, and $\forall X.|c|_X \mathbin{,,} |c|_\emptyset$ is a space-efficient value coercion.

□

Lemma I.44 (Value Coercion Composition, Consecutively Applied)**.** Let $n > 0$. For any i such that $n \geq i > 0$, if $\Sigma \mid \emptyset \vdash_S vs_i : A_{i-1} \rightsquigarrow A_i$, then there exists some vs such that $vs = vs_1 \fatsemi \cdots \fatsemi vs_n$ and $\Sigma \mid \emptyset \vdash_S vs : A_0 \rightsquigarrow A_n$.

Proof. By induction on n.

Case $n = 1$: Obvious.

Case $n = k + 1 (k \geq 1)$: By the IH,

$$vs' = vs_1 \fatsemi \cdots \fatsemi vs_k, \quad \Sigma \mid \emptyset \vdash_S vs' : A_0 \rightsquigarrow A_k \quad (\exists vs') \ .$$

Therefore, by Lemma I.42, there exists some vs such that $vs = vs' \fatsemi vs_{k+1}$ and $\Sigma \mid \emptyset \vdash_S vs : A_0 \rightsquigarrow A_{k+1}$. Hence,

$$vs = vs' \fatsemi vs_{k+1} \\ = vs_1 \fatsemi \cdots \fatsemi vs_k \fatsemi vs_{k+1} \ ,$$

so we have the conclusion.

□

Lemma I.45 (Replacement of Coercion Arguments)**.** Let $n > 0$. If neither M_1 nor M'_1 is a coercion application, and $\Sigma \mid \emptyset \vdash M_1\langle c_1\rangle \cdots \langle c_n\rangle \approx M'_1\langle s\rangle : A_{n+1}$, $\Sigma \mid \emptyset \vdash\ c_i : A_i \rightsquigarrow A_{i+1}$ (for $1 \leq i \leq n$), $\Sigma \mid \emptyset \vdash M_1 \approx M'_1 : A_1$ and $\Sigma \mid \emptyset \vdash M_2 \approx M'_2 : A_1$, then $\Sigma \mid \emptyset \vdash M_2\langle c_1\rangle \cdots \langle c_n\rangle \approx M'_2\langle s\rangle : A_{n+1}$.

Proof. By induction on n.

Case $n = 1$: By case analysis on the rule applied last to derive $\Sigma \mid \emptyset \vdash M_1\langle c_1\rangle \approx M'_1\langle s\rangle : A_2$, which is either of (Bs_CrcId), (Bs_CrcIdL), (Bs_Crc), or (Bs_CrcMore).

Case (Bs_CrcId): We are given

$$s = |\mathsf{id}_{A_2}|_\emptyset, \quad \Sigma \mid \emptyset \vdash_C \mathsf{id}_{A_2} : A_2 \rightsquigarrow A_2, \quad \Sigma \mid \emptyset \vdash M_1\langle c_1\rangle \approx M'_1 : A_2 \ .$$

Since M'_1 is not a coercion application, $\Sigma \mid \emptyset \vdash M_1\langle c_1\rangle \approx M'_1 : A_{n+1}$ is derived by (Bs_CrcIdL). Therefore, we have

$$c_1 = c_1^I, \quad \Sigma \mid \emptyset \vdash M_1 \approx M'_1 : A_1, \quad \Sigma \mid \emptyset \vdash\ c_1^I : A_1 \rightsquigarrow A_2 \quad (\exists c_1^I) \ .$$

Hence, by (Bs_CrcIdL) and (Bs_CrdId), we have $\Sigma \mid \emptyset \vdash M_2\langle c_1^I\rangle \approx M'_2\langle |\mathsf{id}_{A_2}|_\emptyset\rangle : A_2$.

Case (Bs_CrcIdL): Similary to the case (Bs_CrcId).

Case (Bs_Crc): We are given

$$s = |c_1|_\emptyset, \quad \Sigma \mid \emptyset \vdash M_1 \approx M'_1 : A_1, \quad \Sigma \mid \emptyset \vdash_C\ c_1 : A_1 \rightsquigarrow A_2 \ .$$

Hence, by (Bs_Crc), we have $\Sigma \mid \emptyset \vdash M_2\langle c_1\rangle \approx M'_2\langle |c_1|_\emptyset\rangle : A_2$.

Case (Bs_CrcMore): We are given

$$s = s' \mathbin{;} |c_1|_\emptyset, \quad \Sigma \mid \emptyset \vdash M_1 \approx M'_1\langle s'\rangle : A_1, \quad \Sigma \mid \emptyset \vdash_C\ c_1 : A_1 \rightsquigarrow A_2 \quad (\exists s') \ .$$

Because M_1 is not a coercion application, $\Sigma \mid \emptyset \vdash M_1 \approx M'_1\langle s'\rangle : A_1$ is derived by (Bs_CrcId). Therefore, we have $s' = |\mathsf{id}_{A_1}|_\emptyset$. Hence, $\Sigma \mid \emptyset \vdash M_2\langle c_1\rangle \approx M'_2\langle |\mathsf{id}_{A_1}|_\emptyset \mathbin{;} |c_1|_\emptyset\rangle : A_2$ is derived as follows.

$$\dfrac{\dfrac{\Sigma \mid \emptyset \vdash M_2 \approx M'_2 : A_1}{\Sigma \mid \emptyset \vdash M_2 \approx M'_2\langle |\mathsf{id}_{A_1}|_\emptyset\rangle : A_1}\text{(Bs\_CrcId)} \qquad \Sigma \mid \emptyset \vdash_C\ c_1 : A_1 \rightsquigarrow A_2}{\Sigma \mid \emptyset \vdash M_2\langle c_1\rangle \approx M'_2\langle |\mathsf{id}_{A_1}|_\emptyset \mathbin{;} |c_1|_\emptyset\rangle : A_2}\text{(Bs\_CrcMore)}$$

Case $n = k + 1 (k \geq 1)$: By case analysis on the rule applied last to derive $\Sigma \mid \emptyset \vdash M_1\langle c_1\rangle \cdots \langle c_k\rangle \langle c_{k+1}\rangle \approx M'_1\langle s\rangle : A_{k+1}$, which is either of (Bs_CrcId), (Bs_CrcIdL), (Bs_Crc), or (Bs_CrcMore).

Case (Bs_CrcId): We are given

$$s = |\mathsf{id}_{A_{k+1}}|_\emptyset, \quad \Sigma \mid \emptyset \vdash M_1\langle c_1\rangle \cdots \langle c_k\rangle \langle c_{k+1}\rangle \approx M'_1 : A_{k+1} \ .$$

Since M'_1 is not a coercion application, $\Sigma \mid \emptyset \vdash M_1\langle c_1\rangle \cdots \langle c_k\rangle \langle c_{k+1}\rangle \approx M'_1 : A_{k+1}$ is derived by applying (Bs_CrcIdL) k+1 times. Therefore, there exists c_i^I $(1 \leq i \leq ks)$ such that $c_i = c_i^I$. Hence, by (Bs_CrcIdL) and (Bs_CrcId), we have $\Sigma \mid \emptyset \vdash M_2\langle c_1\rangle \cdots \langle c_k\rangle \langle c_{k+1}\rangle \approx M'_2\langle s\rangle : A_{k+1}$.

Case (Bs_CrcIdL): Similary to the case (Bs_CrcId).

Case (Bs_Crc): We are given

$$s = |c_{k+1}|_\emptyset, \quad \Sigma \mid \emptyset \vdash M_1\langle c_1\rangle \cdots \langle c_k\rangle \approx M'_1 : A_k \ .$$

By the IH, $\Sigma \mid \emptyset \vdash M_2\langle c_1\rangle \cdots \langle c_k\rangle \approx M'_2 : A_k$. Therefore, by (Bs_Crc), $\Sigma \mid \emptyset \vdash M_2\langle c_1\rangle \cdots \langle c_k\rangle \langle c_{k+1}\rangle \approx M'_2\langle |c_{k+1}|_\emptyset\rangle : A_{k+1}$.

Case (Bs_CrcMore): We are given

$$s = s' \fatsemi |c_{k+1}|_{\emptyset}, \quad \Sigma \mid \emptyset \vdash M_1\langle c_1\rangle \cdots \langle c_k\rangle \approx M_1'\langle s'\rangle : A_k \quad (\exists s') .$$

By the IH, $\Sigma \mid \emptyset \vdash M_2\langle c_1\rangle \cdots \langle c_k\rangle \approx M_2'\langle s'\rangle : A_k$. Therefore, by (Bs_CrcMore), $\Sigma \mid \emptyset \vdash M_2\langle c_1\rangle \cdots \langle c_k\rangle \langle c_{k+1}\rangle \approx M_2'\langle s' \fatsemi |c_{k+1}|_{\emptyset}\rangle : A_{k+1}$.

□

Lemma I.46 (Intermediate Coercion Composition Determines the Left Coercions)**.** If $s \fatsemi t = i$, then there exists some j such that $s = j$.

Proof. Staraightforward because the contrapositive holds trivially. □

Lemma I.47 (No-op Coercion Types are not Dynamic Types)**.** If $\Sigma \mid \emptyset \vdash_C c^I : A \rightsquigarrow B$, then A and B are non dynamic types.

Proof. Straightforward by induction on the derivation of $\Sigma \mid \emptyset \vdash_C c^I : A \rightsquigarrow B$. □

Lemma I.48 (No-op Coercion Types are Same Under Σ)**.** If $\Sigma \mid \Gamma \vdash_C c^I : A \rightsquigarrow B$, then $\Sigma(B) = \Sigma(A)$.

Proof. Straightforward by induction on the derivation of $\Sigma \mid \Gamma \vdash_C c^I : A \rightsquigarrow B$. □

Lemma I.49 (Space-efficient Value Coercion Composition and Identity Coercion)**.** If $\Sigma \mid \emptyset \vdash_S vs_1 : A \rightsquigarrow B$ and $\Sigma \mid \emptyset \vdash_S vs_2 : B \rightsquigarrow C$ and $vs_1 \fatsemi vs_2 = \mathsf{id}$, then $vs_1 = \mathsf{id}$ and $vs_2 = \mathsf{id}$.

Proof. (1) Suppose that $vs_1 \neq \mathsf{id}$. By case analysis on vs_1.

Case $vs_1 = g_1 \,;\, G_1!$ $(\exists g_1, G_1)$: Since $\Sigma \mid \emptyset \vdash_S g_1 \,;\, G_1! : A \rightsquigarrow B$ is derived by (Ct_Inj_S), we have $B = \star$. By case analysis on the rule applied last to derive $\Sigma \mid \emptyset \vdash_S vs_2 : \star \rightsquigarrow C$.

Case (Ct_Id_S): Contradictory because $vs_1 \fatsemi vs_2 = (g_1 \,;\, G_1!) \fatsemi \mathsf{id} = g_1 \,;\, G_1! \neq \mathsf{id}$.

Case (Ct_Inj_S): We are given

$$vs_2 = g_2 \,;\, G_2!, \quad \Sigma \mid \emptyset \vdash g_2 : \star \rightsquigarrow \Sigma(G_2) \quad (\exists g_2, G_2) .$$

Since $B = \star$, $\Sigma \mid \emptyset \vdash g_2 : \star \rightsquigarrow \Sigma(G_2)$ is derived by (Ct_Id_S), $\star = \Sigma(G_2)$. There is contradiction because $G_2 \neq \star$ and for any $\mathbb{A}$ in $\alpha := \mathbb{A} \in \Sigma$ is not a dynamic type.

Otherwise: Contradictory because B cannot be a dynamic type.

Case $vs_1 = s_1 \to t_1$ $(\exists s_1, t_1)$: Since $\Sigma \mid \emptyset \vdash_S s_1 \to t_1 : A \rightsquigarrow B$ is derived by (Ct_Arrow_S), we have B is a function type. By case analysis on the rule applied last to derive $\Sigma \mid \emptyset \vdash_S vs_2 : B \rightsquigarrow C$.

Case (Ct_Id_S): Contradictory because $vs_1 \fatsemi vs_2 = (s_1 \to t_1) \fatsemi \mathsf{id} = s_1 \to t_1 \neq \mathsf{id}$.

Case (Ct_Inj_S): Contradictory because $vs_1 \fatsemi vs_2 = (s_1 \to t_1) \fatsemi (g_2 \,;\, G_2!) = ((s_1 \to t_1) \fatsemi g_2) \,;\, G_2! \neq \mathsf{id}$.

Case (Ct_Arrow_S): We are given $vs_2 = s_2 \to t_2$ $(\exists s_2, t_2)$. Contradictory because $vs_1 \fatsemi vs_2 = (s_1 \to t_1) \fatsemi (s_2 \to t_2) = (s_2 \fatsemi s_1) \to (t_1 \fatsemi t_2) \neq \mathsf{id}$.

Otherwise: Contradictory because B cannot be a function type.

Case $vs_1 = \forall X.s_1 \,,,\, t_1$ $(\exists X, s_1, t_1)$: Since $\Sigma \mid \emptyset \vdash_S \forall X.s_1 \,,,\, t_1 : A \rightsquigarrow B$ is derived by (Ct_All_S), we have $B = \forall X.B_1$ $(\exists B_1)$. By case analysis on the rule applied last to derive $\Sigma \mid \emptyset \vdash_S vs_2 : B \rightsquigarrow C$.

Case (Ct_Id_S): Contradictory because $vs_1 \fatsemi vs_2 = (\forall X.s_1 \,,,\, t_1) \fatsemi \mathsf{id} = \forall X.s_1 \,,,\, t_1 \neq \mathsf{id}$.

Case (Ct_Inj_S): Contradictory because $vs_1 \fatsemi vs_2 = (\forall X.s_1 \,,,\, t_1) \fatsemi (g_2 \,;\, G_2!) = ((\forall X.s_1 \,,,\, t_1) \fatsemi g_2) \,;\, G_2! \neq \mathsf{id}$.

Case (Ct_All_S): We are given $vs_2 = \forall X.s_2 \,,,\, t_2$ $(\exists s_2, t_2)$. Contradictory because $vs_1 \fatsemi vs_2 = (\forall X.s_1 \,,,\, t_1) \fatsemi (\forall X.s_2 \,,,\, t_2) = \forall X.(s_1 \fatsemi s_2) \,,,\, (t_1 \fatsemi t_2) \neq \mathsf{id}$.

Otherwise: Contradictory because B cannot be a polymorphic type.

(2) Suppose that $vs_2 \neq \mathsf{id}$. By case analysis on vs_1.

Case $vs_1 = \mathsf{id}$: Contradictory because $\mathsf{id} = vs_1 \fatsemi vs_2 = \mathsf{id} \fatsemi vs_2 = vs_2$.

Case $vs_1 = g_1 \, ; G_1!$ $(\exists g_1, G_1)$: Since $\Sigma \mid \emptyset \vdash_S g_1 \, ; G_1! : A \rightsquigarrow B$ is derived by (Ct_Inj_S), we have $B = \star$. By case analysis on the rule applied last to derive $\Sigma \mid \emptyset \vdash_S vs_2 : \star \rightsquigarrow C$.

Case (Ct_Inj_S): We are given

$$vs_2 = g_2 \, ; G_2!, \quad \Sigma \mid \emptyset \vdash g_2 : \star \rightsquigarrow \Sigma(G_2) \quad (\exists g_2, G_2) \; .$$

Since $B = \star$, $\Sigma \mid \emptyset \vdash g_2 : \star \rightsquigarrow \Sigma(G_2)$ is derived by (Ct_Id_S), $\star = \Sigma(G_2)$. There is contradiction because $G_2 \neq \star$ and for any $\mathbb{A}$ in $\alpha := \mathbb{A} \in \Sigma$ is not a dynamic type.

Otherwise: Contradictory because B cannot be a dynamic type.

Case $vs_1 = s_1 \to t_1$ $(\exists s_1, t_1)$: Since $\Sigma \mid \emptyset \vdash_S s_1 \to t_1 : A \rightsquigarrow B$ is derived by (Ct_Arrow_S), we have B is a function type. By case analysis on the rule applied last to derive $\Sigma \mid \emptyset \vdash_S vs_2 : B \rightsquigarrow C$.

Case (Ct_Inj_S): Contradictory because $vs_1 \fatsemi vs_2 = (s_1 \to t_1) \fatsemi (g_2 \, ; G_2!) = ((s_1 \to t_1) \fatsemi g_2) \, ; G_2! \neq \mathsf{id}$.

Case (Ct_Arrow_S): We are given $vs_2 = s_2 \to t_2$ $(\exists s_2, t_2)$. Contradictory because $vs_1 \fatsemi vs_2 = (s_1 \to t_1) \fatsemi (s_2 \to t_2) = (s_2 \fatsemi s_1) \to (t_1 \fatsemi t_2) \neq \mathsf{id}$.

Otherwise: Contradictory because B cannot be a function type.

Case $vs_1 = \forall X.s_1 \,,, t_1$ $(\exists X, s_1, t_1)$: Since $\Sigma \mid \emptyset \vdash_S \forall X.s_1 \,,, t_1 : A \rightsquigarrow B$ is derived by (Ct_All_S), we have $B = \forall X.B_1$ $(\exists B_1)$. By case analysis on the rule applied last to derive $\Sigma \mid \emptyset \vdash_S vs_2 : B \rightsquigarrow C$.

Case (Ct_Inj_S): Contradictory because $vs_1 \fatsemi vs_2 = (\forall X.s_1 \,,, t_1) \fatsemi (g_2 \, ; G_2!) = ((\forall X.s_1 \,,, t_1) \fatsemi g_2) \, ; G_2! \neq \mathsf{id}$.

Case (Ct_All_S): We are given $vs_2 = \forall X.s_2 \,,, t_2$ $(\exists s_2, t_2)$. Contradictory because $vs_1 \fatsemi vs_2 = (\forall X.s_1 \,,, t_1) \fatsemi (\forall X.s_2 \,,, t_2) = \forall X.(s_1 \fatsemi s_2) \,,, (t_1 \fatsemi t_2) \neq \mathsf{id}$.

Otherwise: Contradictory because B cannot be a polymorphic type.

□

Lemma I.50 (Intermediate Coercions are Bisimilar to Values)**.** If $\Sigma \mid \emptyset \vdash V \approx M'\langle s \rangle : A$, then there exists some i such that $s = i$.

Proof. By induction on the derivation of $\Sigma \mid \emptyset \vdash V \approx M'\langle s \rangle : A$. We perform case analysis on the rule applied last to derive $\Sigma \mid \emptyset \vdash V \approx M'\langle s \rangle : A$, which is either (Bs_CrcId), (Bs_Crc), (Bs_CrcMore) or (Bs_CrcIdL).

Case (Bs_CrcId): We are given $s = |\mathsf{id}_A|_\emptyset$. By the definition of the translation, $|\mathsf{id}_A|_\emptyset$ is a ground coercion, that is, intermediate coercion, so we have the conclusion.

Case (Bs_Crc): Since $V_1\langle c \rangle$ is a value, we are given

$$V = V_1\langle vc \rangle, \quad s = |vc|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_1 \approx M' : B, \quad \Sigma \mid \emptyset \vdash_C c : B \rightsquigarrow A \quad (\exists B, vc, V_1) \; .$$

Moreover, by Lemma I.43,

$$s = vs = |vc|_\emptyset, \quad \Sigma \mid \emptyset \vdash_S vs : \Sigma(B) \rightsquigarrow \Sigma(A) \quad (\exists vs) \; .$$

Since vs is an intermediate coercion, we finish the case.

Case (Bs_CrcMore): We are given

$$V = V_1\langle c \rangle, \quad s = s' \fatsemi |c|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_1 \approx M'\langle s' \rangle : B, \quad \Sigma \mid \emptyset \vdash_C c : B \rightsquigarrow A \quad (\exists B, c, s', V_1) \; .$$

By the IH, there exists some i' such that $s' = i'$. Furthermore, Lemma I.22 implies $\Sigma \mid \emptyset \vdash_S M'\langle i' \rangle : \Sigma(B)$. Because this judgment is derived by (T_Crc_S), we have $\Sigma \mid \emptyset \vdash_S i' : C \rightsquigarrow \Sigma(B)$ $(\exists C)$. Moreover, by Lemma I.3, $\Sigma \mid \emptyset \vdash_S |c|_\emptyset : \Sigma(B) \rightsquigarrow \Sigma(A)$. By case analysis on c.

Case $c = G!$ $(\exists G)$: Since $\Sigma \mid \emptyset \vdash_C c : B \rightsquigarrow A$, G is not a type variable and $|G!|_\emptyset = |\mathsf{id}_G|_\emptyset \, ; G!$. Therefore,

$$\begin{aligned} s &= i' \fatsemi |G!|_\emptyset \\ &= i' \fatsemi (|\mathsf{id}_G|_\emptyset \, ; G!) \\ &= (i' \fatsemi |\mathsf{id}_G|_\emptyset) \, ; G! \; . \end{aligned}$$

Case $c = \alpha^-$ $(\exists \alpha)$: Similar to the case of $c = G!$.

Case $c = c' \to d'$ $(\exists c', d')$: Because $\Sigma \mid \emptyset \vdash_C c' \to d' : B \rightsquigarrow A$ is derived by (Ct_Arrow_C), there exist some A', B' such that $B = A' \to B'$. Because $\Sigma \mid \emptyset \vdash_S i' : C \rightsquigarrow (\Sigma(A') \to \Sigma(B'))$ is derived by (Ct_Arrow_S), there exist some s'', t'' such that $i' = s'' \to t''$. Therefore,

$$\begin{aligned} s &= (s'' \to t'') \mathbin{⨟} |c' \to d'|_\emptyset \\ &= (s'' \to t'') \mathbin{⨟} (|c'|_\emptyset \to |d'|_\emptyset) \\ &= (|c'|_\emptyset \mathbin{⨟} s'') \to (t'' \mathbin{⨟} |d'|_\emptyset) \ . \end{aligned}$$

Case $c = \forall X.c'$ $(\exists X, c')$: Similarl to the case of $c = c' \to d'$.

Otherwise: Contradictory with the assumption that $V_1\langle c\rangle$ is a value.

Case (Bs_CrcIdL): We are given

$$V = V_1\langle c^I\rangle, \quad \Sigma \mid \emptyset \vdash V_1 \approx M'\langle s\rangle : B, \quad \Sigma \mid \emptyset \vdash_C c^I : B \rightsquigarrow A \quad (\exists B, c^I, V_1) \ .$$

By the IH, there exists some i' such that $s = i'$. We conclude the case by (Bs_CrcIdL).

□

Lemma I.51 (Coercion Reduction Preserves Bisimilarity (Single Coercion))**.** If $\Sigma \mid \emptyset \vdash V\langle c\rangle \approx U\langle s\rangle : A$, then either of the following holds:

(1) there exists some V_1 such that $\Sigma \triangleright V\langle c\rangle \longrightarrow_C^* \Sigma \triangleright V_1$ and $\Sigma \mid \emptyset \vdash V_1 \approx U\langle s\rangle : A$; or

(2) there exists some p such that $\Sigma \triangleright V\langle c\rangle \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p$ and $s = \bot^p$.

Proof. By induction on c. We perform case analysis on the last rule to derive $\Sigma \mid \emptyset \vdash V\langle c\rangle \approx U\langle s\rangle : A$.

Case (Bs_Crc): We are given

$$s = |c|_\emptyset, \quad \Sigma \mid \emptyset \vdash V \approx U : B, \quad \Sigma \mid \emptyset \vdash_C c : B \rightsquigarrow A \quad (\exists B) \ .$$

By case analysis on c.

Case $c = \mathsf{id}_{A'}$ $(\exists A')$: By (R_Id_C), $\Sigma \triangleright V\langle \mathsf{id}_{A'}\rangle \longrightarrow_C \Sigma \triangleright V$. Because $\Sigma \mid \emptyset \vdash_C \mathsf{id}_{A'} : B \rightsquigarrow A$ is derived by (Ct_Id_C), we have $A' = B = A$. Therefore, it suffices to show that

$$\Sigma \mid \emptyset \vdash V \approx U\langle |\mathsf{id}_A|_\emptyset\rangle : A \ ,$$

which is given by (Bs_CrcId).

Case $c = G!$ $(\exists G)$, $c = \alpha^-$ $(\exists \alpha)$, $c = c' \to d'$ $(\exists c', d')$, $c = \forall X.c'$ $(\exists X, c')$: Because $V\langle c\rangle$ is a value, we have the conclusion by letting $V_1 = V\langle c\rangle$.

Case $c = G?^p$ $(\exists p, G)$: Lemma I.22 implies $\Sigma \mid \emptyset \vdash_C V\langle G?^p\rangle : A$. By inversion of the derivation of this judgment, we have $A = G$ and $\Sigma \mid \emptyset \vdash_C V : \star$. Therefore, Lemma E.2 implies that there exist some H, V_1 such that $V = V_1\langle H!\rangle$. Now, we have $\Sigma \mid \emptyset \vdash V_1\langle H!\rangle \approx U : B$. However, there is contradiction because there are no rules to derive $\Sigma \mid \emptyset \vdash V_1\langle H!\rangle \approx U : B$.

Case $c = \alpha^+$ $(\exists \alpha)$: We are given $|\alpha^+|_\emptyset = \mathsf{id}$. Lemma I.22 implies $\Sigma \mid \emptyset \vdash_C V\langle \alpha^+\rangle : A$. By inversion of the derivation of this judgment, we have $A = \mathbb{A}$ $(\exists \mathbb{A})$, $B = \alpha$ and $\Sigma \mid \emptyset \vdash_C V : \alpha$. Therefore, Lemma E.2 implies that there exist some V_1 such that $V = V_1\langle \alpha^-\rangle$. Furthermore, since $\Sigma \mid \emptyset \vdash V_1\langle \alpha^-\rangle \approx U : \alpha$ is derived by (Bs_CrcIdL), we have

$$\Sigma \mid \emptyset \vdash V_1 \approx U : \mathbb{B}, \quad \Sigma \mid \emptyset \vdash_C \alpha^- : \mathbb{B} \rightsquigarrow \alpha \quad (\exists \mathbb{B}) \ .$$

Since $\Sigma \mid \emptyset \vdash_C \alpha^- : \mathbb{B} \rightsquigarrow \alpha$ is derived by (Ct_Conceal_C), we have $\mathbb{B} = \mathbb{A}$. Furthermore, by (R_Remove_C),

$$\begin{aligned} V\langle c\rangle &= V_1\langle \alpha^-\rangle\langle \alpha^+\rangle \\ &\longrightarrow_C^* V_1 \ . \end{aligned}$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash V_1 \approx U\langle \mathsf{id}\rangle : \mathbb{A}$, which is given by (Bs_CrcId).

Case $c = c_1 \,;\, c_2$ $(\exists c_1, c_2)$: We are given $|c_1 \,;\, c_2|_\emptyset = |c_1|_\emptyset \,\fatsemi\, |c_2|_\emptyset$. By (R_Split_C), $\Sigma \triangleright V\langle c_1 \,;\, c_2\rangle \longrightarrow_C \Sigma \triangleright V\langle c_1\rangle\langle c_2\rangle$. Because $\Sigma \mid \emptyset \vdash_C c_1 \,;\, c_2 : B \rightsquigarrow A$ is derived by (Ct_Seq_C), we have

$$\Sigma \mid \emptyset \vdash_C c_1 : B \rightsquigarrow B', \quad \Sigma \mid \emptyset \vdash_C c_2 : B' \rightsquigarrow A \quad (\exists B') .$$

Therefore, by Lemma I.3, we have $\Sigma \mid \emptyset \vdash_S |c_1|_\emptyset : \Sigma(B) \rightsquigarrow \Sigma(B')$ and $\Sigma \mid \emptyset \vdash_S |c_2|_\emptyset : \Sigma(B') \rightsquigarrow \Sigma(A)$. Furthermore, by applying (Bs_Crc) and (Bs_CrcMore) to $\Sigma \mid \emptyset \vdash V \approx U : B$,

$$\Sigma \mid \emptyset \vdash V\langle c_1\rangle\langle c_2\rangle \approx U\langle |c_1|_\emptyset \,\fatsemi\, |c_2|_\emptyset\rangle : A .$$

Case $c = \perp^p_{A' \rightsquigarrow B'}$ $(\exists p, A', B')$: By (R_Fail_C), $\Sigma \triangleright V\langle \perp^p_{A' \rightsquigarrow B'}\rangle \longrightarrow_C \Sigma \triangleright \mathsf{blame}\, p$. Therefore, it suffices to show that $s = \perp^p$, which is given by

$$\begin{aligned} s &= |\perp^p_{A' \rightsquigarrow B'}|_\emptyset \\ &= \perp^p . \end{aligned}$$

Case (Bs_CrcMore): We are given

$$s = s' \,\fatsemi\, |c|_\emptyset, \quad \Sigma \mid \emptyset \vdash V \approx U\langle s'\rangle : B, \quad \Sigma \mid \emptyset \vdash_C c : B \rightsquigarrow A \quad (\exists B, s') .$$

Therefore, Lemma I.22 implies $\Sigma \mid \emptyset \vdash_S U\langle s'\rangle : \Sigma(B)$. Because this judgment is derived by (T_Crc_S), we have $\Sigma \mid \emptyset \vdash_S s' : C \rightsquigarrow \Sigma(B)$ $(\exists C)$. By case analysis on c.

Case $c = \mathsf{id}_{A'}$ $(\exists A')$: By (R_Id_C), $\Sigma \triangleright V\langle \mathsf{id}_{A'}\rangle \longrightarrow_C \Sigma \triangleright V$. Because $\Sigma \mid \emptyset \vdash_C \mathsf{id}_{A'} : B \rightsquigarrow A$ is derived by (Ct_Id_C), we have $A' = B = A$. Therefore, it suffices to show that

$$\Sigma \mid \emptyset \vdash V \approx U\langle s' \,\fatsemi\, |\mathsf{id}_A|_\emptyset\rangle : A .$$

Now, we have $\Sigma \mid \emptyset \vdash_S s' : C \rightsquigarrow \Sigma(A)$, so by Lemma I.11, $s' \,\fatsemi\, |\mathsf{id}_A|_\emptyset = s'$. Therefore, it suffices to show that

$$\Sigma \mid \emptyset \vdash V \approx U\langle s'\rangle : A ,$$

which holds already.

Case $c = G!$ $(\exists G)$, $c = \alpha^-$ $(\exists \alpha)$, $c = c' \to d'$ $(\exists c', d')$, $c = \forall X.c'$ $(\exists X, c')$: Because $V\langle c\rangle$ is a value, we have the conclusion by letting $V_1 = V\langle c\rangle$.

Case $c = G?^p$ $(\exists p, G)$: Lemma I.22 implies $\Sigma \mid \emptyset \vdash_C V\langle G?^p\rangle : A$. By inversion of the derivation of this judgment, we have $A = G$ and $\Sigma \mid \emptyset \vdash_C V : \star$. Therefore, Lemma E.2 implies that there exist some H, V_1 such that $V = V_1\langle H!\rangle$. Now, we have $\Sigma \mid \emptyset \vdash V_1\langle H!\rangle \approx U\langle s'\rangle : B$. We perform case analysis on the last rule to derive $\Sigma \mid \emptyset \vdash V_1\langle H!\rangle \approx U\langle s'\rangle : B$.

Case (Bs_Crc): We are given

$$s' = |H!|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_1 \approx U : B', \quad \Sigma \mid \emptyset \vdash_C H! : B' \rightsquigarrow B \quad (\exists B') .$$

Because $\Sigma \mid \emptyset \vdash_C H! : B' \rightsquigarrow B$ is derived by (Ct_Inj_C), we have $B' = H$ and $B = \star$. Furthermore, by Lemma I.3, $\Sigma \mid \emptyset \vdash_S |H!|_\emptyset : \Sigma(H) \rightsquigarrow \Sigma(\star)$ and $\Sigma \mid \emptyset \vdash_S |G?^p|_\emptyset : \Sigma(\star) \rightsquigarrow \Sigma(G)$. By case analysis on whether $H = G$ or not.

Case $H = G$: By (R_Collapse_C), $\Sigma \triangleright V_1\langle G!\rangle\langle G?^p\rangle \longrightarrow_C \Sigma \triangleright V_1$. Therefore, it suffices to show that

$$\Sigma \mid \emptyset \vdash V_1 \approx U\langle |G!| \,\fatsemi\, |G?^p|\rangle : G .$$

Lemma I.23 implies $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash G$. Therefore, by (Ct_Id_C), $\Sigma \mid \emptyset \vdash_C \mathsf{id}_G : G \rightsquigarrow G$. Hence, by Lemma I.3, $\Sigma \mid \emptyset \vdash_S |\mathsf{id}_G|_\emptyset : \Sigma(G) \rightsquigarrow \Sigma(G)$. Furthermore, by Lemma I.11, $|\mathsf{id}_G|_\emptyset \,\fatsemi\, |\mathsf{id}_G|_\emptyset = |\mathsf{id}_G|_\emptyset$. By Lemma I.11, $|\mathsf{id}_G|_\emptyset \,\fatsemi\, |\mathsf{id}_G|_\emptyset = |\mathsf{id}_G|_\emptyset$. Since $\Sigma \mid \emptyset \vdash_C G?^p : \star \rightsquigarrow G$, G is not a type variable. Thus, $|G!|_\emptyset = |\mathsf{id}_G|_\emptyset \,;\, G!$ and $|G?^p|_\emptyset = G?^p \,;\, |\mathsf{id}_G|_\emptyset$. Therefore,

$$\begin{aligned} |G!|_\emptyset \,\fatsemi\, |G?^p|_\emptyset &= (|\mathsf{id}_G|_\emptyset \,;\, G!) \,\fatsemi\, (G?^p \,;\, |\mathsf{id}_G|_\emptyset) \\ &= |\mathsf{id}_G|_\emptyset \,\fatsemi\, |\mathsf{id}_G|_\emptyset \\ &= |\mathsf{id}_G|_\emptyset . \end{aligned}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash V_1 \approx U\langle |\mathsf{id}_G|_\emptyset\rangle : G$, which is given by (Bs_CrcId).

Case $H \neq G$: By (R_CONFLICT_C), $\Sigma \triangleright V_1\langle H!\rangle\langle G?^p\rangle \longrightarrow_C \Sigma \triangleright \mathsf{blame}\, p$. Therefore, it suffices to show that $s = \bot^p$. Therefore,

$$s = |H!|_\emptyset \,;\, |G?^p|_\emptyset \ .$$

Since $\Sigma \mid \emptyset \vdash_C G?^p : \star \rightsquigarrow G$ and $\Sigma \mid \emptyset \vdash_C H! : H \rightsquigarrow \star$, G and H are not type variables. Thus, $|H!|_\emptyset = |\mathsf{id}_H|_\emptyset \,; H!$ and $|G?^p|_\emptyset = G?^p \,; |\mathsf{id}_G|_\emptyset$. Hence,

$$\begin{aligned} |H!|_\emptyset \,;\, |G?^p|_\emptyset &= (|\mathsf{id}_H|_\emptyset \,; H!) \,;\, (G?^p \,; |\mathsf{id}_G|_\emptyset) \\ &= \bot^p \ . \end{aligned}$$

Case (BS_CRCMORE): We are given

$$s' = s'' \,;\, |H!|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_1 \approx U\langle s''\rangle : B', \quad \Sigma \mid \emptyset \vdash_C H! : B' \rightsquigarrow B \quad (\exists B', s'') \ .$$

Because $\Sigma \mid \emptyset \vdash_C H! : B' \rightsquigarrow B$ is derived by (CT_INJ_C), we have $B' = H$. Therefore, Lemma I.22 implies $\Sigma \mid \emptyset \vdash_S U\langle s''\rangle : \Sigma(H)$. Because this judgment is derived by (T_CRC_S), we have $\Sigma \mid \emptyset \vdash_S s'' : C \rightsquigarrow \Sigma(H)$ $(\exists C)$. Furthermore, by Lemma I.3, $\Sigma \mid \emptyset \vdash_S |H!|_\emptyset : \Sigma(H) \rightsquigarrow \Sigma(B)$ and $\Sigma \mid \emptyset \vdash_S |G?^p|_\emptyset : \Sigma(B) \rightsquigarrow \Sigma(G)$. Therefore, by Lemma I.7

$$(s'' \,;\, |H!|_\emptyset) \,;\, |G?^p|_\emptyset = s'' \,;\, (|H!|_\emptyset \,;\, |G?^p|_\emptyset) \ .$$

By case analysis on whether $H = G$.

Case $H = G$: By (R_COLLAPSE_C), $\Sigma \triangleright V_1\langle G!\rangle\langle G?^p\rangle \longrightarrow_C \Sigma \triangleright V_1$. Therefore, it suffices to show that

$$\Sigma \mid \emptyset \vdash V_1 \approx U\langle (s'' \,;\, |G!|) \,;\, |G?^p|\rangle : G \ .$$

Lemma I.23 implies $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash G$. Therefore, by (CT_ID_C), $\Sigma \mid \emptyset \vdash_C \mathsf{id}_G : G \rightsquigarrow G$. Hence, by Lemma I.3, $\Sigma \mid \emptyset \vdash_S |\mathsf{id}_G|_\emptyset : \Sigma(G) \rightsquigarrow \Sigma(G)$. Furthermore, by Lemma I.11, $|\mathsf{id}_G|_\emptyset \,;\, |\mathsf{id}_G|_\emptyset = |\mathsf{id}_G|_\emptyset$ and $s'' \,;\, |\mathsf{id}_G|_\emptyset = s''$. Therefore,

$$(s'' \,;\, |G!|_\emptyset) \,;\, |G?^p|_\emptyset = s'' \,;\, (|G!|_\emptyset \,;\, |G?^p|_\emptyset)$$

We perform a case analysis on whether $G = X$ $(\exists X)$ or not.

Case $G = X$ $(\exists X)$:

$$\begin{aligned} s'' \,;\, (|X!|_\emptyset \,;\, |X?^p|_\emptyset) &= s'' \,;\, (\mathsf{id} \,;\, \mathsf{id}) \\ &= s'' \,;\, \mathsf{id} \\ &= s'' \ . \end{aligned}$$

Case $G \neq X$ $(\exists X)$: By Lemma I.11, $|\mathsf{id}_G|_\emptyset \,;\, |\mathsf{id}_G|_\emptyset = |\mathsf{id}_G|_\emptyset$ and $s'' \,;\, |\mathsf{id}_G|_\emptyset = s''$. Therefore,

$$\begin{aligned} s'' \,;\, (|G!|_\emptyset \,;\, |G?^p|_\emptyset) &= s'' \,;\, ((|\mathsf{id}_G|_\emptyset \,; G!) \,;\, (G?^p \,; |\mathsf{id}_G|_\emptyset)) \\ &= s'' \,;\, (|\mathsf{id}_G|_\emptyset \,;\, |\mathsf{id}_G|_\emptyset) \\ &= s'' \,;\, |\mathsf{id}_G|_\emptyset \\ &= s'' \ . \end{aligned}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash V_1 \approx U\langle s''\rangle : G$, which holds already.

Case $H \neq G$: By (R_CONFLICT_C), $\Sigma \triangleright V_1\langle H!\rangle\langle G?^p\rangle \longrightarrow_C \Sigma \triangleright \mathsf{blame}\, p$. Therefore, it suffices to show that $s = \bot^p$. Lemma I.50 implies that there exists some i such that $s'' = i$. Therefore,

$$\begin{aligned} s &= (s'' \,;\, |H!|_\emptyset) \,;\, |G?^p|_\emptyset \\ &= s'' \,;\, (|H!|_\emptyset \,;\, |G?^p|_\emptyset) \\ &= i \,;\, (|H!|_\emptyset \,;\, |G?^p|_\emptyset) \end{aligned}$$

Assume that there exists a type variable X such that $G = X$. Thus, $|X?^p|_\emptyset = \mathsf{id}$. Hence, $\Sigma \mid \emptyset \vdash_S \mathsf{id} : \Sigma(B) \rightsquigarrow \Sigma(X)$. However, it is contradictory because it cannot be derived any rules. Therefore, G is not a type variable. Also, H is not a type variable. Hence,

$$\begin{aligned} i \fatsemi (|H!|_\emptyset \fatsemi |G?^p|_\emptyset) &= i \fatsemi ((|\mathsf{id}_H|_\emptyset \,;\, H!) \fatsemi (G?^p \,;\, |\mathsf{id}_G|_\emptyset)) \\ &= i \fatsemi \bot^p \\ &= \bot^p \ . \end{aligned}$$

Case (Bs_CrcId) and (Bs_CrcIdL): Contradictory because $H!$ is not a no-op coercion.

Case $c = \alpha^+$ ($\exists \alpha$): Since $\Sigma \mid \emptyset \vdash_C \alpha^+ : B \rightsquigarrow A$ is derived by (Ct_Reveal_C), we have

$$B = \alpha, \quad A = \mathbb{A} \quad \alpha := \mathbb{A} \in \Sigma \quad (\exists \mathbb{A}) \ .$$

Lemma I.22 implies $\Sigma \mid \emptyset \vdash_C V\langle \alpha^+ \rangle : \mathbb{A}$. Since $\Sigma \mid \emptyset \vdash_C V\langle \alpha^+ \rangle : \mathbb{A}$ is derived by (T_Crc_C), we have

$$\Sigma \mid \emptyset \vdash_C V : \alpha \ .$$

Therefore, Lemma E.2 implies that there exist some V_1 such that $V = V_1\langle \alpha^- \rangle$. By (R_Remove_C),

$$\begin{aligned} V\langle c \rangle &= V_1\langle \alpha^- \rangle \langle \alpha^+ \rangle \\ &\longrightarrow_C^* V_1 \ . \end{aligned}$$

Moreover, since

$$\begin{aligned} s &= s' \fatsemi |c|_\emptyset \\ &= s' \fatsemi |\alpha^+|_\emptyset \\ &= s' \fatsemi \mathsf{id} \\ &= s' \ , \end{aligned}$$

it suffices to show that $\Sigma \mid \emptyset \vdash V_1 \approx U\langle s' \rangle : \mathbb{A}$. Now, we have $\Sigma \mid \emptyset \vdash V_1\langle \alpha^- \rangle \approx U\langle s' \rangle : \alpha$. We perform case analysis on the last rule to derive $\Sigma \mid \emptyset \vdash V_1\langle \alpha^- \rangle \approx U\langle s' \rangle : \alpha$, which is either of (Bs_CrcId), (Bs_CrcIdL), (Bs_Crc), or (Bs_CrcMore).

Case (Bs_CrcId): We are given

$$s' = |\mathsf{id}_\alpha|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_1\langle \alpha^- \rangle \approx U : \alpha, \quad \Sigma \mid \emptyset \vdash_C \mathsf{id}_\alpha : \alpha \rightsquigarrow \alpha \ .$$

Since $\Sigma \mid \emptyset \vdash V_1\langle \alpha^- \rangle \approx U : \alpha$ is derived by (Bs_CrcIdL) and (Ct_Conceal_C), we have

$$\Sigma \mid \emptyset \vdash V_1 \approx U : \mathbb{A}, \quad \Sigma \mid \emptyset \vdash_C \alpha^- : \mathbb{A} \rightsquigarrow \alpha \ .$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash V_1 \approx U\langle |\mathsf{id}_\alpha|_\emptyset \rangle : \mathbb{A}$, which is given by (Bs_CrcId).

Case (Bs_CrcIdL): We are given

$$\Sigma \mid \emptyset \vdash V_1 \approx U\langle s' \rangle : C, \quad \Sigma \mid \emptyset \vdash_C \alpha^- : C \rightsquigarrow \alpha \ .$$

Since $\Sigma \mid \emptyset \vdash_C \alpha^- : C \rightsquigarrow \alpha$ is derived by (Ct_Conceal_C), we have $C = \mathbb{A}$.

Case (Bs_Crc): We are given

$$s' = |\alpha^-|_\emptyset = \mathsf{id}, \quad \Sigma \mid \emptyset \vdash V_1 \approx U : B', \quad \Sigma \mid \emptyset \vdash_C \alpha^- : B' \rightsquigarrow \alpha \quad (\exists B') \ .$$

Because $\Sigma \mid \emptyset \vdash_C \alpha^- : B' \rightsquigarrow \alpha$ is derived by (Ct_Conceal_C), we have $B' = \mathbb{A}$. Here, we have

$$\begin{aligned} s = s' &= \mathsf{id} \\ &= |\mathsf{id}_\mathbb{A}|_\emptyset \ . \end{aligned}$$

It suffices to show that $\Sigma \mid \emptyset \vdash V_1 \approx U\langle |\mathsf{id}_\mathbb{A}|_\emptyset \rangle : \mathbb{A}$. By Lemma I.23, $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash \mathbb{A}$. Therefore, by (Ct_Id_C), $\Sigma \mid \emptyset \vdash_C \mathsf{id}_\mathbb{A} : \mathbb{A} \rightsquigarrow \mathbb{A}$. Hence, by (Bs_CrcId), we finish the case.

Case (Bs_CrcMore): We are given

$$s' = s'' \fatsemi |\alpha^-|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_1 \approx U\langle s'' \rangle : B', \quad \Sigma \mid \emptyset \vdash_C \alpha^- : B' \rightsquigarrow \alpha \quad (\exists B', s'') .$$

Because $\Sigma \mid \emptyset \vdash_C \alpha^- : B' \rightsquigarrow \alpha$ is derived by (Ct_Conceal_C), we have $B' = \mathbb{A}$. Furthermore,

$$\begin{aligned} s = s' &= s'' \fatsemi |\alpha^-|_\emptyset \\ &= s'' \fatsemi \mathsf{id} \\ &= s'' . \end{aligned}$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash V_1 \approx U\langle s'' \rangle : \mathbb{A}$, which already holds.

Case $c = c_1 \mathbin{;} c_2$ $(\exists c_1, c_2)$: We are given $|c_1 ; c_2|_\emptyset = |c_1 ; c_2|_\emptyset = |c_1|_\emptyset \fatsemi |c_2|_\emptyset$. By (R_Split_C), $\Sigma \triangleright V\langle c_1 ; c_2 \rangle \longrightarrow_C \Sigma \triangleright V\langle c_1 \rangle \langle c_2 \rangle$. Because $\Sigma \mid \emptyset \vdash_C c_1 \mathbin{;} c_2 : B \rightsquigarrow A$ is derived by (Ct_Seq_C), we have

$$\Sigma \mid \emptyset \vdash_C c_1 : B \rightsquigarrow B', \quad \Sigma \mid \emptyset \vdash_C c_2 : B' \rightsquigarrow A \quad (\exists B') .$$

Therefore, by Lemma I.3, we have $\Sigma \mid \emptyset \vdash_S |c_1|_\emptyset : \Sigma(B) \rightsquigarrow \Sigma(B')$ and $\Sigma \mid \emptyset \vdash_S |c_2|_\emptyset : \Sigma(B') \rightsquigarrow \Sigma(A)$. Therefore, by Lemma I.7,

$$(s' \fatsemi |c_1|_\emptyset) \fatsemi |c_2|_\emptyset = s' \fatsemi (|c_1|_\emptyset \fatsemi |c_2|_\emptyset) .$$

Furthermore, by applying (Bs_CrcMore) to $\Sigma \mid \emptyset \vdash V \approx U\langle s' \rangle : B$,

$$\Sigma \mid \emptyset \vdash V\langle c_1 \rangle \approx U\langle s' \fatsemi |c_1|_\emptyset \rangle : B' .$$

Then, we can apply the IH. We perform case analysis on the result.

Case (1): We are given

$$\Sigma \triangleright V\langle c_1 \rangle \longrightarrow_C^* \Sigma \triangleright V_1, \quad \Sigma \mid \emptyset \vdash V_1 \approx U\langle s' \fatsemi |c_1|_\emptyset \rangle : B' \quad (\exists V_1) .$$

By applying (R_Ctx_C) repeatedly, $\Sigma \triangleright V\langle c_1 \rangle \langle c_2 \rangle \longrightarrow_C^* \Sigma \triangleright V_1\langle c_2 \rangle$. Furthermore, by (Bs_CrcMore),

$$\Sigma \mid \emptyset \vdash V_1\langle c_2 \rangle \approx U\langle (s' \fatsemi |c_1|_\emptyset) \fatsemi |c_2|_\emptyset \rangle : A .$$

Therefore, we can apply the IH. By case analysis on the result.

Case (1): We are given

$$\Sigma \triangleright V_1\langle c_2 \rangle \longrightarrow_C^* \Sigma \triangleright V_2, \quad \Sigma \mid \emptyset \vdash V_2 \approx U\langle (s' \fatsemi |c_1|_\emptyset) \fatsemi |c_2|_\emptyset \rangle : A \quad (\exists V_2) .$$

Therefore,

$$\Sigma \triangleright V\langle c_1 \mathbin{;} c_2 \rangle \longrightarrow_C \Sigma \triangleright V\langle c_1 \rangle \langle c_2 \rangle \longrightarrow_C^* \Sigma \triangleright V_1\langle c_2 \rangle \longrightarrow_C^* \Sigma \triangleright V_2 .$$

Furthermore, because $\Sigma \mid \emptyset \vdash V_2 \approx U\langle (s' \fatsemi |c_1|_\emptyset) \fatsemi |c_2|_\emptyset \rangle : A$ and $(s' \fatsemi |c_1|_\emptyset) \fatsemi |c_2|_\emptyset = s' \fatsemi (|c_1|_\emptyset \fatsemi |c_2|_\emptyset)$, we have

$$\Sigma \mid \emptyset \vdash V_2 \approx U\langle s' \fatsemi (|c_1|_\emptyset \fatsemi |c_2|_\emptyset) \rangle : A .$$

Case (2): We are given

$$\Sigma \triangleright V_1\langle c_2 \rangle \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p_2, \quad (s' \fatsemi |c_1|_\emptyset) \fatsemi |c_2|_\emptyset = \bot^{p_2} \quad (\exists p_2) .$$

Therefore,

$$\Sigma \triangleright V\langle c_1 \mathbin{;} c_2 \rangle \longrightarrow_C \Sigma \triangleright V\langle c_1 \rangle \langle c_2 \rangle \longrightarrow_C^* \Sigma \triangleright V_1\langle c_2 \rangle \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p_2 .$$

Furthermore,

$$\begin{aligned} s &= s' \fatsemi (|c_1|_\emptyset \fatsemi |c_2|_\emptyset) \\ &= (s' \fatsemi |c_1|_\emptyset) \fatsemi |c_2|_\emptyset \\ &= \bot^{p_2} . \end{aligned}$$

Case (2): We are given

$$\Sigma \triangleright V\langle c_1 \rangle \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p_1, \quad s' \,\mathbin{;}\, |c_1|_\emptyset = \bot^{p_1} \quad (\exists p_1)\ .$$

Therefore, by (R_Ctx_C) and (R_Blame),

$$\Sigma \triangleright V\langle c_1 \mathbin{;} c_2 \rangle \longrightarrow_C \Sigma \triangleright V\langle c_1\rangle\langle c_2 \rangle \longrightarrow_C^* \Sigma \triangleright (\mathsf{blame}\, p_1)\langle c_2 \rangle \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p_1\ .$$

Furthermore,

$$\begin{aligned} s &= s' \,\mathbin{;}\, (|c_1|_\emptyset \,\mathbin{;}\, |c_2|_\emptyset) \\ &= (s' \,\mathbin{;}\, |c_1|_\emptyset) \,\mathbin{;}\, |c_2|_\emptyset \\ &= \bot^{p_1} \,\mathbin{;}\, |c_2|_\emptyset \\ &= \bot^{p_1}\ . \end{aligned}$$

Case $c = \bot^{p}_{A' \rightsquigarrow B'}$ $(\exists p, A', B')$: By (R_Fail_C), $\Sigma \triangleright V\langle \bot^{p}_{A' \rightsquigarrow B'} \rangle \longrightarrow_C \Sigma \triangleright \mathsf{blame}\, p$. Therefore, it suffices to show that $s = \bot^p$. By Lemma I.50, there exists some i such that $s' = i$. Therefore,

$$\begin{aligned} s &= s' \,\mathbin{;}\, |\bot^{p}_{A' \rightsquigarrow B'}| \\ &= s' \,\mathbin{;}\, \bot^{p} \\ &= i \,\mathbin{;}\, \bot^{p} \\ &= \bot^{p}\ . \end{aligned}$$

Case (Bs_CrcId): We are given

$$s = |\mathsf{id}_A|_\emptyset, \quad \Sigma \mid \emptyset \vdash V\langle c \rangle \approx U : A\ .$$

By inversion of the derivation of $\Sigma \mid \emptyset \vdash V\langle c \rangle \approx U : A$, we have

$$c = c^I, \quad \Sigma \mid \emptyset \vdash V \approx U : B, \quad \Sigma \mid \emptyset \vdash c^I : B \rightsquigarrow A \quad (\exists c^I, B)\ .$$

By Lemma I.56, there exists a value V_2 such that $\Sigma \triangleright V\langle c^I \rangle \longrightarrow_C^* \Sigma \triangleright V_2$ and $\Sigma \mid \emptyset \vdash V_2 \approx U : A$.

Case (Bs_CrcIdL): We are given

$$c = c^I, \quad \Sigma \mid \emptyset \vdash V \approx U\langle s \rangle : B, \quad \Sigma \mid \emptyset \vdash_C c^I : B \rightsquigarrow A \quad (\exists c^I, B)\ .$$

By Lemma I.56, there exists a value V_2 such that $\Sigma \triangleright V\langle c^I \rangle \longrightarrow_C^* \Sigma \triangleright V_2$ and $\Sigma \mid \emptyset \vdash V_2 \approx U\langle s \rangle : A$.

□

Lemma I.52 (Coercion Reduction Preserves Bisimilarity)**.** Let $n > 0$. If $\Sigma \mid \emptyset \vdash V\langle c_1 \rangle \cdots \langle c_n \rangle \approx U\langle s \rangle : A$, then either of the following holds:

(1) there exists some V_1 such that $\Sigma \triangleright V\langle c_1 \rangle \cdots \langle c_n \rangle \longrightarrow_C^* \Sigma \triangleright V_1$ and $\Sigma \mid \emptyset \vdash V_1 \approx U\langle s \rangle : A$.

(2) there exists some p such that $\Sigma \triangleright V\langle c_1 \rangle \cdots \langle c_n \rangle \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p$ and $s = \bot^p$.

Proof. By induction on n.

Case $n = 1$: By Lemma I.51.

Case $n = k + 1 (k \geq 1)$: We perform case analysis on the last rule to derive $\Sigma \mid \emptyset \vdash V\langle c_1 \rangle \cdots \langle c_k \rangle \langle c_{k+1} \rangle \approx U\langle s \rangle : A$.

Case (Bs_Crc): We are given

$$s = |c_{k+1}|_\emptyset, \quad \Sigma \mid \emptyset \vdash V\langle c_1 \rangle \cdots \langle c_k \rangle \approx U : B, \quad \Sigma \mid \emptyset \vdash_C c_{k+1} : B \rightsquigarrow A \quad (\exists B)\ .$$

By Lemma I.40, we have

$$\Sigma \mid \emptyset \vdash V \approx U : A_0, \quad c_i = c_i^I, \quad A_k = B,$$
$$\Sigma \mid \emptyset \vdash_C c_i^I : A_{i-1} \rightsquigarrow A_i \quad (\exists V_1, A_0, \ldots, A_k, c_1^I, \ldots, c_k^I) \ .$$

Therefore, by Lemma I.56,

$$\Sigma \triangleright V\langle c_1^I \rangle \longrightarrow_C^* \Sigma \triangleright V_1, \quad \Sigma \mid \emptyset \vdash V_1 \approx U : A_1 \quad (\exists V_1).$$

By (Bs_CrcIdL), $\Sigma \mid \emptyset \vdash V_1\langle c_2^I \rangle \approx U : A_2$. Similarly, by applying Lemma I.56 and (Bs_CrcIdL) repeatedly, there exists V_k such that

$$\Sigma \triangleright V\langle c_1^I \rangle \cdots \langle c_k^I \rangle \longrightarrow_C^* \Sigma \triangleright V_k, \quad \Sigma \mid \emptyset \vdash V_k \approx U : A_k \ .$$

By (Bs_Crc),

$$\Sigma \mid \emptyset \vdash V_k\langle c_{k+1} \rangle \approx U\langle |c_{k+1}|_\emptyset \rangle : A \ .$$

Therefore, by Lemma I.51, it suffices to consider the following two cases.

Case (1) in Lemma I.51: We are given

$$\Sigma \triangleright V_k\langle c_{k+1} \rangle \longrightarrow_C^* \Sigma \triangleright V_2, \quad \Sigma \mid \emptyset \vdash V_2 \approx U\langle |c_{k+1}|_\emptyset \rangle : A \quad (\exists V_2) \ .$$

Therefore,

$$\begin{aligned} \Sigma \triangleright V\langle c_1 \rangle \cdots \langle c_k \rangle \langle c_{k+1} \rangle &\longrightarrow_C^* \Sigma \triangleright V_k\langle c_{k+1} \rangle \\ &\longrightarrow_C^* \Sigma \triangleright V_2 \ . \end{aligned}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash V_2 \approx U\langle |c_{k+1}|_\emptyset \rangle : A$, which holds already.

Case (2) in Lemma I.51: We are given

$$\Sigma \triangleright V_k\langle c_{k+1} \rangle \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p, \quad |c_{k+1}|_\emptyset = \bot^p \quad (\exists p) \ .$$

Hence,

$$\begin{aligned} \Sigma \triangleright V\langle c_1 \rangle \cdots \langle c_k \rangle \langle c_{k+1} \rangle &\longrightarrow_C^* \Sigma \triangleright V_k\langle c_{k+1} \rangle \\ &\longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p \ . \end{aligned}$$

Furthermore,

$$\begin{aligned} s &= |c_{k+1}|_\emptyset \\ &= \bot^p \ . \end{aligned}$$

Case (Bs_CrcMore): We are given

$$s = s' \mathbin{;} |c_{k+1}|_\emptyset, \quad \Sigma \mid \emptyset \vdash V\langle c_1 \rangle \cdots \langle c_k \rangle \approx U\langle s' \rangle : B, \quad \Sigma \mid \emptyset \vdash_C c_{k+1} : B \rightsquigarrow A \quad (\exists B, s') \ .$$

Therefore, we can apply the IH, and perform case analysis on the result.

Case (1): We are given

$$\Sigma \triangleright V\langle c_1 \rangle \cdots \langle c_k \rangle \longrightarrow_C^* \Sigma \triangleright V_1, \quad \Sigma \mid \emptyset \vdash V_1 \approx U\langle s' \rangle : B \quad (\exists V_1) \ .$$

Therefore, by applying (R_Ctx_C) repeatedly, we have

$$\Sigma \triangleright V\langle c_1 \rangle \cdots \langle c_k \rangle \langle c_{k+1} \rangle \longrightarrow_C^* \Sigma \triangleright V_1\langle c_{k+1} \rangle \ .$$

By (Bs_CrcMore),

$$\Sigma \mid \emptyset \vdash V_1\langle c_{k+1} \rangle \approx U\langle s' \mathbin{;} |c_{k+1}|_\emptyset \rangle : A \ .$$

Therefore, by Lemma I.51, it suffices to consider the following two cases.

Case (1) in Lemma I.51: We are given

$$\Sigma \triangleright V_1\langle c_{k+1}\rangle \longrightarrow_C^* \Sigma \triangleright V_2, \quad \Sigma \mid \emptyset \vdash V_2 \approx U\langle s' \mathbin{;} |c_{k+1}|_\emptyset\rangle : A \quad (\exists V_2)\ .$$

Therefore,

$$\begin{aligned}\Sigma \triangleright V\langle c_1\rangle \cdots \langle c_k\rangle\langle c_{k+1}\rangle &\longrightarrow_C^* \Sigma \triangleright V_1\langle c_{k+1}\rangle \\ &\longrightarrow_C^* \Sigma \triangleright V_2\ .\end{aligned}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash V_2 \approx U\langle s' \mathbin{;} |c_{k+1}|_\emptyset\rangle : A$, which holds already.

Case (2) in Lemma I.51: We are given

$$\Sigma \triangleright V_1\langle c_{k+1}\rangle \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p, \quad s' \mathbin{;} |c_{k+1}|_\emptyset = \bot^p \quad (\exists p)\ .$$

Hence,

$$\begin{aligned}\Sigma \triangleright V\langle c_1\rangle \cdots \langle c_k\rangle\langle c_{k+1}\rangle &\longrightarrow_C^* \Sigma \triangleright V_1\langle c_{k+1}\rangle \\ &\longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p\ .\end{aligned}$$

Furthermore,

$$\begin{aligned}s &= s' \mathbin{;} |c_{k+1}|_\emptyset \\ &= \bot^p\ .\end{aligned}$$

Case (2): We are given

$$\Sigma \triangleright V\langle c_1\rangle \cdots \langle c_k\rangle \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p, \quad s' = \bot^p \quad (\exists p)\ .$$

Therefore, by (R_Ctx_C) and (R_Blame_C),

$$\begin{aligned}\Sigma \triangleright V\langle c_1\rangle \cdots \langle c_k\rangle\langle c_{k+1}\rangle &\longrightarrow_C^* \Sigma \triangleright (\mathsf{blame}\, p)\langle c_{k+1}\rangle \\ &\longrightarrow_C \Sigma \triangleright \mathsf{blame}\, p\ .\end{aligned}$$

Furthermore,

$$\begin{aligned}s &= s' \mathbin{;} |c_{k+1}|_\emptyset \\ &= \bot^p \mathbin{;} |c_{k+1}|_\emptyset \\ &= \bot^p\ .\end{aligned}$$

Case (Bs_CrcId): We are given

$$s = |\mathsf{id}_A|_\emptyset, \quad \Sigma \mid \emptyset \vdash V\langle c_1\rangle \cdots \langle c_k\rangle \approx U : A, \quad \Sigma \mid \emptyset \vdash_C c_{k+1} : B \rightsquigarrow A\ .$$

By inversion of the derivation of $\Sigma \mid \emptyset \vdash V\langle c_1\rangle \cdots \langle c_k\rangle \approx U : A$, we have for all $0 \le i \le k$

$$c_i = c_i^I, \quad \Sigma \mid \emptyset \vdash V \approx U : A_1, \quad \Sigma \mid \emptyset \vdash_C c_i : A_i \rightsquigarrow A_{i+1}, \quad A_{k+1} = A \quad (\exists c_1^I, ..., c_k^I, A_1, ..., A_{k+1})\ .$$

By applying Lemma I.56 repeatedly, there exists a value V_2 such that $\Sigma \triangleright V\langle c_1^I\rangle \cdots \langle c_k^I\rangle \longrightarrow_C^* \Sigma \triangleright V_2$ and $\Sigma \mid \emptyset \vdash V_2 \approx U : A$.

Case (Bs_CrcIdL): We are given

$$c_{k+1} = c_{k+1}^I, \quad \Sigma \mid \emptyset \vdash V\langle c_1\rangle \cdots \langle c_k\rangle \approx U\langle s\rangle : B, \quad \Sigma \mid \emptyset \vdash_C c_{k+1}^I : B \rightsquigarrow A\ .$$

Therefore, we can apply the IH, and perform case analysis on the result.

Case (1): We are given

$$\Sigma \triangleright V\langle c_1\rangle \cdots \langle c_k\rangle \longrightarrow_C^* \Sigma \triangleright V_1, \quad \Sigma \mid \emptyset \vdash V_1 \approx U\langle s\rangle : B \quad (\exists V_1) \ .$$

Therefore, by applying (R_Ctx_C) repeatedly, we have

$$\Sigma \triangleright V\langle c_1\rangle \cdots \langle c_k\rangle\langle c^I_{k+1}\rangle \longrightarrow_C^* \Sigma \triangleright V_1\langle c^I_{k+1}\rangle \ .$$

By (Bs_CrcIdL),

$$\Sigma \mid \emptyset \vdash V_1\langle c^I_{k+1}\rangle \approx U\langle s\rangle : A \ .$$

Therefore, by Lemma I.51, it suffices to consider the following two cases.

Case (1) in Lemma I.51: We are given

$$\Sigma \triangleright V_1\langle c^I_{k+1}\rangle \longrightarrow_C^* \Sigma \triangleright V_2, \quad \Sigma \mid \emptyset \vdash V_2 \approx U\langle s\rangle : A \quad (\exists V_2) \ .$$

Therefore,

$$\begin{aligned}\Sigma \triangleright V\langle c_1\rangle \cdots \langle c_k\rangle\langle c^I_{k+1}\rangle &\longrightarrow_C^* \Sigma \triangleright V_1\langle c^I_{k+1}\rangle \\ &\longrightarrow_C^* \Sigma \triangleright V_2 \ .\end{aligned}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash V_2 \approx U\langle s\rangle : A$, which holds already.

Case (2) in Lemma I.51: We are given

$$\Sigma \triangleright V_1\langle c^I_{k+1}\rangle \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p, \quad s = \bot^p \quad (\exists p) \ .$$

Hence,

$$\begin{aligned}\Sigma \triangleright V\langle c_1\rangle \cdots \langle c_k\rangle\langle c^I_{k+1}\rangle &\longrightarrow_C^* \Sigma \triangleright V_1\langle c^I_{k+1}\rangle \\ &\longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p \ .\end{aligned}$$

Furthermore, we already have $s = \bot^p$.

Case (2): We are given

$$\Sigma \triangleright V\langle c_1\rangle \cdots \langle c_k\rangle \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p, \quad s = \bot^p \quad (\exists p) \ .$$

Therefore, by (R_Ctx_C) and (R_Blame_C),

$$\begin{aligned}\Sigma \triangleright V\langle c_1\rangle \cdots \langle c_k\rangle\langle c_{k+1}\rangle &\longrightarrow_C^* \Sigma \triangleright (\mathsf{blame}\, p)\langle c_{k+1}\rangle \\ &\longrightarrow_C \Sigma \triangleright \mathsf{blame}\, p \ .\end{aligned}$$

Furthermore, we already have $s = \bot^p$.

□

Lemma I.53 (Coercion Reduction Preserves Bisimilarity (for Function Coercions))**.** Let $n > 0$. If V_1 is not a coercion application, and $\Sigma \mid \emptyset \vdash V_1\langle c_1 \to d_1\rangle \cdots \langle c_n \to d_n\rangle \approx U'_1\langle s\rangle : A' \to B'$ and $\Sigma \mid \emptyset \vdash V_2 \approx U'_2\langle s'\rangle : A'$, then either of the following holds:

(1) there exists some M_3, A_1, and A_2 such that $\Sigma \triangleright (V_1\langle c_1 \to d_1\rangle \cdots \langle c_n \to d_n\rangle)\, V_2 \longrightarrow_C^* \Sigma \triangleright (V_1\, M_3)\langle d_1\rangle \cdots \langle d_n\rangle$ and $\Sigma \mid \emptyset \vdash M_3 \approx U'_2\langle s' \mathbin{⨟} |c_n|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |c_1|_\emptyset\rangle : A_1$ and $\Sigma \mid \emptyset \vdash_C c_1 : A_2 \rightsquigarrow A_1$; or

(2) there exists some p such that $\Sigma \triangleright (V_1\langle c_1 \to d_1\rangle \cdots \langle c_n \to d_n\rangle)\, V_2 \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p$ and $s' \mathbin{⨟} |c_n|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |c_1|_\emptyset = \bot^p$.

Proof. By induction on n.

Case $n = 1$: By (R_Wrap_C), $\Sigma \triangleright (V_1\langle c_1 \to d_1\rangle)\, V_2 \longrightarrow_C \Sigma \triangleright (V_1\,(V_2\langle c_1\rangle))\langle d_1\rangle$. Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash V_2\langle c_1\rangle \approx U'_2\langle s' \mathbin{⨟} |c_1|_\emptyset\rangle : A$. We perform case analysis on the last rule to derive $\Sigma \mid \emptyset \vdash V_1\langle c_1 \to d_1\rangle \approx U'_1\langle s\rangle : A' \to B'$.

Case (Bs_Crc): We are given

$$s = |c_1 \to d_1|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_1 \approx U'_1 : C' \quad \Sigma \mid \emptyset \vdash_C c_1 \to d_1 : C' \rightsquigarrow (A' \to B') \quad (\exists C') \ .$$

Because $\Sigma \mid \emptyset \vdash_C c_1 \to d_1 : C' \rightsquigarrow (A' \to B')$ is derived by (Ct_Arrow_C), we have

$$C' = A \to B, \quad \Sigma \mid \emptyset \vdash_C c_1 : A' \rightsquigarrow A, \quad \Sigma \mid \emptyset \vdash_C d_1 : B \rightsquigarrow B' \quad (\exists A, B) \ .$$

Therefore, by (Bs_CrcMore), $\Sigma \mid \emptyset \vdash V_2\langle c_1 \rangle \approx U'_2\langle s' \mathbin{;} |c_1|_\emptyset \rangle : A$.

Case (Bs_CrcMore): We are given

$$s = s_0 \mathbin{;} |c_1 \to d_1|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_1 \approx U'_1\langle s_0 \rangle : C' \quad \Sigma \mid \emptyset \vdash_C c_1 \to d_1 : C' \rightsquigarrow (A' \to B') \quad (\exists C', s_0) \ .$$

Because $\Sigma \mid \emptyset \vdash_C c_1 \to d_1 : C' \rightsquigarrow (A' \to B')$ is derived by (Ct_Arrow_C), we have

$$C' = A \to B, \quad \Sigma \mid \emptyset \vdash_C c_1 : A' \rightsquigarrow A, \quad \Sigma \mid \emptyset \vdash_C d_1 : B \rightsquigarrow B' \quad (\exists A, B) \ .$$

Therefore, by (Bs_CrcMore), $\Sigma \mid \emptyset \vdash V_2\langle c_1 \rangle \approx U'_2\langle s' \mathbin{;} |c_1|_\emptyset \rangle : A$.

Case (Bs_CrcId): We are given

$$s = |\mathsf{id}_{A' \to B'}|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_1\langle c_1 \to d_1 \rangle \approx U'_1 : A' \to B',$$
$$\Sigma \mid \emptyset \vdash_C \mathsf{id}_{A' \to B'} : (A' \to B') \rightsquigarrow (A' \to B') \ .$$

Because $\Sigma \mid \emptyset \vdash V_1\langle c_1 \to d_1 \rangle \approx U'_1 : A' \to B'$ is derived by (Bs_CrcIdL), we have

$$c_1 = c_1^I, \quad , d_1 = d_1^I, \quad \Sigma \mid \emptyset \vdash V_1 \approx U'_1 : C', \quad \Sigma \mid \emptyset \vdash_C c_1^I \to d_1^I : C' \rightsquigarrow (A' \to B') \quad (\exists C', c_1^I, d_1^I) \ .$$

Because $\Sigma \mid \emptyset \vdash_C c_1^I \to d_1^I : C' \rightsquigarrow (A' \to B')$ is derived by (Ct_Arrow_C), we have

$$C' = A \to B, \quad \Sigma \mid \emptyset \vdash_C c_1^I : A' \rightsquigarrow A, \quad \Sigma \mid \emptyset \vdash_C d_1^I : B \rightsquigarrow B' \quad (\exists A, B) \ .$$

Therefore, by (Bs_CrcMore), $\Sigma \mid \emptyset \vdash V_2\langle c_1^I \rangle \approx U'_2\langle s' \mathbin{;} |c_1^I|_\emptyset \rangle : A$.

Case (Bs_CrcIdL): Similarly to the case (Bs_CrcId).

Case $n = k + 1 (k \geq 1)$: We perform case analysis on the last rule to derive $\Sigma \mid \emptyset \vdash V_1\langle c_1 \to d_1 \rangle \cdots \langle c_k \to d_k \rangle \langle c_{k+1} \to d_{k+1} \rangle \approx U'_1\langle s \rangle : A' \to B'$.

Case (Bs_Crc): We are given

$$s = |c_{k+1} \to d_{k+1}|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_1\langle c_1 \to d_1 \rangle \cdots \langle c_k \to d_k \rangle \approx U'_1 : C'$$
$$\Sigma \mid \emptyset \vdash_C c_{k+1} \to d_{k+1} : C' \rightsquigarrow (A' \to B') \quad (\exists C') \ .$$

Because $\Sigma \mid \emptyset \vdash_C c_{k+1} \to d_{k+1} : C' \rightsquigarrow (A' \to B')$ is derived by (Ct_Arrow_C), we have

$$C' = A'' \to B'', \quad \Sigma \mid \emptyset \vdash_C c_{k+1} : A' \rightsquigarrow A'', \quad \Sigma \mid \emptyset \vdash_C d_{k+1} : B'' \rightsquigarrow B' \quad (\exists A'', B'') \ .$$

Therefore, by (Bs_CrcMore), $\Sigma \mid \emptyset \vdash V_2\langle c_{k+1} \rangle \approx U'_2\langle s' \mathbin{;} |c_{k+1}|_\emptyset \rangle : A''$. Hence, by Lemma I.51, we consider the following two cases.

Case (1) in Lemma I.51: We are given

$$\Sigma \triangleright V_2\langle c_{k+1} \rangle \longrightarrow_C^* \Sigma \triangleright V_3, \quad \Sigma \mid \emptyset \vdash V_3 \approx U'_2\langle s' \mathbin{;} |c_{k+1}|_\emptyset \rangle : A'' \quad (\exists V_3) \ .$$

By the IH, we consider the following two cases.

Case (1): We are given

$$\Sigma \triangleright (V_1\langle c_1 \to d_1 \rangle \cdots \langle c_k \to d_k \rangle)\ V_3 \longrightarrow_C^* \Sigma \triangleright (V_1\ M_4)\langle d_1 \rangle \cdots \langle d_k \rangle,$$
$$\Sigma \mid \emptyset \vdash M_4 \approx U'_2\langle (s' \mathbin{;} |c_{k+1}|_\emptyset) \mathbin{;} |c_k|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_1|_\emptyset \rangle : A \quad (\exists M_4) \ .$$

By (R_WRAP_C) and (R_CTX_C),

$$\begin{aligned}\Sigma \triangleright (V_1\langle c_1 \to d_1\rangle \cdots \langle c_k \to d_k\rangle\langle c_{k+1} \to d_{k+1}\rangle)\ V_2 &\longrightarrow_C \Sigma \triangleright ((V_1\langle c_1 \to d_1\rangle \cdots \langle c_k \to d_k\rangle)\,(V_2\langle c_{k+1}\rangle))\langle d_{k+1}\rangle \\ &\longrightarrow_C^* \Sigma \triangleright ((V_1\langle c_1 \to d_1\rangle \cdots \langle c_k \to d_k\rangle)\ V_3)\langle d_{k+1}\rangle \\ &\longrightarrow_C^* \Sigma \triangleright (V_1\ M_4)\langle d_1\rangle \cdots \langle d_k\rangle\langle d_{k+1}\rangle\ .\end{aligned}$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash M_4 \approx U_2'\langle s' \,;\, |c_{k+1}|_\emptyset \,;\, |c_k|_\emptyset \,;\, \cdots \,;\, |c_1|_\emptyset\rangle : A$, which holds already.

Case (2): We are given

$$\Sigma \triangleright (V_1\langle c_1 \to d_1\rangle \cdots \langle c_k \to d_k\rangle)\ V_3 \ \longrightarrow_C^*\ \Sigma \triangleright \mathsf{blame}\, p, \quad (s' \,;\, |c_{k+1}|_\emptyset) \,;\, |c_k|_\emptyset \,;\, \cdots \,;\, |c_1|_\emptyset = \bot^p \quad (\exists p)\ .$$

By (R_WRAP_C), (R_CTX_C), and (R_BLAME_C),

$$\begin{aligned}\Sigma \triangleright (V_1\langle c_1 \to d_1\rangle \cdots \langle c_k \to d_k\rangle\langle c_{k+1} \to d_{k+1}\rangle)\ V_2 &\longrightarrow_C \Sigma \triangleright ((V_1\langle c_1 \to d_1\rangle \cdots \langle c_k \to d_k\rangle)\,(V_2\langle c_{k+1}\rangle))\langle d_{k+1}\rangle \\ &\longrightarrow_C^* \Sigma \triangleright ((V_1\langle c_1 \to d_1\rangle \cdots \langle c_k \to d_k\rangle)\ V_3)\langle d_{k+1}\rangle \\ &\longrightarrow_C^* \Sigma \triangleright (\mathsf{blame}\, p)\langle d_{k+1}\rangle \\ &\longrightarrow_C \Sigma \triangleright \mathsf{blame}\, p\ .\end{aligned}$$

Therefore, it suffices to show that $s' \,;\, |c_{k+1}|_\emptyset \,;\, |c_k|_\emptyset \,;\, \cdots \,;\, |c_1|_\emptyset = \bot^p$, which holds already.

Case (2) in Lemma I.51: We are given

$$\Sigma \triangleright V_2\langle c_{k+1}\rangle \ \longrightarrow_C^*\ \Sigma \triangleright \mathsf{blame}\, p, \quad s' \,;\, |c_{k+1}|_\emptyset = \bot^p \quad (\exists p)\ .$$

By (R_WRAP_C), (R_CTX_C), and (R_BLAME_C),

$$\begin{aligned}\Sigma \triangleright (V_1\langle c_1 \to d_1\rangle \cdots \langle c_k \to d_k\rangle\langle c_{k+1} \to d_{k+1}\rangle)\ V_2 &\longrightarrow_C \Sigma \triangleright ((V_1\langle c_1 \to d_1\rangle \cdots \langle c_k \to d_k\rangle)\,(V_2\langle c_{k+1}\rangle))\langle d_{k+1}\rangle \\ &\longrightarrow_C^* \Sigma \triangleright ((V_1\langle c_1 \to d_1\rangle \cdots \langle c_k \to d_k\rangle)\,(\mathsf{blame}\, p))\langle d_{k+1}\rangle \\ &\longrightarrow_C^* \Sigma \triangleright (\mathsf{blame}\, p)\langle d_{k+1}\rangle \\ &\longrightarrow_C \Sigma \triangleright \mathsf{blame}\, p\ .\end{aligned}$$

Furthermore, since

$$\begin{aligned}s' \,;\, |c_{k+1}|_\emptyset \,;\, |c_k|_\emptyset \,;\, \cdots \,;\, |c_1|_\emptyset &= \bot^p \,;\, |c_k|_\emptyset \,;\, \cdots \,;\, |c_1|_\emptyset \\ &= \bot^p\ ,\end{aligned}$$

we finish the case.

Case (BS_CRCMORE): We are given

$$\begin{gathered}s = s_0 \,;\, |c_{k+1} \to d_{k+1}|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_1\langle c_1 \to d_1\rangle \cdots \langle c_k \to d_k\rangle \approx U_1'\langle s_0\rangle : C' \\ \Sigma \mid \emptyset \vdash_C\ c_{k+1} \to d_{k+1} : C' \rightsquigarrow (A' \to B') \quad (\exists C', s_0)\ .\end{gathered}$$

Because $\Sigma \mid \emptyset \vdash_C\ c_{k+1} \to d_{k+1} : C' \rightsquigarrow (A' \to B')$ is derived by (CT_ARROW_C), we have

$$C' = A'' \to B'', \quad \Sigma \mid \emptyset \vdash_C\ c_{k+1} : A' \rightsquigarrow A'', \quad \Sigma \mid \emptyset \vdash_C\ d_{k+1} : B'' \rightsquigarrow B' \quad (\exists A'', B'')\ .$$

Therefore, by (BS_CRCMORE), $\Sigma \mid \emptyset \vdash V_2\langle c_{k+1}\rangle \approx U_2'\langle s' \,;\, |c_{k+1}|_\emptyset\rangle : A''$. Hence, by Lemma I.51, we consider the following two cases.

Case (1) in Lemma I.51: We are given

$$\Sigma \triangleright V_2\langle c_{k+1}\rangle \ \longrightarrow_C^*\ \Sigma \triangleright V_3, \quad \Sigma \mid \emptyset \vdash V_3 \approx U_2'\langle s' \,;\, |c_{k+1}|_\emptyset\rangle : A'' \quad (\exists V_3)\ .$$

By the IH, we consider the following two cases.

Case (1): We are given

$$\Sigma \triangleright (V_1\langle c_1 \rightarrow d_1\rangle \cdots \langle c_k \rightarrow d_k\rangle)\ V_3 \longrightarrow_C^* \Sigma \triangleright (V_1\ M_4)\langle d_1\rangle \cdots \langle d_k\rangle,$$
$$\Sigma \mid \emptyset \vdash M_4 \approx U_2'\langle (s' \mathbin{⨾} |c_{k+1}|_\emptyset) \mathbin{⨾} |c_k|_\emptyset \mathbin{⨾} \cdots \mathbin{⨾} |c_1|_\emptyset\rangle : A \quad (\exists M_4)\ .$$

By (R_Wrap_C) and (R_Ctx_C),

$$\begin{aligned}\Sigma \triangleright (V_1\langle c_1 \rightarrow d_1\rangle \cdots \langle c_k \rightarrow d_k\rangle\langle c_{k+1} \rightarrow d_{k+1}\rangle)\ V_2 &\longrightarrow_C \Sigma \triangleright ((V_1\langle c_1 \rightarrow d_1\rangle \cdots \langle c_k \rightarrow d_k\rangle)\ (V_2\langle c_{k+1}\rangle))\langle d_{k+1}\rangle \\ &\longrightarrow_C^* \Sigma \triangleright ((V_1\langle c_1 \rightarrow d_1\rangle \cdots \langle c_k \rightarrow d_k\rangle)\ V_3)\langle d_{k+1}\rangle \\ &\longrightarrow_C^* \Sigma \triangleright (V_1\ M_4)\langle d_1\rangle \cdots \langle d_k\rangle\langle d_{k+1}\rangle\ .\end{aligned}$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash M_4 \approx U_2'\langle s' \mathbin{⨾} |c_{k+1}|_\emptyset \mathbin{⨾} |c_k|_\emptyset \mathbin{⨾} \cdots \mathbin{⨾} |c_1|_\emptyset\rangle : A$, which holds already.

Case (2): We are given

$$\Sigma \triangleright (V_1\langle c_1 \rightarrow d_1\rangle \cdots \langle c_k \rightarrow d_k\rangle)\ V_3 \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\,p, \quad (s' \mathbin{⨾} |c_{k+1}|_\emptyset) \mathbin{⨾} |c_k|_\emptyset \mathbin{⨾} \cdots \mathbin{⨾} |c_1|_\emptyset = \bot^p \quad (\exists p)\ .$$

By (R_Wrap_C), (R_Ctx_C), and (R_Blame_C),

$$\begin{aligned}\Sigma \triangleright (V_1\langle c_1 \rightarrow d_1\rangle \cdots \langle c_k \rightarrow d_k\rangle\langle c_{k+1} \rightarrow d_{k+1}\rangle)\ V_2 &\longrightarrow_C \Sigma \triangleright ((V_1\langle c_1 \rightarrow d_1\rangle \cdots \langle c_k \rightarrow d_k\rangle)\ (V_2\langle c_{k+1}\rangle))\langle d_{k+1}\rangle \\ &\longrightarrow_C^* \Sigma \triangleright ((V_1\langle c_1 \rightarrow d_1\rangle \cdots \langle c_k \rightarrow d_k\rangle)\ V_3)\langle d_{k+1}\rangle \\ &\longrightarrow_C^* \Sigma \triangleright (\mathsf{blame}\,p)\langle d_{k+1}\rangle \\ &\longrightarrow_C \Sigma \triangleright \mathsf{blame}\,p\ .\end{aligned}$$

Therefore, it suffices to show that $s' \mathbin{⨾} |c_{k+1}|_\emptyset \mathbin{⨾} |c_k|_\emptyset \mathbin{⨾} \cdots \mathbin{⨾} |c_1|_\emptyset = \bot^p$, which holds already.

Case (2) in Lemma I.51: We are given

$$\Sigma \triangleright V_2\langle c_{k+1}\rangle \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\,p, \quad s' \mathbin{⨾} |c_{k+1}|_\emptyset = \bot^p \quad (\exists p)\ .$$

By (R_Wrap_C), (R_Ctx_C), and (R_Blame_C),

$$\begin{aligned}\Sigma \triangleright (V_1\langle c_1 \rightarrow d_1\rangle \cdots \langle c_k \rightarrow d_k\rangle\langle c_{k+1} \rightarrow d_{k+1}\rangle)\ V_2 &\longrightarrow_C \Sigma \triangleright ((V_1\langle c_1 \rightarrow d_1\rangle \cdots \langle c_k \rightarrow d_k\rangle)\ (V_2\langle c_{k+1}\rangle))\langle d_{k+1}\rangle \\ &\longrightarrow_C^* \Sigma \triangleright ((V_1\langle c_1 \rightarrow d_1\rangle \cdots \langle c_k \rightarrow d_k\rangle)\ (\mathsf{blame}\,p))\langle d_{k+1}\rangle \\ &\longrightarrow_C^* \Sigma \triangleright (\mathsf{blame}\,p)\langle d_{k+1}\rangle \\ &\longrightarrow_C \Sigma \triangleright \mathsf{blame}\,p\ .\end{aligned}$$

Furthermore, since

$$\begin{aligned}s' \mathbin{⨾} |c_{k+1}|_\emptyset \mathbin{⨾} |c_k|_\emptyset \mathbin{⨾} \cdots \mathbin{⨾} |c_1|_\emptyset &= \bot^p \mathbin{⨾} |c_k|_\emptyset \mathbin{⨾} \cdots \mathbin{⨾} |c_1|_\emptyset \\ &= \bot^p\ ,\end{aligned}$$

we finish the case.

Case (Bs_CrcId): We are given

$$s = |\mathsf{id}_{A' \rightarrow B'}|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_1\langle c_1 \rightarrow d_1\rangle \cdots \langle c_{k+1} \rightarrow d_{k+1}\rangle \approx U_1' : A' \rightarrow B'\ .$$

Because $\Sigma \mid \emptyset \vdash V_1\langle c_1 \rightarrow d_1\rangle \cdots \langle c_{k+1} \rightarrow d_{k+1}\rangle \approx U_1' : A' \rightarrow B'$ is derived by (Bs_CrcIdL), we have

$$c_i = c_i^I, \quad d_i = d_i^I, \quad \Sigma \mid \emptyset \vdash_C c_i : A_{i+1}' \rightsquigarrow A_i', \quad \Sigma \mid \emptyset \vdash_C d_i : B_i' \rightsquigarrow B_{i+1}'$$
$$A_{k+2}' = A', \quad B_{k+2}' = B' \quad (\exists c_i^I, d_i^I, A_i'', B_i'')\ (1 \leq \forall i \leq k{+}1)\ .$$

Therefore, by (Bs_CrcMore), $\Sigma \mid \emptyset \vdash V_2\langle c_{k+1}^I\rangle \approx U_2'\langle s' \mathbin{⨾} |c_{k+1}^I|_\emptyset\rangle : A_{k+1}'$. Hence, similarly to the case (Bs_CrcMore).

Case (Bs_CrcIdL): We are given

$$c_{k+1} = c^I_{k+1}, \quad d_{k+1} = d^I_{k+1}, \quad \Sigma \mid \emptyset \vdash V_1 \langle c_1 \to d_1 \rangle \cdots \langle c_k \to d_k \rangle \approx U'_1 \langle s \rangle : C'$$
$$\Sigma \mid \emptyset \vdash_C c^I_{k+1} \to d^I_{k+1} : C' \rightsquigarrow (A' \to B') \quad (\exists c^I_{k+1}, d^I_{k+1}, C') \ .$$

Because $\Sigma \mid \emptyset \vdash_C c^I_{k+1} \to d^I_{k+1} : C' \rightsquigarrow (A' \to B')$ is derived by (Ct_Arrow_C), we have

$$C' = A \to B, \quad \Sigma \mid \emptyset \vdash_C c^I_{k+1} : A' \rightsquigarrow A, \quad \Sigma \mid \emptyset \vdash_C d^I_{k+1} : B \rightsquigarrow B' \quad (\exists A, B) \ .$$

Hence, by the IH, we consider the cases similarly to the case (Bs_CrcMore).

□

Lemma I.54 (Inversion of Bisimilar Identity Coercion Applications)**.** If $\Sigma \mid \emptyset \vdash V \langle vc_1 \rangle \cdots \langle vc_n \rangle \approx U \langle \mathsf{id} \rangle : A$ and V is not a coercion application, then $\Sigma \mid \emptyset \vdash V \langle vc_1 \rangle \cdots \langle vc_n \rangle \approx U : A$.

Proof. By induction on the derivation of $\Sigma \mid \emptyset \vdash V \langle vc_1 \rangle \cdots \langle vc_n \rangle \approx U \langle \mathsf{id} \rangle : A$.

Case $n = 0$: Since V and U are not coercion aplications, $\Sigma \mid \emptyset \vdash V \approx U \langle \mathsf{id} \rangle : A$ is derived by (Bs_CrcId). Therefore, we have

$$|\mathsf{id}_A|_\emptyset = \mathsf{id}, \quad \Sigma \mid \emptyset \vdash V \approx U : B, \quad \Sigma \mid \emptyset \vdash_C \mathsf{id}_A : B \rightsquigarrow A \quad (\exists B) \ .$$

Hence, it suffices to show that $B = A$. Since $\Sigma \mid \emptyset \vdash_C \mathsf{id}_A : B \rightsquigarrow A$ is derived by (Ct_Id_C), we have $B = A$.

Case $n > 0$: We perform case analysis on the last rule to derive $\Sigma \mid \emptyset \vdash V \langle vc_1 \rangle \cdots \langle vc_n \rangle \approx U \langle \mathsf{id} \rangle : A$, which is either (Bs_CrcId), (Bs_CrcIdL), (Bs_Crc), or (Bs_CrcMore).

Case (Bs_CrcId): We are given

$$\Sigma \mid \emptyset \vdash V \langle vc_1 \rangle \cdots \langle vc_n \rangle \approx U : B, \quad \Sigma \mid \emptyset \vdash_C \mathsf{id}_A : B \rightsquigarrow A \ .$$

It suffices to show that $B = A$. Since $\Sigma \mid \emptyset \vdash_C \mathsf{id}_A : B \rightsquigarrow A$ is derived by (Ct_Id_C), we have $B = A$.

Case (Bs_CrcIdL): We are given vc_n ia a no-op coercion and

$$vc_n = vc^I_n, \quad \Sigma \mid \emptyset \vdash V \langle vc_1 \rangle \cdots \langle vc_{n-1} \rangle \approx U \langle \mathsf{id} \rangle : B, \quad \Sigma \mid \emptyset \vdash_C vc^I_n : B \rightsquigarrow A \quad (\exists vc^I_n, B) \ .$$

Therefore, by the IH, we have $\Sigma \mid \emptyset \vdash V \langle vc_1 \rangle \cdots \langle vc_{n-1} \rangle \approx U : B$. Hence, by (Bs_CrcIdL), we have $\Sigma \mid \emptyset \vdash V \langle vc_1 \rangle \cdots \langle vc_{n-1} \rangle \langle vc^I_n \rangle \approx U : A$.

Case (Bs_Crc): We are given

$$|vc_n|_\emptyset = \mathsf{id}, \quad \Sigma \mid \emptyset \vdash V \langle vc_1 \rangle \cdots \langle vc_{n-1} \rangle \approx U : B, \quad \Sigma \mid \emptyset \vdash_C vc_n : B \rightsquigarrow A \quad (\exists B) \ .$$

We perform case analysis on vc_n.

Case $vc_n = G!$ $(\exists G)$: Since $\Sigma \mid \emptyset \vdash_C G! : B \rightsquigarrow A$, $G!$ does not contain any free type variables. Therefore, there is contradiction because $|vc_n|_\emptyset = |G!|_\emptyset = |\mathsf{id}_G|_\emptyset \, ; G! \neq \mathsf{id}$.

Case $vc_n = c \to d$ $(\exists c, d)$: There is a contradiction because $|vc_n|_\emptyset = |c \to d|_\emptyset = |c|_\emptyset \to |d|_\emptyset \neq \mathsf{id}$.

Case $vc_n = \forall X.c$ $(\exists X, c)$: There is a contradiction because $|vc_n|_\emptyset = |\forall X.c|_\emptyset = \forall X.|c|_\emptyset \neq \mathsf{id}$.

Case $vc_n = \beta^-$ $(\exists \beta)$: By (Bs_CrcIdL), we have $\Sigma \mid \emptyset \vdash V \langle vc_1 \rangle \cdots \langle vc_{n-1} \rangle \langle \beta^- \rangle \approx U : \beta$.

Case (Bs_CrcMore): We are given

$$s \,\mathring{;}\, |vc_n|_\emptyset = \mathsf{id}, \quad \Sigma \mid \emptyset \vdash V \langle vc_1 \rangle \cdots \langle vc_{n-1} \rangle \approx U \langle s \rangle : B,$$
$$\Sigma \mid \emptyset \vdash_C vc_n : B \rightsquigarrow A \quad (\exists B, s) \ .$$

By Lemma I.50, there exists some intermediate coercion i such that $s = i$. By Lemma I.22, $\Sigma \mid \emptyset \vdash_S U \langle i \rangle : \Sigma(B)$. Since $\Sigma \mid \emptyset \vdash_S U \langle i \rangle : \Sigma(B)$ is derived by (T_Crc_S), we have $\Sigma \mid \emptyset \vdash_S i : C \rightsquigarrow \Sigma(B)$ $(\exists C)$. We perform case analysis on vc_n.

Case $vc_n = G!$ ($\exists G$): Since $\Sigma \mid \emptyset \vdash_C G! : B \rightsquigarrow A$, $G!$ does not contain any free type variables. By Lemma I.11, $i \mathbin{⨾} |\mathsf{id}_G|_\emptyset = i$. Therefore,

$$\begin{aligned} i \mathbin{⨾} |vc_n|_\emptyset &= i \mathbin{⨾} |G!|_\emptyset \\ &= i \mathbin{⨾} (|\mathsf{id}_G|_\emptyset \,;\, G!) \\ &= (i \mathbin{⨾} |\mathsf{id}_G|_\emptyset) \,;\, G! \\ &= i \,;\, G! \end{aligned}$$

There is contradiction because $i \,;\, G! \neq \mathsf{id}$.

Case $vc_n = c \to d$ ($\exists c, d$): Since $\Sigma \mid \emptyset \vdash_C c \to d : B \rightsquigarrow A$ is derived by (Ct_Arrow_C), we have

$$B = B_1 \to B_2, \quad A = A_1 \to A_2, \quad \Sigma \vdash \emptyset, \quad \Sigma \mid \emptyset \vdash G \,.$$

By the definition, $\Sigma(B_1 \to B_2) = \Sigma(B_1) \to \Sigma(B_2)$. We perform case analysis on i.

Case $i = g_2 \,;\, G_2!$ ($\exists g_2, G_2$), $i = \forall X. s_2$ ($\exists s_2$): Contradictory because the return type of i cannot be a function type.

Case $i = s_1 \to s_2$ ($\exists s_1, s_2$):

$$\begin{aligned} s \mathbin{⨾} |c \to d|_\emptyset &= (s_1 \to s_2) \mathbin{⨾} |c \to d|_\emptyset \\ &= (s_1 \to s_2) \mathbin{⨾} (|c|_\emptyset \to |d|_\emptyset) \\ &= (|c|_\emptyset \mathbin{⨾} s_1) \to (s_2 \mathbin{⨾} |d|_\emptyset) \,. \end{aligned}$$

However, there is a contradiction because $(|c|_\emptyset \mathbin{⨾} s_1) \to (s_2 \mathbin{⨾} |d|_\emptyset) \neq \mathsf{id}$.

Case $i = \mathsf{id}$:

$$\begin{aligned} s \mathbin{⨾} |c \to d|_\emptyset &= \mathsf{id} \mathbin{⨾} |c \to d|_\emptyset \\ &= |c \to d|_\emptyset \\ &= |c|_\emptyset \to |d|_\emptyset \,. \end{aligned}$$

However, there is a contradiction because $|c|_\emptyset \to |d|_\emptyset \neq \mathsf{id}$.

Case $vc_n = \forall X. c$ ($\exists X, c$): Similary to the case $vc_n = c \to d$.

Case $vc_n = \beta^-$ ($\exists \beta$): We are given $\mathsf{id} = s \mathbin{⨾} |\beta^-|_\emptyset = s \mathbin{⨾} \mathsf{id}$ and $s \mathbin{⨾} \mathsf{id} = s$ by Lemma I.6. By the IH, we have $\Sigma \mid \emptyset \vdash V\langle vc_1 \rangle \cdots \langle vc_{n-1} \rangle \approx U : B$. Hence, by (Bs_CrcIdL), we have $\Sigma \mid \emptyset \vdash V\langle vc_1 \rangle \cdots \langle vc_{n-1} \rangle \langle \beta^- \rangle \approx U : A$.

□

Lemma I.55 (Inversion of Bisimilar Coercion Applications)**.** If $\Sigma \mid \emptyset \vdash V\langle G! \rangle \approx U\langle \mathsf{id} \,;\, G! \rangle : \star$ and $\Sigma(A') = \Sigma(G)$, then $\Sigma \mid \emptyset \vdash V \approx U : G$.

Proof. By Lemma I.36, there exist some V_1 that is not a coercion application, $n > 0$, $c_1, \dots, c_n$ such that $V\langle G! \rangle = V_1\langle c_1 \rangle \cdots \langle c_n \rangle$ and $c_n = G!$. Furthermore, because $V_1\langle c_1 \rangle \cdots \langle c_n \rangle$ is a value, there exist some $vc_1, \dots, vc_n$ such that, for any i such that $n \geq i > 0$, $c_i = vc_i$. Therefore,

$$\Sigma \mid \emptyset \vdash V_1\langle vc_1 \rangle \cdots \langle vc_{n-1} \rangle \langle G! \rangle \approx U\langle \mathsf{id} \,;\, G! \rangle : \star \,.$$

By case analysis on n.

Case $n = 1$: We have $V = V_1$. By case analysis on the last rule to derive $\Sigma \mid \emptyset \vdash V\langle G! \rangle \approx U\langle \mathsf{id} \,;\, G! \rangle : \star$.

Case (Bs_CrcId): Contradictory because there is no identity corcion id_A such that $|\mathsf{id}_A|_\emptyset = \mathsf{id} \,;\, G!$.

Case (Bs_CrcIdL): Contradictory because $G!$ is not a no-op coercion.

Case (Bs_Crc): We are given

$$|G!|_\emptyset = \mathsf{id} \,;\, G!, \quad \Sigma \mid \emptyset \vdash V \approx U : B, \quad \Sigma \mid \emptyset \vdash_C G! : B \rightsquigarrow \star \quad (\exists B) \,.$$

Since $\Sigma \mid \emptyset \vdash_C G! : B \rightsquigarrow \star$ is derived by (Ct_Inj_C), $B = G$.

Case (Bs_CrcMore): There exist some A, s such that $\Sigma \mid \emptyset \vdash V \approx U\langle s\rangle : A$. Because V is not a coercion application, $\Sigma \mid \emptyset \vdash V \approx U\langle s\rangle : A$ is derived by (Bs_CrcId). Hence, $\Sigma \mid \emptyset \vdash V \approx U : G$.

Case $n > 1$: By case analysis on the last rule to derive $\Sigma \mid \emptyset \vdash V_1\langle vc_1\rangle \cdots \langle vc_{n-1}\rangle \langle G!\rangle \approx U\langle \mathsf{id} \mathbin{;} G!\rangle : \star$.

Case (Bs_CrcId): Contradictory because there is no identity corcion id_A such that $|\mathsf{id}_A|_\emptyset = \mathsf{id} \mathbin{;} G!$.

Case (Bs_CrcIdL): Contradictory because $G!$ is not a no-op coercion.

Case (Bs_Crc): We are given

$$\mathsf{id} \mathbin{;} G! = |G!|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_1\langle vc_1\rangle \cdots \langle vc_{n-1}\rangle \approx U : A, \quad \Sigma \mid \emptyset \vdash_C G! : A \rightsquigarrow \star \quad (\exists A) .$$

Because $\Sigma \mid \emptyset \vdash_C G! : A \rightsquigarrow \star$ is derived by (Ct_Inj_C), we have $A = G$.

Case (Bs_CrcMore): We are given

$$\mathsf{id} \mathbin{;} G! = s \mathbin{⨟} |G!|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_1\langle vc_1\rangle \cdots \langle vc_{n-1}\rangle \approx U\langle s\rangle : A, \quad \Sigma \mid \emptyset \vdash_C G! : A \rightsquigarrow \star \quad (\exists A, s) .$$

By Lemma I.46, there exists some i such that $s = i$. Because $\Sigma \mid \emptyset \vdash_C G! : A \rightsquigarrow \star$ is derived by (Ct_Inj_C), we have $A = G$. Therefore, by Lemma I.22, we have $\Sigma \mid \emptyset \vdash_S U\langle i\rangle : \Sigma(G)$. Because this judgment is derived by (T_Crc_S), we have $\Sigma \mid \emptyset \vdash_S i : B \rightsquigarrow \Sigma(G)$ $(\exists B)$. Hence, by Lemma I.11, we have $i \mathbin{⨟} |\mathsf{id}_{A'}|_\emptyset = i$. Therefore, we have

$$\begin{aligned} \mathsf{id} \mathbin{;} G! &= s \mathbin{⨟} |G!|_\emptyset \\ &= i \mathbin{⨟} |G!|_\emptyset \\ &= i \mathbin{⨟} (|\mathsf{id}_G|_\emptyset \mathbin{;} G!) \\ &= (i \mathbin{⨟} |\mathsf{id}_G|_\emptyset) \mathbin{;} G! \\ &= i \mathbin{;} G! \ . \end{aligned}$$

Therefore, $s = i = \mathsf{id}$. Hence, by $\Sigma \mid \emptyset \vdash V_1\langle vc_1\rangle \cdots \langle vc_{n-1}\rangle \approx U\langle \mathsf{id}\rangle : G$ and Lemma I.39, for all $1 \leq i \leq n-1$ there exists some A_i such that

$$\Sigma \mid \emptyset \vdash V_1 \approx U : A_1, \quad \Sigma \mid \emptyset \vdash_C vc_i : A_i \rightsquigarrow A_{i+1}, \quad A_n = G \ .$$

Furthermore, there exists some j such that $0 \leq j \leq n-1$ and

$$vc_i = c^I_i \ (1 \leq \forall i \leq j), \quad \mathsf{id} = |\mathsf{id}_{A_{j+1}}|_\emptyset \mathbin{⨟} |vc_{j+1}|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |vc_n|_\emptyset \quad (\exists c^I_1, \ldots, c^I_j) \ .$$

Therefore, by Lemma I.43, for all $1 \leq i \leq n-1$

$$vs_i = |vc_i|_\emptyset, \quad \Sigma \mid \emptyset \vdash_S vs_i : \Sigma(A_i) \rightsquigarrow \Sigma(A_{i+1}) \quad (\exists vs_1, \ldots, vs_{n-1}) \ .$$

Hence, by Lemma I.44, there exists some vs such that $vs = vs_{j+1} \mathbin{⨟} \cdots \mathbin{⨟} vs_n$. Therefore,

$$\begin{aligned} \mathsf{id} &= |\mathsf{id}_{A_{i+1}}|_\emptyset \mathbin{⨟} |vc_{j+1}|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |vc_n|_\emptyset \\ &= |\mathsf{id}_{A_{i+1}}|_\emptyset \mathbin{⨟} (vs_{j+1} \mathbin{⨟} \cdots \mathbin{⨟} vs_n) \\ &= |\mathsf{id}_{A_{i+1}}|_\emptyset \mathbin{⨟} vs \\ &= vs \ . \end{aligned}$$

Furthermore, V_1 is not a cercion aplication, by Lemma I.54, we have $\Sigma \mid \emptyset \vdash V_1\langle vc_1\rangle \cdots \langle vc_{n-1}\rangle \approx U : G$.

□

Lemma I.56 (No-op coercion Preserves Bisimularity for Values)**.** If $\Sigma \mid \emptyset \vdash V \approx V' : A$ and $\Sigma \mid \emptyset \vdash_C c^I : A \rightsquigarrow B$, then there exists a value V_1 such that $\Sigma \triangleright V\langle c^I\rangle \longrightarrow^*_C \Sigma \triangleright V_1$ and $\Sigma \mid \emptyset \vdash V_1 \approx V' : B$.

Proof. By induction on c^I. By case analysis on c^I.

Case $c^I = \mathsf{id}_A$: We are given $B = A$. By $\Sigma \triangleright V\langle \mathsf{id}_A\rangle \longrightarrow_C \Sigma \triangleright V$, it suffices to show that $\Sigma \mid \emptyset \vdash V \approx V' : A$, which has been assumed.

Case $c^I = \alpha^+$ ($\exists \alpha$): We are given

$$A = \alpha, \quad B = \mathbb{B}, \quad \Sigma \mid \emptyset \vdash_C \alpha^+ : \alpha \rightsquigarrow \mathbb{B} \quad (\exists \mathbb{B}) .$$

By Lemma I.22, $\Sigma \mid \emptyset \vdash_C V : \alpha$. By Lemma E.2, it must be the case that $\Sigma \mid \emptyset \vdash_C V : \alpha$ is derived by (T_Crc_C) and (Ct_Conceal_C); we are given

$$V = V_2\langle \alpha^- \rangle, \quad \Sigma \mid \emptyset \vdash \alpha^- : \mathbb{B} \rightsquigarrow \alpha, \quad \Sigma \mid \emptyset \vdash V_2 : \mathbb{B}, \quad \Sigma \mid \emptyset \vdash V_2\langle \alpha^- \rangle \approx V' : \alpha \quad (\exists V_2) .$$

We perform case analysis on V'.

Case $V' = U_1'$ ($\exists U_1'$): It must be the case that $\Sigma \mid \emptyset \vdash V_2\langle \alpha^- \rangle \approx U_1' : \alpha$ is derived by (Bs_CrcIdL); we are given

$$\Sigma \mid \emptyset \vdash V_2 \approx U_1' : \mathbb{B} .$$

Since $\Sigma \triangleright V_2\langle \alpha^- \rangle\langle \alpha^+ \rangle \longrightarrow_C \Sigma \triangleright V_2$, it suffices to show that $\Sigma \mid \emptyset \vdash V_2 \approx U_1' : \mathbb{B}$, which has been shown.

Case $V' = U_1'\langle s \rangle$ ($\exists U_1', s$): We perform case analysis on the rule applied last to derive $\Sigma \mid \emptyset \vdash V_2\langle \alpha^- \rangle \approx U_1'\langle s \rangle : \alpha$, which is either (Bs_CrcId), (Bs_CrcIdL), (Bs_Crc), or (Bs_CrcMore).

Case (Bs_CrcId): We are given

$$s = |\mathsf{id}_\alpha|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_2\langle \alpha^- \rangle \approx U_1' : \alpha .$$

Since $\Sigma \mid \emptyset \vdash V_2\langle \alpha^- \rangle \approx U_1' : \alpha$ is derived by (Bs_CrcIdL), we are given

$$\Sigma \mid \emptyset \vdash V_2 \approx U_1' : \mathbb{B} .$$

Therefore, since $\Sigma \triangleright V_2\langle \alpha^- \rangle\langle \alpha^+ \rangle \longrightarrow_C \Sigma \triangleright V_2$, it suffices to show that $\Sigma \mid \emptyset \vdash V_2 \approx U_1'\langle s \rangle : \mathbb{B}$, which is given by (Bs_CrcId).

Case (Bs_CrcIdL): We are given

$$\Sigma \mid \emptyset \vdash V_2 \approx U_1'\langle s \rangle : \mathbb{B} .$$

Therefore, since $\Sigma \triangleright V_2\langle \alpha^- \rangle\langle \alpha^+ \rangle \longrightarrow_C \Sigma \triangleright V_2$, it suffices to show that $\Sigma \mid \emptyset \vdash V_2 \approx U_1'\langle s \rangle : \mathbb{B}$, which has been shown.

Case (Bs_Crc): We are given

$$s = |\alpha^-|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_2 \approx U_1' : \mathbb{B} .$$

$s = |\alpha^-|_\emptyset = \mathsf{id}$. Therefore, since $\Sigma \triangleright V_2\langle \alpha^- \rangle\langle \alpha^+ \rangle \longrightarrow_C \Sigma \triangleright V_2$, it suffices to show that $\Sigma \mid \emptyset \vdash V_2 \approx U_1'\langle s \rangle : \mathbb{B}$, which is given by (Bs_CrcId).

Case (Bs_CrcMore): We are given

$$s = s' \fatsemi |\alpha^-|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_2 \approx U_1'\langle s' \rangle : \mathbb{B} \quad (\exists s').$$

By Lemma I.11, $s = s' \fatsemi |\alpha^-|_\emptyset = s'$. Therefore, since $\Sigma \triangleright V_2\langle \alpha^- \rangle\langle \alpha^+ \rangle \longrightarrow_C \Sigma \triangleright V_2$, it suffices to show that $\Sigma \mid \emptyset \vdash V_2 \approx U_1'\langle s' \rangle : \mathbb{B}$, which has been shown.

Case $c^I = c_1^I \,;\, c_2^I$ ($\exists c_1^I, c_2^I$): We are given

$$\Sigma \mid \emptyset \vdash c_1^I : A \rightsquigarrow C, \quad \Sigma \mid \emptyset \vdash c_2^I : C \rightsquigarrow B \quad (\exists C) .$$

We have $\Sigma \triangleright V\langle c_1^I \,;\, c_2^I \rangle \longrightarrow_C \Sigma \triangleright V\langle c_1^I \rangle\langle c_2^I \rangle$. Furthermore, by (T_Crc_C), $\Sigma \mid \emptyset \vdash V\langle c_1^I \rangle : C$. Hence, by the IH, there exists a value V_2 such that $\Sigma \triangleright V\langle c_1^I \rangle \longrightarrow_C \Sigma \triangleright V_2$ and $\Sigma \mid \emptyset \vdash V_2 \approx V' : C$. Therefore, by Lemma I.22, $\Sigma \mid \emptyset \vdash V_2 : C$. By (T_Crc_C), $\Sigma \mid \emptyset \vdash V_2\langle c_2^I \rangle : B$. Moreover, by the IH, there exists a value V_3 such that $\Sigma \triangleright V_2\langle c_2^I \rangle \longrightarrow_C \Sigma \triangleright V_3$ and $\Sigma \mid \emptyset \vdash V_3 \approx V' : B$. Therefore, we have $\Sigma \triangleright V\langle c_1^I \,;\, c_2^I \rangle \longrightarrow_C^* \Sigma \triangleright V_3$ and $\Sigma \mid \emptyset \vdash V_3 \approx V' : B$.

Case $c^I = \alpha^-$, $c^I = c_1^I \to c_2^I$, and $c^I = \forall X.c_1^I$: Since $V\langle c^I\rangle$ is a value, it suffices to show that $\Sigma \mid \emptyset \vdash V\langle c^I\rangle \approx V' : B$, which is given by (Bs_CrcIdL).

□

Lemma I.57 (Bisimulation and Composition (Value on the Left))**.** If $\Sigma \mid \emptyset \vdash V\langle vc_1\rangle \cdots \langle vc_n\rangle \approx U'\langle s\rangle : A_n$ for some $n > 0$, then there exist a nonnegative integer n, j and V_1, A_0, $\ldots A_n$ such that:

- $\Sigma \mid \emptyset \vdash V \approx U' : A_0$, whose derivation is a subderivation of $\Sigma \mid \emptyset \vdash V\langle vc_1\rangle \cdots \langle vc_n\rangle \approx U'\langle s\rangle : A_n$;
- $\Sigma \mid \emptyset \vdash_C vc_i : A_{i-1} \rightsquigarrow A_i$ $(1 \le i \le n)$;
- $j \le n$
- the first j coercions vc_1, …, vc_j are no-op value ceorcions; and
- $s = |\mathsf{id}_{A_{j+1}}|_\emptyset \mathbin{⨟} |vc_{j+1}|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |vc_n|_\emptyset$.

Furthermore,

1. If $A = B_n \to C_n$ for some B_n, C_n, then there exists B_i and C_i for $i \in [0..n]$, and c_i and d_i for $i \in [1..n]$ such that

$$A_0 = B_0 \to C_0, \quad A_i = B_i \to C_i, \quad vc_i = c_i \to d_i,$$
$$\Sigma \mid \Gamma \vdash c_i : B_i \rightsquigarrow B_{i-1}, \quad \Sigma \mid \Gamma \vdash d_i : C_{i-1} \rightsquigarrow C_i \ (1 \le i \le n) \ .$$

2. If $A = \forall X.B_n$ for some X, B_n, then there exists B_i for $i \in [0..n]$ and c_i for $i \in [1..n]$ such that

$$A_0 = \forall X.B_0, \quad A_i = \forall X.B_i, \quad vc_i = \forall X.c_i, \quad \Sigma \mid \Gamma, X \vdash_C c_i : B_{i-1} \rightsquigarrow B_i \ (1 \le i \le n) \ .$$

Proof. By Lemma I.39, we have

- $\Sigma \mid \emptyset \vdash V \approx U' : A_0$, whose derivation is a subderivation of $\Sigma \mid \emptyset \vdash V\langle vc_1\rangle \cdots \langle vc_n\rangle \approx U'\langle s\rangle : A_n$;
- $\Sigma \mid \emptyset \vdash_C vc_i : A_{i-1} \rightsquigarrow A_i$ $(1 \le i \le n)$ and $A_n = A$;
- $j \le n$
- the first j coercions vc_1, …, vc_j are no-op value ceorcions; and
- $s = |\mathsf{id}_{A_{j+1}}|_\emptyset \mathbin{⨟} |vc_{j+1}|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |vc_n|_\emptyset$.

Furthermore,

Case $A = B_n \to C_n$ $(\exists B_n, C_n)$: Since $\Sigma \mid \emptyset \vdash_C vc_n : A_{n-1} \rightsquigarrow (B_n \to C_n)$ is derived by (Ct_Arrow_C), we have

$$vc_n = c_n \to d_n, \quad A_{n-1} = B_{n-1} \to C_{n-1},$$
$$\Sigma \mid \emptyset \vdash_C c_n : B_n \rightsquigarrow B_{n-1}, \quad \Sigma \mid \emptyset \vdash_C d_n : C_{n-1} \rightsquigarrow C_n \quad (\exists c_n, d_n, B_{n-1}, C_{n-1}) \ .$$

Similarly, since $\Sigma \mid \emptyset \vdash_C vc_i : A_{i-1} \rightsquigarrow (B_i \to C_i)$ is derived by (Ct_Arrow_C), we have

$$vc_i = c_i \to d_i, \quad A_i = B_i \to C_i,$$
$$\Sigma \mid \emptyset \vdash_C c_i : B_i \rightsquigarrow B_{i-1}, \quad \Sigma \mid \emptyset \vdash_C d_i : C_{i-1} \rightsquigarrow C_i \quad (\exists c_i, d_i, B_i, C_i) \ (1 \le i \le n) \ .$$

Case $A = \forall X.B_n$ $(\exists B_n)$: Since $\Sigma \mid \emptyset \vdash_C vc_n : A_{n-1} \rightsquigarrow \forall X.B_n$ is derived by (Ct_All_C), we have

$$vc_n = \forall X.c_n, \quad A_{n-1} = \forall X.B_{n-1}, \quad \Sigma \mid \emptyset \vdash_C c_n : B_{n-1} \rightsquigarrow B_n \quad (\exists c_n, B_{n-1}) \ .$$

Similarly, since $\Sigma \mid \emptyset \vdash_C vc_i : A_{i-1} \rightsquigarrow \forall X.B_i$ is derived by (Ct_All_C), we have

$$vc_i = \forall X.c_i, \quad A_i = \forall X.B_i, \quad \Sigma \mid \emptyset \vdash_C c_i : B_{i-1} \rightsquigarrow B_i \quad (\exists c_i, B_i) \ (1 \le i \le n) \ .$$

□

Lemma I.58 (Uncoerced Values are Bisimilar to Values (Value on the Left))**.** If $\Sigma \mid \Gamma \vdash V \approx U : A$, then there exists V_1 that is not a coercion application, a nonnegative integer n, A_i for $0 \in [0..n]$ and vc_i^I for $0 \in [1..n]$ such that

$$V = V_1\langle vc_1^I\rangle \cdots \langle vc_n^I\rangle, \quad A_n = A, \quad \Sigma \mid \Gamma \vdash V_1 \approx U : A_0, \quad \Sigma \mid \Gamma \vdash_C vc_i^I : A_{i-1} \rightsquigarrow A_i \ (1 \le i \le n) \ .$$

Furthermore,

1. If $A = \iota$ for some ι, then we have
$$n = 0, \quad A = ty(k) \ .$$
2. If $A = B_n \to C_n$ for some B_n, C_n, then there exists B_i and C_i for $i \in [0..n]$, and c_i^I and d_i^I for $i \in [1..n]$ such that
$$A_0 = B_0 \to C_0,$$
$$A_i = B_i \to C_i, \quad vc_i^I = c_i^I \to d_i^I, \quad \Sigma \mid \Gamma \vdash c_i^I : B_i \rightsquigarrow B_{i-1}, \quad \Sigma \mid \Gamma \vdash d_i^I : C_{i-1} \rightsquigarrow C_i \ (1 \le i \le n) \ .$$
3. If $A = \forall X.B_n$ for some X, B_n, then there exists B_i for $i \in [0..n]$ and c_i^I for $i \in [1..n]$ such that
$$A_0 = \forall X.B_0,$$
$$A_i = \forall X.B_i, \quad vc_i^I = \forall X.c_i^I, \quad \Sigma \mid \Gamma, X \vdash_C c_i^I : B_{i-1} \rightsquigarrow B_i \ (1 \le i \le n) \ .$$

Proof. By Lemma I.40, there exists some term V_1 such that V_1 is not a coercion application and

$$V = V_1\langle c_1^I\rangle \cdots \langle c_n^I\rangle, \quad A_n = A, \quad \Sigma \mid \Gamma \vdash V_1 \approx U : A_0,$$
$$\Sigma \mid \Gamma \vdash_C c_i^I : A_{i-1} \rightsquigarrow A_i \ (1 \le i \le n) \quad (\exists A_0, \dots, A_n, c_1^I, \dots, c_n^I) \ .$$

Furthermore, $V = V_1\langle c_1^I\rangle \cdots \langle c_n^I\rangle$ is a value, there exists vc_i^I such that $c_i^I = vc_i^I$ ($i \in [1..n]$). Moreover,

Case $A = \iota = ty(A)$ ($\exists \iota$): By Lemma I.40, we have

$$A_0 = ty(k), \quad \Sigma \mid \Gamma \vdash k \approx k' : \iota \ .$$

If we suppose $n > 0$, then there is a contradiction because there is no rule to derive $\Sigma \mid \Gamma \vdash_C vc_n^I : A_{n-1} \rightsquigarrow \iota$. Hence, $n = 0$.

Case $A = B_n \to C_n$ ($\exists B_n, C_n$): By Lemma I.40, we have

$$A_0 = B_0 \to C_0, \quad \Sigma \mid \Gamma \vdash M_2 \approx M_2' : C_0 \quad (\exists M_2, C_0) \ .$$

Since $\Sigma \mid \Gamma \vdash_C vc_n^I : A_{n-1} \rightsquigarrow (B_n \to C_n)$ is derived by (Ct_Arrow_C), we have

$$A_{n-1} = B_{n-1} \to C_{n-1}, \quad vc_n^I = c_n^I \to d_n^I,$$
$$\Sigma \mid \Gamma \vdash_C c_n^I : B_n \rightsquigarrow B_{n-1}, \quad \Sigma \mid \Gamma \vdash_C d_n^I : C_{n-1} \rightsquigarrow C_n \ (\exists B_{n-1}, C_{n-1}, c_n^I, d_n^I) \ .$$

Similarly, since $\Sigma \mid \Gamma \vdash_C vc_i^I : A_{i-1} \rightsquigarrow (B_i \to C_i)$ is derived by (Ct_Arrow_C) for $i \in [1..n]$, there exists B_i and C_i for $i \in [0..n]$ and c_i^I and d_i^I for $i \in [1..n]$ such that

$$A_i = B_i \to C_i, \quad vc_i^I = c_i^I \to d_i^I, \quad \Sigma \mid \Gamma \vdash c_i^I : B_i \rightsquigarrow B_{i-1}, \quad \Sigma \mid \Gamma \vdash d_i^I : C_{i-1} \rightsquigarrow C_i \quad (1 \le i \le n) \ .$$

Case $A = \forall X.B_n$ ($\exists X, B_n$): By Lemma I.40, we have

$$A_0 = \forall X.B_0 \quad (\exists B_0) \ .$$

Since $\Sigma \mid \Gamma \vdash_C vc_n^I : A_{n-1} \rightsquigarrow \forall X.B_n$ is derived by (Ct_All_C), we have

$$A_{n-1} = \forall X.B_{n-1}, \quad vc_n^I = \forall X.c_n^I, \quad \Sigma \mid \Gamma, X \vdash_C c_n^I : B_{n-1} \rightsquigarrow B_n \quad (\exists B_{n-1}, c_n^I) \ .$$

Similarly, since $\Sigma \mid \Gamma \vdash_C vc_i^I : A_{i-1} \rightsquigarrow \forall X.B_i$ is derived by (Ct_All_C) for $i \in [1..n]$, there exists B_i for $i \in [0..n]$ and c_i^I for $i \in [1..n]$ such that

$$A_i = \forall X.B_i, \quad vc_i^I = \forall X.c_i^I, \quad \Sigma \mid X, \Gamma \vdash_C c_i^I : B_{i-1} \rightsquigarrow B_i \quad (1 \le i \le n) \ .$$

□

Lemma I.59 (Bisimulation up to Reduction (Lemma 4.6 of the paper))**.** Suppose that $\Sigma \mid \emptyset \vdash M \approx M' : A$.

(1) If $\Sigma \triangleright M \longrightarrow_C \Sigma_1 \triangleright M_1$, then there exists some Σ_2, M_2, M_2' such that $\Sigma_1 \triangleright M_1 \longrightarrow_C^* \Sigma_2 \triangleright M_2$ and $\Sigma \triangleright M' \longrightarrow_S^* \Sigma_2 \triangleright M_2'$ and $\Sigma_2 \mid \emptyset \vdash M_2 \approx M_2' : A$.

(2) If $\Sigma \triangleright M' \longrightarrow_S \Sigma_1 \triangleright M_1'$, then there exist some Σ_2, M_2, M_2' such that $\Sigma_1 \triangleright M_1' \longrightarrow_S^* \Sigma_2 \triangleright M_2'$ and $\Sigma \triangleright M \longrightarrow_C^* \Sigma_2 \triangleright M_2$ and $\Sigma_2 \mid \emptyset \vdash M_2 \approx M_2' : A$.

(3) If $M = V$, then there exists some V' such that $\Sigma \triangleright M' \longrightarrow_S^* \Sigma \triangleright V'$ and $\Sigma \mid \emptyset \vdash V \approx V' : A$.

(4) If $M' = V'$, then there exists some V such that $\Sigma \triangleright M \longrightarrow_C^* \Sigma \triangleright V$ and $\Sigma \mid \emptyset \vdash V \approx V' : A$.

(5) If $M = \mathsf{blame}\, p$, then $\Sigma \triangleright M' \longrightarrow_S^* \Sigma \triangleright \mathsf{blame}\, p$.

(6) If $M' = \mathsf{blame}\, p$, then $\Sigma \triangleright M \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p$.

Proof. We prove the cases (3), (5), (1), (4), (6), (2) in order.

(3) By induction on the derivation of $\Sigma \mid \emptyset \vdash V \approx M' : A$. We perform case analysis on the rule applied last to derive $\Sigma \mid \emptyset \vdash V \approx M' : A$.

Case (Bs_Const), (Bs_Abs), (Bs_Tyabs): Because M' is a value, we have the conclusion by letting $V' = M'$.

Case (Bs_Crc): We are given

$$V = V_1\langle c\rangle, \quad M' = M_1'\langle |c|_\emptyset\rangle, \quad \Sigma \mid \emptyset \vdash V_1 \approx M_1' : B, \quad \Sigma \mid \emptyset \vdash_C c : B \rightsquigarrow A \quad (\exists B, c, M_1', V_1) .$$

Furthermore, since $V_1\langle c\rangle$ is a value, there exists a value coercion vc such that $c = vc$. By the IH,

$$\Sigma \triangleright M_1' \longrightarrow_S^* \Sigma \triangleright V_1', \quad \Sigma \mid \emptyset \vdash V_1 \approx V_1' : B \quad (\exists V_1') .$$

We perform case analysis on whether V_1' is coercion application or not.

Case $V_1' = U_1'$ $(\exists U_1')$: By (Bs_Crc),

$$\Sigma \mid \emptyset \vdash V_1\langle vc\rangle \approx U_1'\langle |vc|_\emptyset\rangle : A .$$

By case analysis on vc.

Case $vc = G!$ $(\exists G)$: Since $\Sigma \mid \emptyset \vdash_C G! : B \rightsquigarrow A$, $G!$ does not contain any free type variables. Therefore, we have $|G!|_\emptyset = |\mathsf{id}_G|_\emptyset \,;\, G!$. Since $\Sigma \triangleright M_1' \longrightarrow_S^* \Sigma \triangleright U_1'$ and $U_1'\langle |\mathsf{id}_G|_\emptyset \,;\, G!\rangle$ is a value, by Lemma I.14, we have

$$\Sigma \triangleright M_1'\langle(|\mathsf{id}_G|_\emptyset \,;\, G!)\rangle \longrightarrow_S^* \Sigma \triangleright U_1'\langle |\mathsf{id}_G|_\emptyset \,;\, G!\rangle .$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash V_1\langle G!\rangle \approx U_1'\langle |\mathsf{id}_G|_\emptyset \,;\, G!\rangle : A$, which has been shown.

Case $vc = \alpha^-$ $(\exists \alpha)$: Because $\Sigma \mid \emptyset \vdash_C \alpha^- : B \rightsquigarrow A$ is derived by (Ct_Conceal_C), we have

$$A = \alpha, \quad B = \mathbb{B}, \quad \alpha := \mathbb{B} \in \Sigma \quad (\exists \mathbb{B})$$

Also, we have $|\alpha^-|_\emptyset = \mathsf{id}$. Hence, by (R_Id_S), we have $\Sigma \triangleright U_1'\langle \mathsf{id}\rangle \longrightarrow_S^* \Sigma \triangleright U_1'$. Since U_1' is a value, by Lemma I.14, we have

$$\begin{aligned} \Sigma \triangleright M'\langle |\alpha^-|_\emptyset\rangle &= \Sigma \triangleright M'\langle \mathsf{id}\rangle \\ &\longrightarrow_S^* \Sigma \triangleright U_1' . \end{aligned}$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash V_1\langle \alpha^-\rangle \approx U_1' : A$, which is given by (Bs_CrcIdL).

Case $vc = c' \to d'$ $(\exists c', d')$: We have $|c' \to d'|_\emptyset = |c'|_\emptyset \to |d'|_\emptyset$. Because $U_1'\langle |c'|_\emptyset \to |d'|_\emptyset\rangle$ is a value, we can prove this case similarly to the case of $c = G!$.

Case $vc = \forall X.c'$ $(\exists X, c')$: We have $|\forall X.c'|_\emptyset = \forall X.|c'|_{\emptyset,X} \mathbin{,,} |c'|_\emptyset$. Because $U_1'\langle \forall X.|c'|_{\emptyset,X} \mathbin{,,} |c'|_\emptyset\rangle$ is a value, we can prove this case similarly to the case of $c = G!$.

Case $V_1' = U_1'\langle t\rangle$ $(\exists t, U_1')$: By Lemma I.22, $\Sigma \mid \emptyset \vdash_S U_1'\langle t\rangle : \Sigma(B)$. Since $\Sigma \mid \emptyset \vdash_S U_1'\langle t\rangle : \Sigma(B)$ is derived by (T_Crc_S), we have

$$\Sigma \mid \emptyset \vdash_S t : A' \rightsquigarrow \Sigma(B) \quad (\exists A') .$$

By Lemma I.43, $|vc|_\emptyset$ is a space-efficient value coercion. Also, since $U_1'\langle t\rangle$ is a value, t is a space-efficient value coercion and $t \neq \mathsf{id}$. Therefore, by Lemma I.42, $t \mathbin{;} |vc|_\emptyset$ is a space-efficient value coercion. We perform case analysis whether $t \mathbin{;} |vc|_\emptyset = \mathsf{id}$ or not.

Case $t \mathbin{;} |vc|_\emptyset = \mathsf{id}$: By Lemma I.49, $t = \mathsf{id}$ and $|vc|_\emptyset = \mathsf{id}$. However, there is contradiction because $t \neq \mathsf{id}$.

Case $t \mathbin{;} |vc|_\emptyset \neq \mathsf{id}$: By the definition of space-efficient value coercions, $U_1'\langle t \mathbin{;} |vc|_\emptyset\rangle$ is a value. By (R_Merge_S), $\Sigma \triangleright U_1'\langle t\rangle\langle |vc|_\emptyset\rangle \longrightarrow_S \Sigma \triangleright U_1'\langle t \mathbin{;} |vc|_\emptyset\rangle$. Therefore, Lemma I.18 implies

$$\Sigma \triangleright M_1\langle s \mathbin{;} |vc|_\emptyset\rangle \longrightarrow_S^* \Sigma \triangleright U_1'\langle t \mathbin{;} |vc|_\emptyset\rangle .$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash V_1\langle vc\rangle \approx U_1'\langle t \mathbin{;} |vc|_\emptyset\rangle : A$, which is given by (Bs_CrcMore).

Case (Bs_CrcId): We are given

$$M' = M_1'\langle |\mathsf{id}_A|_\emptyset\rangle, \quad \Sigma \mid \emptyset \vdash V \approx M_1' : A, \quad \Sigma \mid \emptyset \vdash_C \mathsf{id}_A : A \rightsquigarrow A \quad (\exists M_1') .$$

By the IH,

$$\Sigma \triangleright M_1' \longrightarrow_S^* \Sigma \triangleright V_1', \quad \Sigma \mid \emptyset \vdash V \approx V_1' : A \quad (\exists V_1') .$$

Lemma I.22 implies $\Sigma \mid \emptyset \vdash_S M_1' : \Sigma(A)$ and $\Sigma \mid \emptyset \vdash_S M_1'\langle |\mathsf{id}_A|_\emptyset\rangle : \Sigma(A)$. Because this judgment is derived by (T_Crc_S), we have

$$\Sigma \mid \emptyset \vdash_S |\mathsf{id}_A|_\emptyset : \Sigma(A) \rightsquigarrow \Sigma(A) .$$

By case analysis on V_1'.

Case $V_1' = U_1'$ $(\exists U_1')$: By case analysis on A:

Case $A = A' \to B'$ $(\exists A', B')$: We have $|\mathsf{id}_{A'\to B'}|_\emptyset = |\mathsf{id}_{A'\to B'}|_\emptyset = |\mathsf{id}_{A'}|_\emptyset \to |\mathsf{id}_{B'}|_\emptyset$. Furthermore, $U_1'\langle |\mathsf{id}_{A'}|_\emptyset \to |\mathsf{id}_{B'}|_\emptyset\rangle$ is a value. Therefore, Lemma I.14 implies

$$\Sigma \triangleright M_1'\langle |\mathsf{id}_{A'}|_\emptyset \to |\mathsf{id}_{B'}|_\emptyset\rangle \longrightarrow_S^* \Sigma \triangleright U_1'\langle |\mathsf{id}_{A'}|_\emptyset \to |\mathsf{id}_{B'}|_\emptyset\rangle .$$

Moreover, by (Bs_CrcId),

$$\Sigma \mid \emptyset \vdash V \approx U_1'\langle |\mathsf{id}_{A'}|_\emptyset \to |\mathsf{id}_{B'}|_\emptyset\rangle : A' \to B' .$$

Case $A = \forall X.A'$ $(\exists X, A')$: $|\mathsf{id}_{\forall X.A'}|_\emptyset = |\mathsf{id}_{\forall X.A'}|_\emptyset = \forall X.|\mathsf{id}_{A'}|_{\emptyset,X} \mathbin{,,} |\mathsf{id}_{A'}|_\emptyset$, and $U_1'\langle \forall X.|\mathsf{id}_{A'}|_{\emptyset,X} \mathbin{,,} |\mathsf{id}_{A'}|_\emptyset\rangle$ is a value. Therefore, this case is provable similarly to the case of $A = A' \to B'$.

Otherwise: Since $\Sigma \mid \emptyset \vdash_C \mathsf{id}_A : A \rightsquigarrow A$, A does not have any free type variables. Therefore, we have $|\mathsf{id}_A|_\emptyset = \mathsf{id}$. By (R_Id_S), $\Sigma \triangleright U_1'\langle \mathsf{id}\rangle \longrightarrow_S \Sigma \triangleright U_1'$. Therefore, Lemma I.14 implies

$$\Sigma \triangleright M_1'\langle \mathsf{id}\rangle \longrightarrow_S^* \Sigma \triangleright U_1' .$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash V \approx U_1' : A$, which has been shown.

Case $V_1' = U_1'\langle s\rangle$ $(\exists s, U_1')$: By Lemma I.22 and Corollary E.46, $\Sigma \mid \emptyset \vdash_S U_1'\langle s\rangle : \Sigma(A)$. Because this judgment is derived by (T_Crc_S), we have $\Sigma \mid \emptyset \vdash_S s : A' \rightsquigarrow \Sigma(A)$ $(\exists A')$. Therefore, Lemma I.11 implies $s \mathbin{;} |\mathsf{id}_A|_\emptyset = s$. Therefore, by Lemma I.14

$$\Sigma \triangleright M_1'\langle |\mathsf{id}_A|_\emptyset\rangle \longrightarrow_S^* \Sigma \triangleright U_1'\langle s \mathbin{;} |\mathsf{id}_A|_\emptyset\rangle = \Sigma \triangleright U_1'\langle s\rangle$$

(note that $U_1'\langle s\rangle$ is a value). Hence, it suffices to show that $\Sigma \mid \emptyset \vdash V \approx U_1'\langle s\rangle : A$, which has been shown.

Case (Bs_CrcMore): We are given

$$V = V_1\langle c\rangle, \quad M' = M'_1\langle s \fatsemi |c|_\emptyset\rangle, \quad \Sigma \mid \emptyset \vdash V_1 \approx M'_1\langle s\rangle : B, \quad \Sigma \mid \emptyset \vdash_C c : B \rightsquigarrow A \quad (\exists B, c, s, M'_1, V_1) .$$

Furthermore, since $V_1\langle c\rangle$ is a value, there exists a value coercion vc such that $c = vc$. By the IH,

$$\Sigma \triangleright M'_1\langle s\rangle \longrightarrow^*_S \Sigma \triangleright V'_1, \quad \Sigma \mid \emptyset \vdash V_1 \approx V'_1 : B \quad (\exists V'_1) .$$

We also have, by Lemma I.22, $\Sigma \mid \emptyset \vdash_C V_1 : B$ and $\Sigma \mid \emptyset \vdash_S M'_1\langle s\rangle : \Sigma(B)$. By Lemma E.10, we have $\vdash \Sigma$, $\Sigma \vdash \emptyset$, and $\Sigma \mid \emptyset \vdash B$. Therefore, by (Ct_Id_C), we have $\Sigma \mid \emptyset \vdash_C \mathsf{id}_B : B \rightsquigarrow B$.

$$\Sigma \mid \emptyset \vdash_S |vc|_\emptyset : \Sigma(B) \rightsquigarrow \Sigma(A) .$$

Moreover, Corollary E.46 implies $\Sigma \mid \emptyset \vdash_S V'_1 : \Sigma(B)$. By case analysis on V'_1.

Case $V'_1 = U'_1$ $(\exists U'_1)$: By (Bs_CrcId), $\Sigma \mid \emptyset \vdash V_1 \approx U'_1\langle|\mathsf{id}_B|_\emptyset\rangle : B$. Lemma I.11 implies $|\mathsf{id}_B|_\emptyset \fatsemi |vc|_\emptyset = |vc|_\emptyset$. Therefore, by (Bs_CrcMore),

$$\Sigma \mid \emptyset \vdash V_1\langle vc\rangle \approx U'_1\langle|vc|_\emptyset\rangle : A .$$

By case analysis on vc.

Case $vc = G!$ $(\exists G)$: Since $\Sigma \mid \emptyset \vdash_C G! : B \rightsquigarrow A$, $G!$ does not contain any free type variables. Therefore, we have $|G!|_\emptyset = |\mathsf{id}_G|_\emptyset \,;\, G!$. Because $U'_1\langle|\mathsf{id}_G|_\emptyset \,;\, G!\rangle$ is a value, Lemma I.18 implies

$$\Sigma \triangleright M'_1\langle s \fatsemi (|\mathsf{id}_B|_\emptyset \,;\, G!)\rangle \longrightarrow^*_S \Sigma \triangleright U'_1\langle|\mathsf{id}_G|_\emptyset \,;\, G!\rangle .$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash V_1\langle G!\rangle \approx U'_1\langle|\mathsf{id}_G|_\emptyset \,;\, G!\rangle : A$, which has been shown.

Case $vc = \alpha^-$ $(\exists\alpha)$: Because $\Sigma \mid \emptyset \vdash_C \alpha^- : B \rightsquigarrow A$ is derived by (Ct_Conceal_C), we have

$$A = \alpha, \quad B = \mathbb{B}, \quad \alpha := \mathbb{B} \in \Sigma \quad (\exists\mathbb{B})$$

Also, $\Sigma \mid \emptyset \vdash_S M'_1\langle s\rangle : \Sigma(B)$ is derived by (t_Crc_S), we have

$$\Sigma \mid \emptyset \vdash_S M'_1 : A'_1, \quad \Sigma \mid \emptyset \vdash_S s : A \rightsquigarrow \Sigma(B) \quad (A'_1) .$$

By Lemma I.11, $s \fatsemi |\alpha^-|_\emptyset = s$. Therefore,

$$\begin{aligned}\Sigma \triangleright M'\langle s \fatsemi |\alpha^-|_\emptyset\rangle &= \Sigma \triangleright M'\langle s\rangle \\ &\longrightarrow^*_S \Sigma \triangleright U'_1 .\end{aligned}$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash V_1\langle\alpha^-\rangle \approx U'_1 : A$, which is given by (Bs_CrcIdL).

Case $vc = c' \to d'$ $(\exists c', d')$: We have $|c' \to d'|_\emptyset = |c'|_\emptyset \to |d'|_\emptyset$. Because $U'_1\langle|c'|_\emptyset \to |d'|_\emptyset\rangle$ is a value, we can prove this case similarly to the case of $c = G!$.

Case $vc = \forall X.c'$ $(\exists X, c')$: We have $|\forall X.c'|_\emptyset = \forall X.|c'|_{\emptyset,X} \,,,\, |c'|_\emptyset$. Because $U'_1\langle\forall X.|c'|_{\emptyset,X} \,,,\, |c'|_\emptyset\rangle$ is a value, we can prove this case similarly to the case of $c = G!$.

Case $V'_1 = U'_1\langle t\rangle$ $(\exists t, U'_1)$: Because $\Sigma \mid \emptyset \vdash_S U'_1\langle t\rangle : \Sigma(B)$ is derived by (T_Crc_S), we have

$$\Sigma \mid \emptyset \vdash_S t : A' \rightsquigarrow \Sigma(B) \quad (\exists A') .$$

By Lemma I.43 implies that $|vc|_\emptyset$ is a space-efficient value coercion. Also, because $U'_1\langle t\rangle$ is a value, t is a space-efficient value coercion and $t \neq \mathsf{id}$. Therefore, Lemma I.42 implies that $t \fatsemi |vc|_\emptyset$ is a space-efficient value coercion. We perform case analysis whether $t \fatsemi |vc|_\emptyset = \mathsf{id}$ or not.

Case $t \fatsemi |vc|_\emptyset = \mathsf{id}$: By Lemma I.49, $t = \mathsf{id}$ and $|vc|_\emptyset = \mathsf{id}$. However, there is contradiction because $t \neq \mathsf{id}$.

Case $t \fatsemi |vc|_\emptyset \neq \mathsf{id}$: By the definition of space-efficient value coercions, $U'_1\langle t \fatsemi |vc|_\emptyset\rangle$ is a value. By (R_Merge_S), $\Sigma \triangleright U'_1\langle t\rangle\langle|vc|_\emptyset\rangle \longrightarrow_S \Sigma \triangleright U'_1\langle t \fatsemi |vc|_\emptyset\rangle$. Therefore, Lemma I.18 implies

$$\Sigma \triangleright M_1\langle s \fatsemi |vc|_\emptyset\rangle \longrightarrow^*_S \Sigma \triangleright U'_1\langle t \fatsemi |vc|_\emptyset\rangle .$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash V_1\langle vc\rangle \approx U'_1\langle t \fatsemi |vc|_\emptyset\rangle : A$, which is given by (Bs_CrcMore).

Case (Bs_CrcIdL): We are given

$$V = V_1\langle c^I \rangle, \quad \Sigma \mid \emptyset \vdash V_1 \approx M' : B, \quad \Sigma \mid \emptyset \vdash_C c^I : B \rightsquigarrow A \quad (\exists B, c^I, V_1) .$$

By the IH,

$$\Sigma \triangleright M' \longrightarrow_S^* \Sigma \triangleright V_1', \quad \Sigma \mid \emptyset \vdash V_1 \approx V_1' : B \quad (\exists V_1') .$$

By (Bs_CrcIdL), we have

$$\Sigma \mid \emptyset \vdash V_1\langle c^I \rangle \approx V_1' : A .$$

Therefore, we conclude the case by letting $V' = V_1'$.

Case (Bs_Var), (Bs_App), (Bs_Tyapp), (Bs_Blame): Contradictory because M is a value.

(5) By induction on the derivation of $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx M' : A$. We perform case analysis on the rule applied last to derive $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx M' : A$, which is either of (Bs_Blame) or (Bs_CrcId).

Case (Bs_Blame): We have $M' = \mathsf{blame}\, p$. Hence, $\Sigma \triangleright \mathsf{blame}\, p \longrightarrow_S^* \Sigma \triangleright \mathsf{blame}\, p$.

Case (Bs_CrcId): We are given

$$M' = M_1'\langle |\mathsf{id}_A|_\emptyset \rangle, \quad \Sigma \mid \emptyset \vdash_C \mathsf{id}_A : A \rightsquigarrow A, \quad \Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx M_1' : A \quad (\exists M_1') .$$

By the IH, $\Sigma \triangleright M_1' \longrightarrow_S^* \Sigma \triangleright \mathsf{blame}\, p$. By (R_BlameC_S),

$$\Sigma \triangleright (\mathsf{blame}\, p)\langle |\mathsf{id}_A|_\emptyset \rangle \longrightarrow_S^* \Sigma \triangleright \mathsf{blame}\, p .$$

Lemma I.23 implies $\vdash \Sigma$, $\Sigma \vdash \emptyset$, and $\Sigma \mid \emptyset \vdash A$. Therefore, by (Ct_Id_C), $\Sigma \mid \emptyset \vdash_C \mathsf{id}_A : A \rightsquigarrow A$. Hence, by Lemma I.3, we have $\Sigma \mid \emptyset \vdash_S |\mathsf{id}_A|_\emptyset : \Sigma(A) \rightsquigarrow \Sigma(A)$. Therefore, by Lemma I.17, we have $\Sigma \triangleright M_1'\langle |\mathsf{id}_A|_\emptyset \rangle \longrightarrow_S^* \Sigma \triangleright \mathsf{blame}\, p$.

(1) By induction on the derivation of $\Sigma \mid \emptyset \vdash M \approx M' : A$. We perform case analysis on the rule applied last to derive $\Sigma \mid \emptyset \vdash M \approx M' : A$.

Case (Bs_Const), (Bs_Var), (Bs_Abs), (Bs_Tyabs), (Bs_Blame): Contradictory because $\Sigma \triangleright M \longrightarrow_C \Sigma_1 \triangleright M_1$ cannot be derived.

Case (Bs_App): We are given

$$M = M_2\, M_3, \quad M' = M_2'\, M_3', \quad \Sigma \mid \emptyset \vdash M_2 \approx M_2' : B \to A,$$
$$\Sigma \mid \emptyset \vdash M_3 \approx M_3' : B \quad (\exists B, M_2, M_3, M_2', M_3') .$$

Lemma I.22 implies $\Sigma \mid \emptyset \vdash_C M_2\, M_3 : A$ and $\Sigma \mid \emptyset \vdash_S M_2'\, M_3' : \Sigma(A)$. Therefore, Theorem E.19 implies $\Sigma \mid \emptyset \vdash_C M_1 : A$. We perform case analysis on the rule applied last to derive $\Sigma \triangleright M_2\, M_3 \longrightarrow_C \Sigma_1 \triangleright M_1$, which is one of the following rules.

Case (R_Delta_C): We are given

$$M_2 = k_2, \quad M_3 = k_3, \quad M_1 = \delta(k_2, k_3), \quad \Sigma_1 = \Sigma \quad (\exists k_2, k_3) .$$

Because k_2 is a value, by the case (3),

$$\Sigma \triangleright M_2' \longrightarrow_S^* \Sigma \triangleright V_2', \quad \Sigma \mid \emptyset \vdash k_2 \approx V_2' : B \to A \quad (\exists V_2') .$$

Similarly, by the case (3),

$$\Sigma \triangleright M_3' \longrightarrow_S^* \Sigma \triangleright V_3', \quad \Sigma \mid \emptyset \vdash k_3 \approx V_3' : B \quad (\exists V_3') .$$

Therefore, by applying (R_CtxE_S) repeatedly, we have

$$\Sigma \triangleright M_2'\, M_3' \longrightarrow_S^* \Sigma \triangleright V_2'\, V_3' \quad \cdots (*) .$$

Lemma I.22 implies $\Sigma \mid \emptyset \vdash_C k_2 : B \to A$. Because this judgment is derived by (T_Const_C), we have $ty(k_2) = B \to A$. By the definition of ty, there exists some ι such that $B = \iota$. Furthermore, since $M_1 = \delta(k_2, k_3)$, we have $\Sigma \mid \emptyset \vdash_C \delta(k_2, k_3) : A$. Because $\Sigma \mid \emptyset \vdash_C \delta(k_2, k_3) : A$ is derived by (T_Const_C), we have $ty(\delta(k_2, k_3)) = A$. Therefore, by (Bs_Const), $\Sigma \mid \emptyset \vdash \delta(k_2, k_3) \approx \delta(k_2, k_3) : A$. We perform case analysis on the rule applied last to derive $\Sigma \mid \emptyset \vdash k_3 \approx V_3' : \iota$, which is either of (Bs_Const) or (Bs_CrcId).

Case (Bs_Const): We have

$$V_3' = k_3, \quad \vdash \Sigma, \quad \Sigma \vdash \emptyset .$$

By case analysis on the rule applied last to derive $\Sigma \mid \emptyset \vdash k_2 \approx V_2' : \iota \to A$, which is either of (Bs_Const) or (Bs_CrcId).

Case (Bs_Const): We have $V_2' = k_2$. Therefore, by $(*)$ and (R_Delta_S),

$$\Sigma \triangleright M_2' \, M_3' \longrightarrow_S^* \Sigma \triangleright V_2' \, V_3' \longrightarrow_S \Sigma \triangleright \delta(k_2, k_3) .$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash \delta(k_2, k_3) \approx \delta(k_2, k_3) : A$. Now, we have $\Sigma \mid \emptyset \vdash_C \delta(k_2, k_3) : A$. Because $\Sigma \mid \emptyset \vdash_C \delta(k_2, k_3) : A$ is derived by (T_Const_C), we have $ty(\delta(k_2, k_3)) = A$. Therefore, by (Bs_Const), $\Sigma \mid \emptyset \vdash \delta(k_2, k_3) \approx \delta(k_2, k_3) : A$.

Case (Bs_CrcId): We are given

$$V_2' = U_2' \langle |\mathsf{id}_{\iota \to A}|_\emptyset \rangle, \quad \Sigma \mid \emptyset \vdash k_2 \approx U_2' : \iota \to A .$$

Because $\Sigma \mid \emptyset \vdash k_2 \approx U_2' : \iota \to A$ is derived by (Bs_Const), we have $U_2' = k_2$. Furthermore,

$$\begin{aligned} |\mathsf{id}_{\iota \to A}|_\emptyset &= |\mathsf{id}_{\iota \to A}|_\emptyset \\ &= |\mathsf{id}_\iota|_\emptyset \to |\mathsf{id}_A|_\emptyset \\ &= \mathsf{id} \to |\mathsf{id}_A|_\emptyset . \end{aligned}$$

Therefore, by $(*)$, (R_Wrap_S), (R_Id_S), and (R_Delta_S),

$$\begin{aligned} \Sigma \triangleright M_2' \, M_3' &\longrightarrow_S^* \Sigma \triangleright V_2' \, V_3' \\ &= \Sigma \triangleright (k_2 \langle \mathsf{id} \to |\mathsf{id}_A|_\emptyset \rangle) \, k_3 \\ &\longrightarrow_S \Sigma \triangleright (k_2 \, (k_3 \langle \mathsf{id} \rangle)) \langle |\mathsf{id}_A|_\emptyset \rangle \\ &\longrightarrow_S \Sigma \triangleright (k_2 \, k_3) \langle |\mathsf{id}_A|_\emptyset \rangle \\ &\longrightarrow_S \Sigma \triangleright \delta(k_2, k_3) \langle |\mathsf{id}_A|_\emptyset \rangle . \end{aligned}$$

Hence, since $|\mathsf{id}_A|_\emptyset = |\mathsf{id}_A|_\emptyset$, it suffices to show that $\Sigma \mid \emptyset \vdash \delta(k_2, k_3) \approx \delta(k_2, k_3) \langle |\mathsf{id}_A|_\emptyset \rangle : A$. Now, we have $\Sigma \mid \emptyset \vdash_C \delta(k_2, k_3) : A$. Because $\Sigma \mid \emptyset \vdash_C \delta(k_2, k_3) : A$ is derived by (T_Const_C), we have $ty(\delta(k_2, k_3)) = A$. Therefore, by (Bs_Const), $\Sigma \mid \emptyset \vdash \delta(k_2, k_3) \approx \delta(k_2, k_3) : A$. Hence, by (Bs_CrcId), $\Sigma \mid \emptyset \vdash \delta(k_2, k_3) \approx \delta(k_2, k_3) \langle |\mathsf{id}_A|_\emptyset \rangle : A$.

Case (Bs_CrcId): We are given

$$V_3' = U_3' \langle |\mathsf{id}_\iota|_\emptyset \rangle, \quad \Sigma \mid \emptyset \vdash k_3 \approx U_3' : \iota \quad (\exists U_3') .$$

We have $|\mathsf{id}_\iota|_\emptyset = |\mathsf{id}_\iota|_\emptyset = \mathsf{id}$. Therefore, $U_3' \langle \mathsf{id} \rangle$ is not a value, so there is a contradiction.

Case (R_Beta_C): We are given

$$M_2 = \lambda x \! : \! B'.M_4, \quad M_3 = V_3, \quad M_1 = M_4[x := V_3], \quad \Sigma_1 = \Sigma \quad (\exists B', x, M_4, V_3) .$$

Because $\lambda x \! : \! B'.M_4$ is a value, the case (3) implies

$$\Sigma \triangleright M_2' \longrightarrow_S^* \Sigma \triangleright V_2', \quad \Sigma \mid \emptyset \vdash \lambda x \! : \! B'.M_4 \approx V_2' : B \to A \quad (\exists V_2') .$$

Similarly, the case (3) implies

$$\Sigma \triangleright M_3' \longrightarrow_S^* \Sigma \triangleright V_3', \quad \Sigma \mid \emptyset \vdash V_3 \approx V_3' : B \quad (\exists V_3') .$$

Therefore, by applying (R_CtxE_S) repeatedly, we have

$$\Sigma \triangleright M_2' \, M_3' \longrightarrow_S^* \Sigma \triangleright V_2' \, V_3' \quad \cdots (*) \ .$$

Lemma I.22 implies $\Sigma \mid \emptyset \vdash_C \lambda x : B'.M_4 : B \to A$. Because this judgment is derived by (T_Abs_C), we have $B' = B$. We perform case analysis on the rule applied last to derive $\Sigma \mid \emptyset \vdash \lambda x : B.M_4 \approx V_2' : B \to A$, which is either of (Bs_Abs) or (Bs_CrcId).

Case (Bs_Abs): We are given

$$V_2' = \lambda x : B.M_4', \quad \Sigma \mid \emptyset, x : B \vdash M_4 \approx M_4' : A \quad (\exists M_4') \ .$$

Therefore, by $(*)$ and (R_Beta_S),

$$\Sigma \triangleright M_2' \, M_3' \longrightarrow_S^* \Sigma \triangleright (\lambda x : B.M_4') \, V_3' \longrightarrow_S \Sigma \triangleright M_4'[x := V_3'] \ .$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash M_4[x := V_3] \approx M_4'[x := V_3'] : A$, which is given by Lemma I.27.

Case (Bs_CrcId): We are given

$$V_2' = U_2' \langle |\mathsf{id}_{B \to A}|_\emptyset \rangle, \quad \Sigma \mid \emptyset \vdash \lambda x : B.M_4 \approx U_2' : B \to A \quad (\exists U_2') \ .$$

Because $\Sigma \mid \emptyset \vdash \lambda x : B.M_4 \approx U_2' : B \to A$ is derived by (Bs_Abs), we have

$$U_2' = \lambda x : B.M_4', \quad \Sigma \mid \emptyset, x : B \vdash M_4 \approx M_4' : A \quad (\exists M_4') \ .$$

Furthermore, by (Bs_CrcId), $\Sigma \mid \emptyset \vdash V_3 \approx V_3' \langle |\mathsf{id}_B|_\emptyset \rangle : B$. Therefore, by the case (3), we have

$$\Sigma \triangleright V_3' \langle |\mathsf{id}_B|_\emptyset \rangle \longrightarrow_S^* \Sigma \triangleright V_3'', \quad \Sigma \mid \emptyset \vdash V_3 \approx V_3'' : B \quad (\exists V_3'') \ .$$

Hence, since $|\mathsf{id}_{B \to A}|_\emptyset = |\mathsf{id}_{B \to A}|_\emptyset$ and $|\mathsf{id}_A|_\emptyset = |\mathsf{id}_A|_\emptyset$, by $(*)$, (R_Wrap_S), (R_CtxE_S), (R_CtxC_S), and (R_Beta_S), we have

$$\begin{aligned}
\Sigma \triangleright M_2' \, M_3' &\longrightarrow_S^* \Sigma \triangleright (U_2' \langle |\mathsf{id}_{B \to A}|_\emptyset \rangle) \, V_3' \\
&= \quad \Sigma \triangleright (U_2' \langle |\mathsf{id}_B|_\emptyset \to |\mathsf{id}_A|_\emptyset \rangle) \, V_3' \\
&\longrightarrow_S \Sigma \triangleright ((\lambda x : B.M_4') \, (V_3' \langle |\mathsf{id}_B|_\emptyset \rangle)) \langle |\mathsf{id}_A|_\emptyset \rangle \\
&\longrightarrow_S^* \Sigma \triangleright ((\lambda x : B.M_4') \, V_3'') \langle |\mathsf{id}_A|_\emptyset \rangle \\
&\longrightarrow_S \Sigma \triangleright (M_4'[x := V_3'']) \langle |\mathsf{id}_A|_\emptyset \rangle \ .
\end{aligned}$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash M_4[x := V_3] \approx (M_4'[x := V_3'']) \langle |\mathsf{id}_A|_\emptyset \rangle : A$. By Lemma I.27, we have $\Sigma \mid \emptyset \vdash M_4[x := V_3] \approx M_4'[x := V_3''] : A$. Therefore, by (Bs_CrcId), $\Sigma \mid \emptyset \vdash M_4[x := V_3] \approx (M_4'[x := V_3'']) \langle |\mathsf{id}_A|_\emptyset \rangle : A$.

Case (R_Wrap_C): We are given

$$M_2 = V_2 \langle c \to d \rangle, \quad M_3 = V_3, \quad M_1 = (V_2 \, (V_3 \langle c \rangle)) \langle d \rangle, \quad \Sigma_1 = \Sigma \quad (\exists c, d, V_2, V_3) \ .$$

Because $V_2 \langle c \to d \rangle$ is a value, by the case (3) we have

$$\Sigma \triangleright M_2' \longrightarrow_S^* \Sigma \triangleright V_2', \quad \Sigma \mid \emptyset \vdash V_2 \langle c \to d \rangle \approx V_2' : B \to A \quad (\exists V_2') \ .$$

Similarly, by the case (3), we have

$$\Sigma \triangleright M_3' \longrightarrow_S^* \Sigma \triangleright V_3', \quad \Sigma \mid \emptyset \vdash V_3 \approx V_3' : B \quad (\exists V_3') \ .$$

Therefore, by applying (R_CtxE_S) repeatedly, we have

$$\Sigma \triangleright M_2' \, M_3' \longrightarrow_S^* \Sigma \triangleright V_2' \, V_3' \ .$$

By Lemma I.22, we have $\Sigma \mid \emptyset \vdash_C V_2\langle c \to d\rangle : B \to A$. Therefore, by Lemma I.37, there exist some V_4 that is not a coercion application, $n > 0$, and $c_1, \ldots, c_n, d_1, \ldots, d_n$ such that

$$V_2\langle c \to d\rangle = V_4\langle c_1 \to d_1\rangle \cdots \langle c_n \to d_n\rangle, \quad c_n = c, \quad d_n = d \ .$$

Moreover, by Lemma I.38, we have

$$A_{n+1} = A, \quad B_{n+1} = B, \quad \Sigma \mid \emptyset \vdash_C c_i : B_{i+1} \rightsquigarrow B_i, \\ \Sigma \mid \emptyset \vdash_C d_i : A_i \rightsquigarrow A_{i+1} \quad (\exists A_i, B_i)\ (1 \leq \forall i \leq n) \ .$$

We perform case analysis on whether V_2' is a coercion application or not.
Case $V_2' = U_2'\langle s\rangle$ $(\exists U_2', s)$: By Lemma I.39, we have

$$\Sigma \mid \emptyset \vdash_C c_i \to d_i : C_i \rightsquigarrow C_{i+1}\ (1 \leq i \leq n), \quad \Sigma \mid \emptyset \vdash V_4 \approx U_2' : C_1, \\ s = |\mathsf{id}_{C_{j+1}}|_\emptyset \,\mathbin{;}\, |c_{j+1} \to d_{j+1}|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |c_n \to d_n|_\emptyset \quad (\exists C_1, ..., C_{n+1}) \ ,$$

and there exists nonnegative integer j $(1 \leq j < n)$ such that

$$c_i = c_i^I, \quad d_i = d_i^I \quad (\exists c_1^I, ..., c_j^I, d_1^I, ..., d_j^I)\ (1 \leq i \leq j) \ .$$

Furthermore, $\Sigma \mid \emptyset \vdash_C c_i \to d_i : C_i \rightsquigarrow C_{i+1}$ is derived by (Ct_Arrow_C), we have $C_i = B_i \to A_i$. Hence, by Lemma I.3, we have

$$\Sigma \mid \emptyset \vdash_S |c_i|_\emptyset : \Sigma(B_{i+1}) \rightsquigarrow \Sigma(B_i), \quad \Sigma \mid \emptyset \vdash_S |d_i|_\emptyset : \Sigma(A_i) \rightsquigarrow \Sigma(A_{i+1}) \quad (1 \leq \forall i \leq n) \ .$$

Moreover, we have

$$\begin{aligned} |c_{j+1} \to d_{j+1}|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |c_n \to d_n|_\emptyset &= |c_{j+1} \to d_{j+1}|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |c_n \to d_n|_\emptyset \\ &= (|c_{j+1}|_\emptyset \to |d_{j+1}|_\emptyset) \,\mathbin{;}\, \cdots \,\mathbin{;}\, (|c_n|_\emptyset \to |d_n|_\emptyset) \\ &= (|c_{j+1}|_\emptyset \to |d_{j+1}|_\emptyset) \,\mathbin{;}\, \cdots \,\mathbin{;}\, (|c_n|_\emptyset \to |d_n|_\emptyset) \\ &= (|c_n|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |c_{j+1}|_\emptyset) \to (|d_{j+1}|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |d_n|_\emptyset) \ . \end{aligned}$$

Furthermore, by applying Lemma I.11 repeatedly, we have

$$|c_{j+1}|_\emptyset = |c_{j+1}|_\emptyset \,\mathbin{;}\, |c_j^I|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |c_1^I|_\emptyset, \quad |d_{j+1}|_\emptyset = |d_1^I|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |d_j^I|_\emptyset \,\mathbin{;}\, |d_{j+1}|_\emptyset \ .$$

Hence,

$$\begin{aligned} s &= |\mathsf{id}_{B_{j+1} \to A_{j+1}}|_\emptyset \,\mathbin{;}\, |c_{j+1} \to d_{j+1}|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |c_n \to d_n|_\emptyset \\ &= |c_{j+1} \to d_{j+1}|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |c_n \to d_n|_\emptyset \\ &= (|c_{j+1}|_\emptyset \to |d_{j+1}|_\emptyset) \,\mathbin{;}\, \cdots \,\mathbin{;}\, (|c_n|_\emptyset \to |d_n|_\emptyset) \\ &= (|c_n|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |c_{j+1}|_\emptyset) \to (|d_{j+1}|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |d_n|_\emptyset) \\ &= (|c_n|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |c_1|_\emptyset) \to (|d_1|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |d_n|_\emptyset) \ . \end{aligned}$$

By case analysis on V_3'.
Case $V_3' = U_3'$ $(\exists U_3')$: By (Bs_CrcId), $\Sigma \mid \emptyset \vdash V_3 \approx U_3'\langle |\mathsf{id}_B|_\emptyset\rangle : B$. Furthermore, now,

$$\Sigma \mid \emptyset \vdash V_4\langle c_1 \to d_1\rangle \cdots \langle c_n \to d_n\rangle \approx U_2'\langle (|c_n|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |c_1|_\emptyset) \to (|d_1|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |d_n|_\emptyset)\rangle : B \to A \ .$$

Therefore, by Lemma I.53, we consider the following two cases.
Case (1) in Lemma I.53: We are given

$$\begin{aligned} \Sigma \rhd M_2\, M_3 \quad &= \quad \Sigma \rhd (\, V_4\langle c_1 \to d_1\rangle \cdots \langle c_n \to d_n\rangle)\ V_3 \\ &\longrightarrow_C^* \Sigma \rhd (\, V_4\ M_5)\langle d_1\rangle \cdots \langle d_n\rangle \ , \end{aligned}$$

and

$$\Sigma \mid \emptyset \vdash M_5 \approx U_3'\langle |\mathsf{id}_B|_\emptyset \fatsemi |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset \rangle : B_1 \quad (\exists M_5)\ .$$

Furthermore, by (R_WRAP_S), we have

$$\begin{array}{rcl}\Sigma \rhd M_2'\, M_3' & \longrightarrow_S^* & \Sigma \rhd V_2'\, V_3' \\ & = & \Sigma \rhd (U_2'\langle (|c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset) \to (|d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset)\rangle)\, U_3' \\ & \longrightarrow_S & \Sigma \rhd (U_2'\,(U_3'\langle |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset\rangle))\langle |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset\rangle\ .\end{array}$$

Therefore, it suffices to show that

$$\Sigma \mid \emptyset \vdash (V_4\, M_5)\langle d_1\rangle \cdots \langle d_n\rangle \approx (U_2'\,(U_3'\langle |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset\rangle))\langle |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset\rangle : A\ .$$

By Lemma I.23, $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash B$. By (CT_ID_C), $\Sigma \mid \emptyset \vdash_C \mathsf{id}_B : B \rightsquigarrow B$. By Lemma I.11, we have $|\mathsf{id}_B|_\emptyset \fatsemi |c_n|_\emptyset = |c_n|_\emptyset$. Therefore,

$$\Sigma \mid \emptyset \vdash M_5 \approx U_3'\langle |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset\rangle : B_1\ .$$

By (BS_APP),

$$\Sigma \mid \emptyset \vdash V_4\, M_5 \approx U_2'\,(U_3'\langle |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset\rangle) : A_1\ .$$

By Lemma I.23, $\Sigma \mid \emptyset \vdash A_1$. By (CT_ID_C), $\Sigma \mid \emptyset \vdash_C \mathsf{id}_{A_1} : A_1 \rightsquigarrow A_1$. By (BS_CRCID) and (BS_CRCMORE) repeatedly,

$$\Sigma \mid \emptyset \vdash (V_4\, M_5)\langle d_1\rangle \cdots \langle d_n\rangle \approx (U_2'\,(U_3'\langle |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset\rangle))\langle |\mathsf{id}_{A_1}|_\emptyset \fatsemi |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset\rangle : A\ .$$

By Lemma I.11, we have $|\mathsf{id}_{A_1}|_\emptyset \fatsemi |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset = |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset$. Therefore, we finish the case.

Case (2) in Lemma I.53: We are given

$$\begin{array}{rcl}\Sigma \rhd M_2\, M_3 & = & \Sigma \rhd (V_4\langle c_1 \to d_1\rangle \cdots \langle c_n \to d_n\rangle)\, V_3 \\ & \longrightarrow_C^* & \Sigma \rhd \mathsf{blame}\, p\ ,\end{array}$$

and

$$|\mathsf{id}_B|_\emptyset \fatsemi |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset = \bot^p \quad (\exists p)\ .$$

By Lemma I.23, $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash B$. By (CT_ID_C), $\Sigma \mid \emptyset \vdash_C \mathsf{id}_B : B \rightsquigarrow B$. By Lemma I.11 and Lemma I.7, $|\mathsf{id}_B|_\emptyset \fatsemi |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset = |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset$. Therefore, we have

$$|c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset = \bot^p\ .$$

Hence, by (R_WRAP_S), (R_FAIL_S), (R_CTXE_S), (R_CTXC_S), (R_BLAMEE_S), (R_BLAMEC_S), we have

$$\begin{array}{rcl}\Sigma \rhd M_2'\, M_3' & \longrightarrow_S^* & \Sigma \rhd V_2'\, V_3' \\ & = & \Sigma \rhd (U_2'\langle (|c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset) \to (|d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset)\rangle)\, U_3' \\ & \longrightarrow_S & \Sigma \rhd (U_2'\,(U_3'\langle |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset\rangle))\langle |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset\rangle \\ & = & \Sigma \rhd (U_2'\,(U_3'\langle \bot^p\rangle))\langle |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset\rangle \\ & \longrightarrow_S & \Sigma \rhd (U_2'\,(\mathsf{blame}\, p))\langle |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset\rangle \\ & \longrightarrow_S & \Sigma \rhd (\mathsf{blame}\, p)\langle |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset\rangle \\ & \longrightarrow_S & \Sigma \rhd \mathsf{blame}\, p\ .\end{array}$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$. By Lemma I.23, we have $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Therefore, by (BS_BLAME), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case $V'_3 = U'_3\langle t\rangle$ ($\exists t, U'_3$): By Lemma I.22, we have $\Sigma \mid \emptyset \vdash_S U'_3\langle t\rangle : \Sigma(B)$. Since this judgment is derived by (T_Crc_S), we have $\Sigma \mid \emptyset \vdash_S t : C' \rightsquigarrow \Sigma(B)$ ($\exists C'$). Therefore, Lemma I.7 implies $t \mathbin{⨟} (|c_n|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |c_1|_\emptyset) = t \mathbin{⨟} |c_n|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |c_1|_\emptyset$. Here, we have

$$\Sigma \mid \emptyset \vdash V_4\langle c_1 \to d_1\rangle \cdots \langle c_n \to d_n\rangle \approx U'_2\langle(|c_n|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |c_1|_\emptyset) \to (|d_1|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |d_n|_\emptyset)\rangle : B \to A,$$
$$\Sigma \mid \emptyset \vdash V_3 \approx U'_3\langle t\rangle : B\ .$$

Therefore, by Lemma I.53, we consider the following two cases.

Case (1) in Lemma I.53: We are given

$$\begin{aligned}\Sigma \triangleright M_2\, M_3 \quad &= \quad \Sigma \triangleright (\, V_4\langle c_1 \to d_1\rangle \cdots \langle c_n \to d_n\rangle)\, V_3 \\ &\longrightarrow^*_C \Sigma \triangleright (\, V_4\, M_5)\langle d_1\rangle \cdots \langle d_n\rangle \quad (\exists M_5)\ ,\end{aligned}$$

and

$$\Sigma \mid \emptyset \vdash M_5 \approx U'_3\langle t \mathbin{⨟} |c_n|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |c_1|_\emptyset\rangle : B_1\ .$$

Hence, by (R_Wrap_S), (R_Merge_S), (R_CtxE_S), (R_CtxC_S), (R_BlameE_S), (R_BlameC_S), we have

$$\begin{aligned}\Sigma \triangleright M'_2\, M'_3 \longrightarrow^*_S\ & \Sigma \triangleright V'_2\, V'_3 \\ =\ \ & \Sigma \triangleright (\, U'_2\langle(|c_n|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |c_1|_\emptyset) \to (|d_1|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |d_n|_\emptyset)\rangle)\, (\, U'_3\langle t\rangle) \\ \longrightarrow_S\ & \Sigma \triangleright (\, U'_2\, (\, U'_3\langle t\rangle\langle|c_n|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |c_1|_\emptyset\rangle))\langle|d_1|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |d_n|_\emptyset\rangle \\ \longrightarrow_S\ & \Sigma \triangleright (\, U'_2\, (\, U'_3\langle t \mathbin{⨟} (|c_n|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |c_1|_\emptyset)\rangle))\langle|d_1|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |d_n|_\emptyset\rangle \\ =\ \ & \Sigma \triangleright (\, U'_2\, (\, U'_3\langle t \mathbin{⨟} |c_n|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |c_1|_\emptyset\rangle))\langle|d_1|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |d_n|_\emptyset\rangle\ .\end{aligned}$$

Therefore, it suffices to show that

$$\Sigma \mid \emptyset \vdash (\, V_4\, M_5)\langle d_1\rangle \cdots \langle d_n\rangle \approx (\, U'_2\, (\, U'_3\langle t \mathbin{⨟} |c_n|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |c_1|_\emptyset\rangle))\langle|d_1|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |d_n|_\emptyset\rangle : A\ .$$

By (Bs_App),

$$\Sigma \mid \emptyset \vdash V_4\, M_5 \approx U'_2\, (\, U'_3\langle t \mathbin{⨟} |c_n|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |c_1|_\emptyset\rangle) : A_1\ .$$

By Lemma E.9, $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A_1$. By (Ct_Id_C), $\Sigma \mid \emptyset \vdash_C \mathsf{id}_{A_1} : A_1 \rightsquigarrow A_1$. By (Bs_CrcId) and (Bs_CrcMore) repeatedly,

$$\Sigma \mid \emptyset \vdash (\, V_4\, M_5)\langle d_1\rangle \cdots \langle d_n\rangle \approx (\, U'_2\, (\, U'_3\langle t \mathbin{⨟} |c_n|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |c_1|_\emptyset\rangle))\langle|\mathsf{id}_{A_1}|_\emptyset \mathbin{⨟} |d_1|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |d_n|_\emptyset\rangle : A\ .$$

Furthermore, by Lemma I.11 and Lemma I.7, we have $|\mathsf{id}_{A_1}|_\emptyset \mathbin{⨟} |d_1|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |d_n|_\emptyset = |d_1|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |d_n|_\emptyset$. Therefore, we finish the case.

Case (2) in Lemma I.53: We are given

$$\begin{aligned}\Sigma \triangleright M_2\, M_3 \quad &= \quad \Sigma \triangleright (\, V_4\langle c_1 \to d_1\rangle \cdots \langle c_n \to d_n\rangle)\, V_3 \\ &\longrightarrow^*_C \Sigma \triangleright \mathsf{blame}\, p\ ,\end{aligned}$$

and

$$t \mathbin{⨟} |c_n|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |c_1|_\emptyset = \bot^p \quad (\exists p)\ .$$

Furthermore, by (R_Wrap_S), (R_Merge_S) (R_Fail_S), (R_CtxE_S), (R_CtxC_S), (R_BlameE_S), (R_BlameC_S), we have

$$
\begin{aligned}
\Sigma \triangleright M_2'\, M_3' &\longrightarrow_S^* \Sigma \triangleright V_2'\; V_3' \\
&= \;\; \Sigma \triangleright (\, U_2' \langle (|c_n|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |c_1|_\emptyset) \to (|d_1|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |d_n|_\emptyset) \rangle)\, (\, U_3' \langle t \rangle) \\
&\longrightarrow_S \Sigma \triangleright (\, U_2' \,(\, U_3' \langle t \rangle \langle |c_n|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |c_1|_\emptyset \rangle)) \langle |d_1|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |d_n|_\emptyset \rangle \\
&\longrightarrow_S \Sigma \triangleright (\, U_2' \,(\, U_3' \langle t \,\mathbin{;}\, (|c_n|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |c_1|_\emptyset) \rangle)) \langle |d_1|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |d_n|_\emptyset \rangle \\
&= \;\; \Sigma \triangleright (\, U_2' \,(\, U_3' \langle t \,\mathbin{;}\, |c_n|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |c_1|_\emptyset \rangle)) \langle |d_1|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |d_n|_\emptyset \rangle \\
&= \;\; \Sigma \triangleright (\, U_2' \,(\, U_3' \langle \bot^p \rangle)) \langle |d_1|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |d_n|_\emptyset \rangle \\
&\longrightarrow_S \Sigma \triangleright (\, U_2' \,(\mathsf{blame}\, p)) \langle |d_1|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |d_n|_\emptyset \rangle \\
&\longrightarrow_S \Sigma \triangleright (\mathsf{blame}\, p) \langle |d_1|_\emptyset \,\mathbin{;}\, \cdots \,\mathbin{;}\, |d_n|_\emptyset \rangle \\
&\longrightarrow_S \Sigma \triangleright \mathsf{blame}\, p \;.
\end{aligned}
$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$. By Lemma I.23, we have $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Therefore, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case $V_2' = U_2'$ $(\exists U_2')$: By Lemma I.58, we have

$$
\begin{gathered}
\Sigma \mid \emptyset \vdash_C c_i^I : B_{i+1} \rightsquigarrow B_i, \quad \Sigma \mid \emptyset \vdash_C d_i^I : A_i \rightsquigarrow A_i \; (1 \le i \le n), \\
\Sigma \mid \emptyset \vdash V_4 \approx U_2' : B_1 \to A_1 \quad (\exists c_1^I, ..., c_n^I, d_1^I, ..., d_n^I, B_1, ..., B_{n+1}, C_1, ..., C_{n+1}) \;,
\end{gathered}
$$

By Lemma I.56, we have

$$
\Sigma \triangleright V_3 \langle c_n^I \rangle \longrightarrow_C^* \Sigma \triangleright V_3'', \quad \Sigma \mid \emptyset \vdash V_3'' \approx V_3' : B_n
$$

Therefore, by (R_Ctx_C),

$$
\begin{aligned}
\Sigma \triangleright M_2\, M_3 &= \;\; \Sigma \triangleright (\, V_4 \langle c_1^I \to d_1^I \rangle \cdots \langle c_n^I \to d_n^I \rangle)\; V_3 \\
&\longrightarrow_C^* \Sigma \triangleright ((\, V_4 \langle c_1^I \to d_1^I \rangle \cdots \langle c_{n-1}^I \to d_{n-1}^I \rangle)\, (\, V_3 \langle c_n^I \rangle)) \langle d_n^I \rangle \\
&\longrightarrow_C^* \Sigma \triangleright ((\, V_4 \langle c_1^I \to d_1^I \rangle \cdots \langle c_{n-1}^I \to d_{n-1}^I \rangle)\; V_3'') \langle d_n^I \rangle \;.
\end{aligned}
$$

Similarly, by applying (R_Ctx_C) repeatedly, there exists V_3''' such that

$$
\begin{aligned}
\Sigma \triangleright M_2\, M_3 &\longrightarrow_C^* \Sigma \triangleright ((\, V_4 \langle c_1^I \to d_1^I \rangle \cdots \langle c_{n-1}^I \to d_{n-1}^I \rangle)\; V_3'') \langle d_n^I \rangle \\
&\longrightarrow_C^* \Sigma \triangleright (\, V_4\; V_3''') \langle d_1^I \rangle \cdots \langle d_n^I \rangle \;,
\end{aligned}
$$

and

$$
\Sigma \mid \emptyset \vdash V_3''' \approx V_3' : B_1 \;.
$$

Furthermore, we have

$$
\begin{aligned}
\Sigma \triangleright M_2'\, M_3' &\longrightarrow_S^* \Sigma \triangleright V_2'\; V_3' \\
&= \;\; \Sigma \triangleright U_2'\; V_3' \;.
\end{aligned}
$$

Therefore, it suffices to show that

$$
\Sigma \mid \emptyset \vdash (\, V_4\; V_3''') \langle d_1^I \rangle \cdots \langle d_n^I \rangle \approx U_2'\; V_3' : A \;.
$$

By (Bs_App),

$$
\Sigma \mid \emptyset \vdash V_4\; V_3''' \approx U_2'\; V_3' : A_1 \;.
$$

By applying (Bs_CrcIdL) repeatedly, we finish the case by

$$
\Sigma \mid \emptyset \vdash (\, V_4\; V_3''') \langle d_1^I \rangle \cdots \langle d_n^I \rangle \approx U_2'\; V_3' : A \;.
$$

Case (R_Blame_C): We are given

$$M_2\, M_3 = E[\mathsf{blame}\, p], \quad M_1 \ = \ \mathsf{blame}\, p, \quad \Sigma_1 \ = \ \Sigma \quad (\exists p, E)\ .$$

Because $M_2\, M_3 = E[\mathsf{blame}\, p]$, it suffices to consider the following two cases.

Case $E = \Box\, M_3$ and $M_2 \ = \ \mathsf{blame}\, p$: Since $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx M_2' : B \to A$, by the case (5), we have

$$\Sigma \triangleright M_2' \ \longrightarrow_S^* \ \Sigma \triangleright \mathsf{blame}\, p\ .$$

Therefore, by (R_CtxE_S) and (R_BlameE_S),

$$\begin{aligned} \Sigma \triangleright M_2'\, M_3' &\longrightarrow_S^* \Sigma \triangleright (\mathsf{blame}\, p)\, M_3' \\ &\longrightarrow_S \Sigma \triangleright \mathsf{blame}\, p\ . \end{aligned}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$. By Lemma I.23, we have $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Therefore, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case $E = V_2\, \Box$ and $M_2 \ = \ V_2$ and $M_3 \ = \ \mathsf{blame}\, p$ $(\exists V_2)$: Since $\Sigma \mid \emptyset \vdash \ V_2 \approx M_2' : B \to A$, by the case (3), we have

$$\Sigma \triangleright M_2' \ \longrightarrow_S^* \ \Sigma \triangleright V_2' \quad (\exists V_2')\ .$$

Furthermore, since $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx M_3' : B$, by applying the case (5), we have

$$\Sigma \triangleright M_3' \ \longrightarrow_S^* \ \Sigma \triangleright \mathsf{blame}\, p\ .$$

Therefore, by (R_CtxE_S) and (R_BlameE_S),

$$\begin{aligned} \Sigma \triangleright M_2'\, M_3' &\longrightarrow_S^* \Sigma \triangleright V_2'\, M_3' \\ &\longrightarrow_S^* \Sigma \triangleright V_2'\, (\mathsf{blame}\, p) \\ &\longrightarrow_S \Sigma \triangleright \mathsf{blame}\, p\ . \end{aligned}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$. By Lemma I.23, we have $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Therefore, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case (R_Ctx_C): We are given

$$M_2\, M_3 = E[M_4], \quad M_1 \ = \ E[M_5], \quad \Sigma \triangleright M_4 \ \longrightarrow_C \ \Sigma_1 \triangleright M_5 \quad (\exists E, M_4, M_5)\ .$$

Because $M_2\, M_3 = E[M_4]$, it suffices to consider the following two cases.

Case $E = \Box\, M_3$ and $M_2 \ = \ M_4$: We have now $\Sigma \mid \emptyset \vdash M_2 \approx M_2' : B \to A$ and $\Sigma \triangleright M_2 \ \longrightarrow_C \ \Sigma_1 \triangleright M_5$. Therefore, by the IH, we have

$$\Sigma_1 \triangleright M_5 \ \longrightarrow_C^* \ \Sigma_2 \triangleright M_6, \quad \Sigma \triangleright M_2' \ \longrightarrow_S^* \ \Sigma_2 \triangleright M_6', \quad \Sigma_2 \mid \emptyset \vdash M_6 \approx M_6' : B \to A \quad (\exists \Sigma_2, M_6, M_6')\ .$$

Hence, by applying (R_Ctx_C) repeatedly, we have

$$\begin{aligned} \Sigma \triangleright M_2\, M_3 &\longrightarrow_C \Sigma_1 \triangleright M_5\, M_3 \\ &\longrightarrow_C^* \Sigma_2 \triangleright M_6\, M_3\ . \end{aligned}$$

Furthermore, by applying (R_CtxE_S) repeatedly, we have

$$\Sigma \triangleright M_2'\, M_3' \ \longrightarrow_S^* \ \Sigma_2 \triangleright M_6'\, M_3'\ .$$

Therefore, it suffices to show that $\Sigma_2 \mid \emptyset \vdash M_6\, M_3 \approx M_6'\, M_3' : A$. By Lemma E.44 and Lemma I.26, we have $\Sigma_2 \mid \emptyset \vdash M_3 \approx M_3' : B$. Hence, by (Bs_App), $\Sigma_2 \mid \emptyset \vdash M_6\, M_3 \approx M_6'\, M_3' : A$.

Case $E = V_2\,\Box$ and $M_2 = V_2$ and $M_3 = M_4$ ($\exists\, V_2$): Since $\Sigma \mid \emptyset \vdash V_2 \approx M_2' : B \to A$, by the case (3), we have

$$\Sigma \triangleright M_2' \longrightarrow_S^* \Sigma \triangleright V_2', \quad \Sigma \mid \emptyset \vdash V_2 \approx V_2' : B \to A \quad (\exists\, V_2')\ .$$

Furthermore, since $\Sigma \mid \emptyset \vdash M_3 \approx M_3' : B$ and $\Sigma \triangleright M_3 \longrightarrow_C \Sigma_1 \triangleright M_5$, by the IH, we have

$$\Sigma_1 \triangleright M_5 \longrightarrow_C^* \Sigma_2 \triangleright M_6, \quad \Sigma \triangleright M_3' \longrightarrow_S^* \Sigma_2 \triangleright M_6', \quad \Sigma_2 \mid \emptyset \vdash M_6 \approx M_6' : B \quad (\exists \Sigma_2, M_6, M_6')\ .$$

Therefore, by applying (R_Ctx_C) repeatedly, we have

$$\begin{aligned}\Sigma \triangleright V_2\, M_3 &\longrightarrow_C \Sigma_1 \triangleright V_2\, M_5 \\ &\longrightarrow_C^* \Sigma_2 \triangleright V_2\, M_6\ .\end{aligned}$$

Furthermore, by applying (R_CtxE_S) repeatedly, we have

$$\begin{aligned}\Sigma \triangleright M_2'\, M_3' &\longrightarrow_S^* \Sigma \triangleright V_2'\, M_3' \\ &\longrightarrow_S^* \Sigma_2 \triangleright V_2'\, M_6'\ .\end{aligned}$$

Therefore, it suffices to show that $\Sigma_2 \mid \emptyset \vdash V_2\, M_6 \approx V_2'\, M_6' : A$. By Lemma E.44 and Lemma I.26, we have $\Sigma_2 \mid \emptyset \vdash V_2 \approx V_2' : B \to A$. Therefore, by (Bs_App), $\Sigma_2 \mid \emptyset \vdash V_2\, M_6 \approx V_2'\, M_6' : A$.

Case (Bs_Tyapp): We are given

$$A = C[X := B], \quad M = M_2\, B, \quad M' = M_2'\, B, \quad \Sigma \mid \emptyset \vdash M_2 \approx M_2' : \forall X.C, \quad \Sigma \mid \emptyset \vdash B \quad (\exists X, B, C, M_2, M_2')\ .$$

By case analysis on the rule applied last to derive $\Sigma \triangleright M_2\, B \longrightarrow_C \Sigma_1 \triangleright M_1$, which is one of the following three rules.

Case (R_Tybeta_C): We are given

$$\begin{aligned}&B = \mathbb{B}, \quad M_2 = (\Lambda Y.(M_3 : A'))\overline{\langle \forall Y.c\rangle}, \quad M_1 = M_3[Y := \alpha]\overline{\langle c[Y := \alpha]\rangle}\langle \mathit{coerce}_\alpha^+(D[Y := \alpha])\rangle, \\ &\Sigma \vdash \overline{\langle \forall Y.c\rangle} : \forall Y.A' \rightsquigarrow \forall Y.D, \quad \Sigma_1 = \Sigma, \alpha := \mathbb{B} \quad (\exists \alpha, Y, A', D, \overline{\langle c\rangle}, M_3)\ .\end{aligned}$$

Lemma I.22 implies $\Sigma \mid \emptyset \vdash_C (\Lambda Y.(M_3 : A'))\overline{\langle \forall Y.c\rangle} : \forall X.C$. Therefore, Lemma E.3 implies $Y = X$ and $D = C$. Since $(\Lambda X.(M_3 : A'))\overline{\langle \forall X.c\rangle}$ is a value, so by the case (3), we have

$$\Sigma \triangleright M_2' \longrightarrow_S^* \Sigma \triangleright V_2', \quad \Sigma \mid \emptyset \vdash (\Lambda X.(M_3 : A'))\overline{\langle \forall X.c\rangle} \approx V_2' : \forall X.C \quad (\exists\, V_2')\ .$$

Therefore, Lemma I.22 implies $\Sigma \mid \emptyset \vdash_S V_2' : \Sigma(\forall X.C)$. Hence, by Lemma E.25, we consider the following two cases.

Case $V_2' = \Lambda X.M_3'$ ($\exists M_3'$): By (R_CtxE_S) and (R_Tybeta_S),

$$\begin{aligned}\Sigma \triangleright M_2'\, \mathbb{B} &\longrightarrow_S^* \Sigma \triangleright V_2'\, \mathbb{B} \\ &= \Sigma \triangleright (\Lambda X.M_3')\, \mathbb{B} \\ &\longrightarrow_S \Sigma, \alpha := \mathbb{B} \triangleright M_3'[X := \alpha]\ .\end{aligned}$$

Therefore, it suffices to show that

$$\begin{aligned}&\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash \\ &\qquad M_3[X := \alpha]\overline{\langle c[X := \alpha]\rangle}\, \langle \mathit{coerce}_\alpha^+(C[X := \alpha])\rangle \\ &\qquad \approx \\ &\qquad M_3'[X := \alpha] \\ &\qquad\qquad\qquad : C[X := \mathbb{B}] \quad (*)\ .\end{aligned}$$

Because $\Sigma \mid \emptyset \vdash (\Lambda X.(M_3 : A'))\overline{\langle \forall X.c\rangle} \approx \Lambda X.M_3' : \forall X.C$ is derived by (Bs_CrcIdL) and (Bs_TyAbs), we have

$$\overline{\langle c\rangle} = \overline{\langle c^I\rangle}, \quad \Sigma \mid \emptyset, X \vdash M_3 \approx M_3' : C \quad (\exists \overline{\langle c^I\rangle})\ .$$

Therefore, $\overline{\langle c[X := \alpha]\rangle} = \overline{\langle c^I[X := \alpha]\rangle}$ is also a sequence of no-op coercions. Furthermore, by Lemma I.8, $\mathit{coerce}^+_\alpha(C[Y := \alpha])$ is a no-op coercion. Moreover, by Lemma I.33, we have $\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash M_3[X := \alpha] \approx M'_3[X := \alpha] : C[X := \alpha]$. Furthermore, Lemma I.23 implies $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash \forall X.C$. Because $\Sigma \mid \emptyset \vdash \forall X.C$ is derived by (Tw_Poly), we have $\Sigma \mid \emptyset, X \vdash C$. Therefore, Lemma E.17 implies

$$\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash_C \mathit{coerce}^+_\alpha(C[X := \alpha]) : C[X := \alpha] \rightsquigarrow C[X := \mathbb{B}] .$$

Hence, we have $(*)$ by applying (Bs_CrcIdL) repeatedly.

Case $V'_2 = U'_2\langle \forall X.s \,,, t\rangle$ $(\exists s, t, U'_2)$: Because $\Sigma \mid \emptyset \vdash_S U'_2\langle \forall X.s,,t\rangle : \Sigma(\forall X.C)$ is derived by (T_Crc_S), we have

$$\Sigma \mid \emptyset \vdash_S U'_2 : C', \quad \Sigma \mid \emptyset \vdash_S \forall X.s \,,, t : C' \rightsquigarrow \forall X.C \ (\exists C').$$

Since $\Sigma \mid \emptyset \vdash_S \forall X.s \,,, t : C' \rightsquigarrow \forall X.C$ is derived by (Ct_All_S), we have $C' = \forall X.C''$ $(\exists C'')$. Therefore, by Lemma E.25, there exists some M'_3 such that $U'_2 = \Lambda X.M'_3$. Hence, by (R_CtxE_S) and (R_TybetaC_S),

$$\begin{aligned}
\Sigma \triangleright M'_2\,\mathbb{B} &\longrightarrow^*_S \Sigma \triangleright V'_2\,\mathbb{B} \\
&= \quad \Sigma \triangleright (U'_2\langle \forall X.s \,,, t\rangle)\,\mathbb{B} \\
&= \quad \Sigma \triangleright ((\Lambda X.M'_3)\langle \forall X.s \,,, t\rangle)\,\mathbb{B} \\
&\longrightarrow_S \Sigma, \alpha := \mathbb{B} \triangleright (M'_3\langle s\rangle)[X := \alpha] \\
&= \quad \Sigma, \alpha := \mathbb{B} \triangleright M'_3[X := \alpha]\langle s[X := \alpha]\rangle .
\end{aligned}$$

Therefore, it suffices to show that

$$\begin{aligned}
&\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash \\
&\qquad M_3[X := \alpha]\overline{\langle c[X := \alpha]\rangle}\,\langle \mathit{coerce}^+_\alpha(C[X := \alpha])\rangle \\
&\qquad \approx \\
&\qquad M'_3[X := \alpha]\langle s[X := \alpha]\rangle \\
&\qquad\qquad\qquad\qquad : C[X := \mathbb{B}] .
\end{aligned}$$

By case analysis on $\overline{\langle \forall X.c\rangle}$.

Case $\overline{\langle \forall X.c\rangle} = \emptyset$: Because $\Sigma \mid \emptyset \vdash \Lambda X.(M_3 : A') \approx (\Lambda X.M'_3)\langle \forall X.s \,,, t\rangle : \forall X.C$ is derived by (Bs_CrcId), we have

$$\forall X.s \,,, t = |\mathsf{id}_{\forall X.C}|_\emptyset, \quad \Sigma \mid \emptyset \vdash \Lambda X.(M_3 : A') \approx \Lambda X.M'_3 : \forall X.C .$$

Therefore,

$$\begin{aligned}
\forall X.s \,,, t &= |\mathsf{id}_{\forall X.C}|_\emptyset \\
&= \forall X.|\mathsf{id}_C|_{\emptyset,X} \,,, |\mathsf{id}_C|_\emptyset .
\end{aligned}$$

Hence, $s = |\mathsf{id}_C|_{\emptyset,X}$ and $t = |\mathsf{id}_C|_\emptyset$. Furthermore, because $\Sigma \mid \emptyset \vdash \Lambda X.(M_3 : A') \approx \Lambda X.M'_3 : \forall X.C$ is derived by (Bs_Tyabs), we have

$$A' = C, \quad \Sigma \mid \emptyset, X \vdash M_3 \approx M'_3 : A' .$$

Moreover, because $\overline{\langle \forall X.c\rangle} = \emptyset$, we have $\overline{\langle c[X := \alpha]\rangle} = \emptyset$. Therefore, it suffices to show that

$$\begin{aligned}
&\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash \\
&\qquad M_3[X := \alpha]\langle \mathit{coerce}^+_\alpha(C[X := \alpha])\rangle \\
&\qquad \approx \\
&\qquad M'_3[X := \alpha]\langle s[X := \alpha]\rangle \\
&\qquad\qquad\qquad\qquad : C[X := \mathbb{B}] .
\end{aligned}$$

Because $\Sigma \mid \emptyset, X \vdash M_3 \approx M_3' : C$ and $\Sigma \mid \emptyset \vdash \mathbb{B}$, Lemma I.33 implies

$$\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash M_3[X := \alpha] \approx M_3'[X := \alpha] : C[X := \alpha] \ .$$

By (Bs_CrcId),

$$\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash M_3[X := \alpha] \approx M_3'[X := \alpha]\langle |\mathsf{id}_{C[X:=\alpha]}|_\emptyset \rangle : C[X := \alpha] \ .$$

Furthermore, Lemma I.23 implies $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash \forall X. C$ and $\Sigma \vdash \emptyset, X$. Moreover, because $\Sigma \mid \emptyset \vdash \forall X. C$ is derived by (Tw_Poly), we have $\Sigma \mid \emptyset, X \vdash C$. Therefore, Lemma E.17 implies

$$\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash_C \ coerce^+_\alpha(C[X := \alpha]) : C[X := \alpha] \rightsquigarrow C[X := \mathbb{B}] \ .$$

Hence, by (Bs_CrcMore),

$$\begin{aligned} &\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash \\ &\qquad M_3[X := \alpha]\langle coerce^+_\alpha(C[X := \alpha]) \rangle \\ &\qquad \approx \\ &\qquad M_3'[X := \alpha]\langle |\mathsf{id}_{C[X:=\alpha]}|_\emptyset \mathbin{;} |coerce^+_\alpha(C[X := \alpha])|_\emptyset \rangle \\ &\qquad\qquad : C[X := \mathbb{B}] \ . \end{aligned}$$

Therefore, it suffices to show that

$$|\mathsf{id}_{C[X:=\alpha]}|_\emptyset \mathbin{;} |coerce^+_\alpha(C[X := \alpha])|_\emptyset = s[X := \alpha] \ .$$

By Lemma I.8, Lemma I.10, Lemma I.11 and Lemma I.32,

$$\begin{aligned} |\mathsf{id}_{C[X:=\alpha]}|_\emptyset \mathbin{;} |coerce^+_\alpha(C[X := \alpha])|_\emptyset &= |\mathsf{id}_{C[X:=\alpha]}|_\emptyset \mathbin{;} |c^I|_\emptyset \ (\exists c^I) \\ &= |\mathsf{id}_{C[X:=\alpha]}|_\emptyset \\ &= |\mathsf{id}_C[X := \alpha]|_\emptyset \\ &= |\mathsf{id}_C|_{\emptyset, X}[X := \alpha] \\ &= s[X := \alpha] \ . \end{aligned}$$

Case $\overline{\langle \forall X. c \rangle} \neq \emptyset$: Let $n > 0$ such that $\overline{\langle \forall X. c \rangle} = \emptyset, \langle \forall X. c_1 \rangle, \ldots, \langle \forall X. c_n \rangle$. Then, because $\Sigma \vdash \emptyset, \langle \forall X. c_1 \rangle, \ldots, \langle \forall X. c_n \rangle : \forall X. A' \rightsquigarrow \forall X. C$, Lemma I.35 implies

$$A_0 = \forall X. A', \quad A_n = \forall X. C, \quad \Sigma \mid \emptyset \vdash_C \ \forall X. c_i : A_{i-1} \rightsquigarrow A_i \quad (\exists A_0, \ldots, A_n)(1 \leq \forall i \leq n) \ .$$

Each $\Sigma \mid \emptyset \vdash_C \ \forall X. c_i : A_{i-1} \rightsquigarrow A_i$ is derived by (Ct_All_C), we have

$$B_0 = A', \quad B_n = C, \quad A_i = \forall X. B_i, \quad \Sigma \mid \emptyset, X \vdash_C \ c_i : B_{i-1} \rightsquigarrow B_i \quad (\exists B_0, \ldots, B_n)(1 \leq \forall i \leq n) \ .$$

Furthermore, $(\Lambda X.(M_3 : A'))\overline{\langle \forall X. c \rangle} = (\Lambda X.(M_3 : A'))\langle \forall X. c_1 \rangle \cdots \langle \forall X. c_n \rangle$. Therefore,

$$\Sigma \mid \emptyset \vdash (\Lambda X.(M_3 : A'))\langle \forall X. c_1 \rangle \cdots \langle \forall X. c_n \rangle \approx (\Lambda X. M_3')\langle \forall X. s \mathbin{,,} t \rangle : \forall X. C \ .$$

Hence, Lemma I.39 implies that there exists nonnegative j $(1 \leq j \leq n)$ such that

$$\begin{gathered} \forall X. s \mathbin{,,} t = |\mathsf{id}_{A_{j+1}}|_\emptyset \mathbin{;} |\forall X. c_{j+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |\forall X. c_n|_\emptyset, \quad \Sigma \mid \emptyset \vdash \Lambda X.(M_3 : A') \approx \Lambda X. M_3' : C_0, \\ \Sigma \mid \emptyset \vdash_C \ \forall X. c_i : C_{i-1} \rightsquigarrow C_i, \quad C_n = \forall X. C \quad (\exists C_0, \ldots, C_n) \ . \end{gathered}$$

Therefore, because $\Sigma \mid \emptyset \vdash_C \ \forall X. c_i : C_{i-1} \rightsquigarrow C_i$ is derived by (Ct_All_C), we have $A_i = C_i = \forall X. B_i$. Moreover, by Lemma I.11, because

$$\begin{aligned} \forall X. s \mathbin{,,} t &= |\mathsf{id}_{A_{j+1}}|_\emptyset \mathbin{;} (|\forall X. c_{j+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |\forall X. c_n|_\emptyset) \\ &= |\forall X. c_{j+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |\forall X. c_n|_\emptyset \\ &= (\forall X. |c_{j+1}|_{\emptyset, X} \mathbin{,,} |c_{j+1}|_\emptyset) \mathbin{;} \cdots \mathbin{;} (\forall X. |c_n|_{\emptyset, X} \mathbin{,,} |c_n|_\emptyset) \\ &= \forall X. (|c_{j+1}|_{\emptyset, X} \mathbin{;} \cdots \mathbin{;} |c_n|_{\emptyset, X}) \mathbin{,,} (|c_{j+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n|_\emptyset) \ , \end{aligned}$$

we have

$$s = |c_{j+1}|_{\emptyset,X} \mathbin{;} \cdots \mathbin{;} |c_n|_{\emptyset,X}, \quad t = |c_{j+1}|_{\emptyset} \mathbin{;} \cdots \mathbin{;} |c_n|_{\emptyset} \ .$$

Therefore, it suffices to show that

$$\begin{aligned}\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash \\ & M_3[X := \alpha]\langle c_1[X := \alpha]\rangle \cdots \langle c_n[X := \alpha]\rangle\, \langle \mathit{coerce}^{+}_{\alpha}(C[X := \alpha])\rangle \\ & \approx \\ & M'_3[X := \alpha]\langle (|c_{j+1}|_{\emptyset,X} \mathbin{;} \cdots \mathbin{;} |c_n|_{\emptyset,X})[X := \alpha]\rangle \\ & \qquad : C[X := \mathbb{B}] \ .\end{aligned}$$

Now, $\Sigma \mid \emptyset, X \vdash_C c_i : B_{i-1} \rightsquigarrow B_i$ $(1 \leq \forall i \leq n)$. Therefore, Lemma I.3 implies

$$\Sigma \mid \emptyset, X \vdash_S |c_i|_{\emptyset,X} : \Sigma(B_{i-1}) \rightsquigarrow \Sigma(B_i) \quad (1 \leq \forall i \leq n) \ .$$

Hence, by Lemmas I.28 and Lemma I.32,

$$\begin{aligned}(|c_{j+1}|_{\emptyset,X} \mathbin{;} \cdots \mathbin{;} |c_n|_{\emptyset,X})[X := \alpha] &= |c_{j+1}|_{\emptyset,X}[X := \alpha] \mathbin{;} \cdots \mathbin{;} |c_n|_{\emptyset,X}[X := \alpha] \\ &= |c_{j+1}[X := \alpha]|_{\emptyset} \mathbin{;} \cdots \mathbin{;} |c_n[X := \alpha]|_{\emptyset} \ .\end{aligned}$$

Therefore, it suffices to show that

$$\begin{aligned}\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash \\ & M_3[X := \alpha]\langle c_1[X := \alpha]\rangle \cdots \langle c_n[X := \alpha]\rangle\, \langle \mathit{coerce}^{+}_{\alpha}(C[X := \alpha])\rangle \\ & \approx \\ & M'_3[X := \alpha]\langle |c_{j+1}[X := \alpha]|_{\emptyset} \mathbin{;} \cdots \mathbin{;} |c_n[X := \alpha]|_{\emptyset}\rangle \\ & \qquad : C[X := \mathbb{B}] \ .\end{aligned}$$

Because $\Sigma \mid \emptyset \vdash \Lambda X.(M_3 : A') \approx \Lambda X.M'_3 : \forall X.A'$ is derived by (Bs_Tyabs), we have $\Sigma \mid \emptyset, X \vdash M_3 \approx M'_3 : A'$. Therefore, Lemma I.33 implies

$$\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash M_3[X := \alpha] \approx M'_3[X := \alpha] : A'[X := \alpha] \ .$$

Hence, by (Bs_CrcId),

$$\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash M_3[X := \alpha] \approx M'_3[X := \alpha]\langle |\mathsf{id}_{A'[X:=\alpha]}|_{\emptyset}\rangle : A'[X := \alpha] \ .$$

Furthermore, by Lemma E.13, we have

$$\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash_C c_i[X := \alpha] : B_{i-1}[X := \alpha] \rightsquigarrow B_i[X := \alpha] \quad (1 \leq \forall i \leq n) \ .$$

Hence, note that $B_0 = A'$, by (Bs_CrcMore),

$$\begin{aligned}\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash \\ & M_3[X := \alpha]\langle c_1[X := \alpha]\rangle \\ & \approx \\ & M'_3[X := \alpha]\langle |\mathsf{id}_{A'[X:=\alpha]}|_{\emptyset} \mathbin{;} |c_1[X := \alpha]|_{\emptyset}\rangle \\ & \qquad : B_1[X := \alpha] \ .\end{aligned}$$

Furthermore, Lemma I.3 implies

$$\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash_S |c_1[X := \alpha]|_{\emptyset} : \Sigma(A'[X := \alpha]) \rightsquigarrow \Sigma(B_1[X := \alpha]) \ .$$

Therefore, Lemma I.11 implies

$$|\mathsf{id}_{A'[X:=\alpha]}|_{\emptyset} \mathbin{;} |c_1[X := \alpha]|_{\emptyset} = |c_1[X := \alpha]|_{\emptyset} \ .$$

Hence,

$$\begin{array}{l}\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash \\ \quad M_3[X := \alpha]\langle c_1[X := \alpha]\rangle \approx M'_3[X := \alpha]\langle |c_1[X := \alpha]|_\emptyset\rangle : B_1[X := \alpha] \ .\end{array}$$

Therefore, note that $B_n = C$, by applying (Bs_CrcMore) repeatedly, we have

$$\begin{array}{l}\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash \\ \qquad M_3[X := \alpha]\langle c_1[X := \alpha]\rangle \cdots \langle c_n[X := \alpha]\rangle \\ \qquad \approx \\ \qquad M'_3[X := \alpha]\langle |c_1[X := \alpha]|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n[X := \alpha]|_\emptyset\rangle \\ \hfill : C[X := \alpha] \ .\end{array}$$

Furthermore, Lemma I.23 implies $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash \forall X.C$. Because $\Sigma \mid \emptyset \vdash \forall X.C$ is derived by (Tw_Poly), we have $\Sigma \mid \emptyset, X \vdash C$. Therefore, by Lemma E.17, we have

$$\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash_C \ \mathit{coerce}^+_\alpha(C[X := \alpha]) : C[X := \alpha] \rightsquigarrow C[X := \mathbb{B}] \ .$$

Hence, by Lemma I.8, $\mathit{coerce}^+_\alpha(C[X := \alpha])$ is a no-op coercion. Therefore, by (Bs_CrcIdL),

$$\begin{array}{l}\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash \\ \qquad M_3[X := \alpha]\langle c_1[X := \alpha]\rangle \cdots \langle c_n[X := \alpha]\rangle \, \langle \mathit{coerce}^+_\alpha(C[X := \alpha])\rangle \\ \qquad \approx \\ \qquad M'_3[X := \alpha]\langle |c_1[X := \alpha]|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n[X := \alpha]|_\emptyset\rangle \\ \hfill : C[X := \mathbb{B}] \ .\end{array}$$

Case (R_TybetaDyn_C): We are given

$$\begin{array}{c}B = \star, \quad M_2 \ = \ (\Lambda Y.(M_3 : A'))\overline{\langle \forall Y.c\rangle}, \quad M_1 \ = \ (M_3\overline{\langle c\rangle})[Y := \star], \\ \Sigma_1 \ = \ \Sigma \quad (\exists Y, A', \overline{\langle c\rangle}, M_3) \ .\end{array}$$

Lemma I.22 implies $\Sigma \mid \emptyset \vdash_C \ (\Lambda Y.(M_3 : A'))\overline{\langle \forall Y.c\rangle} : \forall X.C$. Therefore, Lemma E.3 implies $Y = X$. Since $(\Lambda X.(M_3 : A'))\overline{\langle \forall X.c\rangle}$ is a value, so by the case (3), we have

$$\Sigma \rhd M'_2 \longrightarrow^*_S \Sigma \rhd V'_2, \quad \Sigma \mid \emptyset \vdash (\Lambda X.(M_3 : A'))\overline{\langle \forall X.c\rangle} \approx V'_2 : \forall X.C \quad (\exists V'_2) \ .$$

Therefore, Lemma I.22 implies $\Sigma \mid \emptyset \vdash_S \ V'_2 : \Sigma(\forall X.C)$. Hence, by Lemma E.25, we consider the following two cases.

Case $V'_2 \ = \ \Lambda X.M'_3$ $(\exists M'_3)$: By (R_CtxE_S) and (R_TybetaDyn_S),

$$\begin{array}{rl}\Sigma \rhd M'_2 \star \longrightarrow^*_S & \Sigma \rhd V'_2 \star \\ = & \Sigma \rhd (\Lambda X.M'_3) \star \\ \longrightarrow_S & \Sigma \rhd M'_3[X := \star] \ .\end{array}$$

Therefore, since $(M_3\overline{\langle c\rangle})[X := \star] \ = \ M_3[X := \star]\overline{\langle c[X := \star]\rangle}$, it suffices to show that

$$\Sigma \mid \emptyset \vdash M_3[X := \star]\overline{\langle c[X := \star]\rangle} \approx M'_3[X := \star] : C[X := \star] \quad (*) \ .$$

Because $\Sigma \mid \emptyset \vdash (\Lambda X.(M_3 : A'))\overline{\langle \forall X.c\rangle} \approx \Lambda X.M'_3 : \forall X.C$ is derived by (Bs_CrcIdL) and (Bs_TyAbs), we have

$$\begin{array}{c}\overline{\langle c\rangle} = \overline{\langle c^I\rangle}, \quad \Sigma \mid \emptyset \vdash_C \ \forall X.c^I_i : C_i \rightsquigarrow C_{i+1}, \quad C_{n+1} = C \\ \Sigma \mid \emptyset, X \vdash M_3 \approx M'_3 : C_1, \quad (\exists \overline{\langle c^I\rangle}, C_i) \ (n \geq i \geq 1) \ .\end{array}$$

Therefore, $\overline{\langle c[X := \star]\rangle} = \overline{\langle c^I[X := \star]\rangle}$ is also a sequence of no-op coercions. Moreover, by Lemma I.33, we have $\Sigma \mid \emptyset \vdash M_3[X := \star] \approx M'_3[X := \star] : C_1[X := \star]$. Hence, we have $(*)$ by applying (Bs_CrcIdL) repeatedly.

Case $V'_2 = U'_2\langle\forall X.s ,, t\rangle$ $(\exists s, t, U'_2)$: Because $\Sigma \mid \emptyset \vdash_S U'_2\langle\forall X.s ,, t\rangle : \forall X.C$ is derived by (T_CRC_S), we have $\Sigma \mid \emptyset \vdash_S U'_2 : \forall X.C'$ $(\exists C')$. Therefore, by Lemma E.25, there exists some M'_3 such that $U'_2 = \Lambda X.M'_3$. Hence, by (R_CTXE_S) and (R_TYBETADYNC_S),

$$\begin{aligned}\Sigma \triangleright M'_2 \star \longrightarrow^*_S\ & \Sigma \triangleright V'_2 \star \\ =\ & \Sigma \triangleright (U'_2\langle\forall X.s ,, t\rangle) \star \\ =\ & \Sigma \triangleright ((\Lambda X.M'_3)\langle\forall X.s ,, t\rangle) \star \\ \longrightarrow_S\ & \Sigma \triangleright M'_3[X := \star]\langle t\rangle\ .\end{aligned}$$

Therefore, since $(M_3\overline{\langle c\rangle})[X := \star] = M_3[X := \star]\overline{\langle c[X := \star]\rangle}$, it suffices to show that

$$\Sigma \mid \emptyset \vdash M_3[X := \star]\overline{\langle c[X := \star]\rangle} \approx M'_3[X := \star]\langle t\rangle : C[X := \star]$$

By case analysis on $\overline{\langle\forall X.c\rangle}$.

Case $\overline{\langle\forall X.c\rangle} = \emptyset$: Because $\Sigma \mid \emptyset \vdash \Lambda X.(M_3 : A') \approx (\Lambda X.M'_3)\langle\forall X.s ,, t\rangle : \forall X.C$ is derived by (BS_CRCID), we have

$$\forall X.s ,, t = |\mathsf{id}_{\forall X.C}|_\emptyset, \quad \Sigma \mid \emptyset \vdash \Lambda X.(M_3 : A') \approx \Lambda X.M'_3 : \forall X.C\ .$$

Furthermore,

$$\begin{aligned}\forall X.s ,, t &= |\mathsf{id}_{\forall X.C}|_\emptyset \\ &= \forall X.|\mathsf{id}_C|_{\emptyset,X} ,, |\mathsf{id}_C|_\emptyset\end{aligned}$$

.

Therefore, we have $s = |\mathsf{id}_C|_{\emptyset,X}$ and $t = |\mathsf{id}_C|_\emptyset$. Furthermore, because $\Sigma \mid \emptyset \vdash \Lambda X.(M_3 : A') \approx \Lambda X.M'_3 : \forall X.C$ is derived by (BS_TYABS), we have

$$A' = C, \quad \Sigma \mid \emptyset, X \vdash M_3 \approx M'_3 : C\ .$$

Moreover, because $\overline{\langle\forall X.c\rangle} = \emptyset$, we have $\overline{\langle c[X := \star]\rangle} = \emptyset$. Therefore, it suffices to show that

$$\Sigma \mid \emptyset \vdash M_3[X := \star] \approx M'_3[X := \star]\langle|\mathsf{id}_C|_\emptyset\rangle : C[X := \star]\ .$$

Because $\Sigma \mid \emptyset, X \vdash M_3 \approx M'_3 : C$, Lemma I.33 implies

$$\Sigma \mid \emptyset \vdash M_3[X := \star] \approx M'_3[X := \star] : C[X := \star]\ .$$

By (BS_CRCID),

$$\Sigma \mid \emptyset \vdash M_3[X := \star] \approx M'_3[X := \star]\langle|\mathsf{id}_{C[X:=\star]}|_\emptyset\rangle : C[X := \star]\ .$$

Also, since $|\mathsf{id}_{C[X:=\star]}|_\emptyset = |\mathsf{id}_C[X := \star]|_\emptyset = |\mathsf{id}_C|_\emptyset$ by Lemma I.30, we finish the case.

Case $\overline{\langle\forall X.c\rangle} \neq \emptyset$: Let $n > 0$ such that $\overline{\langle\forall X.c\rangle} = \emptyset, \langle\forall X.c_1\rangle, \ldots, \langle\forall X.c_n\rangle$. Then, because $\Sigma \vdash \emptyset, \langle\forall X.c_1\rangle, \ldots, \langle\forall X.c_n\rangle : \forall X.A' \rightsquigarrow \forall X.C$, Lemma I.35 implies

$$A_0 = \forall X.A', \quad A_n = \forall X.C, \quad \Sigma \mid \emptyset \vdash_C \forall X.c_i : A_{i-1} \rightsquigarrow A_i \quad (\exists A_0, \ldots, A_n)(1 \leq \forall i \leq n)\ .$$

Each $\Sigma \mid \emptyset \vdash_C \forall X.c_i : A_{i-1} \rightsquigarrow A_i$ is derived by (CT_ALL_C), we have

$$B_0 = A', \quad B_n = C, \quad A_i = \forall X.B_i, \quad \Sigma \mid \emptyset, X \vdash_C c_i : B_{i-1} \rightsquigarrow B_i \quad (\exists B_0, \ldots, B_n)(1 \leq \forall i \leq n)\ .$$

Furthermore, $(\Lambda X.(M_3 : A'))\overline{\langle\forall X.c\rangle} = (\Lambda X.(M_3 : A'))\langle\forall X.c_1\rangle \cdots \langle\forall X.c_n\rangle$. Therefore,

$$\Sigma \mid \emptyset \vdash (\Lambda X.(M_3 : A'))\langle\forall X.c_1\rangle \cdots \langle\forall X.c_n\rangle \approx (\Lambda X.M'_3)\langle\forall X.s\rangle : \forall X.C\ .$$

Hence, Lemma I.39 implies that there exists nonnegative j $(1 \leq j \leq n)$ such that

$$\forall X.s \;,, \; t = |\mathsf{id}_{A_{j+1}}|_\emptyset \mathbin{;} |\forall X.c_{j+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |\forall X.c_n|_\emptyset, \quad \Sigma \mid \emptyset \vdash \Lambda X.(M_3 : A') \approx \Lambda X.M_3' : \forall X.C',$$
$$\Sigma \mid \emptyset \vdash_C \forall X.c_n : C' \rightsquigarrow \forall X.C \quad (\exists C') \; .$$

Moreover, by Lemma I.11, because

$$\begin{aligned}
\forall X.s \;,, \; t &= |\mathsf{id}_{A_{j+1}}|_\emptyset \mathbin{;} (|\forall X.c_{j+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |\forall X.c_n|_\emptyset) \\
&= |\forall X.c_{j+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |\forall X.c_n|_\emptyset \\
&= (\forall X.|c_{j+1}|_{\emptyset,X} \;,, \; |c_{j+1}|_\emptyset) \mathbin{;} \cdots \mathbin{;} (\forall X.|c_n|_{\emptyset,X} \;,, \; |c_n|_\emptyset) \\
&= \forall X.(|c_{j+1}|_{\emptyset,X} \mathbin{;} \cdots \mathbin{;} |c_n|_{\emptyset,X}) \;,, \; (|c_{j+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n|_\emptyset) \; ,
\end{aligned}$$

we have

$$s = |c_{j+1}|_{\emptyset,X} \mathbin{;} \cdots \mathbin{;} |c_n|_{\emptyset,X}, \quad t = |c_{j+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n|_\emptyset \; .$$

Therefore, it suffices to show that

$$\begin{aligned}
&\Sigma \mid \emptyset \vdash \\
&\qquad M_3[X := \star]\langle c_1[X := \star]\rangle \cdots \langle c_n[X := \star]\rangle \\
&\qquad \approx \\
&\qquad M_3'[X := \star]\langle |c_{j+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n|_\emptyset\rangle \\
&\qquad\qquad\qquad : C[X := \star] \; .
\end{aligned}$$

Now, $\Sigma \mid \emptyset, X \vdash_C c_i : B_{i-1} \rightsquigarrow B_i$ $(1 \leq \forall i \leq n)$. Therefore, by Lemma I.3, we have

$$\Sigma \mid \emptyset \vdash_S |c_i|_\emptyset : \Sigma(B_{i-1}[X := \star]) \rightsquigarrow \Sigma(B_i[X := \star]) \quad (1 \leq \forall i \leq n) \; .$$

Hence, by Lemma I.30,

$$|c_1|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n|_\emptyset = |c_1[X := \star]|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n[X := \star]|_\emptyset \; .$$

Therefore, it suffices to show that

$$\begin{aligned}
&\Sigma \mid \emptyset \vdash \\
&\qquad M_3[X := \star]\langle c_1[X := \star]\rangle \cdots \langle c_n[X := \star]\rangle \\
&\qquad \approx \\
&\qquad M_3'[X := \star]\langle |c_1[X := \star]|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n[X := \star]|_\emptyset\rangle \\
&\qquad\qquad\qquad : C[X := \star] \; .
\end{aligned}$$

Because $\Sigma \mid \emptyset \vdash \Lambda X.(M_3 : A') \approx \Lambda X.M_3' : \forall X.C'$ is derived by (Bs_TyAbs), we have

$$A' = C', \quad \Sigma \mid \emptyset, X \vdash M_3 \approx M_3' : C' \; .$$

Therefore, Lemma I.33 implies

$$\Sigma \mid \emptyset \vdash M_3[X := \star] \approx M_3'[X := \star] : C'[X := \star] \; .$$

Hence, by (Bs_CrcId),

$$\Sigma \mid \emptyset \vdash M_3[X := \star] \approx M_3'[X := \star]\langle |\mathsf{id}_{C'[X:=\star]}|_\emptyset\rangle : C'[X := \star] \; .$$

Therefore, Lemma E.15 implies

$$\Sigma \mid \emptyset \vdash_C c_i[X := \star] : B_{i-1}[X := \star] \rightsquigarrow B_i[X := \star] \quad (1 \leq \forall i \leq n) \; .$$

Hence, note that $B_0 = A' = C'$, by (Bs_CrcMore),

$$\Sigma \mid \emptyset \vdash \\ M_3[X := \star]\langle c_1[X := \star]\rangle \\ \approx \\ M'_3[X := \star]\langle |\mathsf{id}_{C'[X:=\star]}|_\emptyset \mathbin{;} |c_1[X := \star]|_\emptyset\rangle \\ : B_1[X := \star] \ .$$

Furthermore, Lemma I.3 implies

$$\Sigma \mid \emptyset \vdash_S |c_1[X := \star]|_\emptyset : \Sigma(C'[X := \star]) \rightsquigarrow \Sigma(B_1[X := \star]) \ .$$

Therefore, Lemma I.11 implies

$$|\mathsf{id}_{C'[X:=\star]}|_\emptyset \mathbin{;} |c_1[X := \star]|_\emptyset = |c_1[X := \star]|_\emptyset \ .$$

Hence,

$$\Sigma \mid \emptyset \vdash M_3[X := \star]\langle c_1[X := \star]\rangle \approx M'_3[X := \star]\langle |c_1[X := \star]|_\emptyset\rangle : B_1[X := \star] \ .$$

Therefore, note that $B_n = C$, by applying (Bs_CrcMore) repeatedly, we have

$$\Sigma \mid \emptyset \vdash \\ M_3[X := \star]\langle c_1[X := \star]\rangle \cdots \langle c_n[X := \star]\rangle \\ \approx \\ M'_3[X := \star]\langle |c_1[X := \star]|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n[X := \star]|_\emptyset\rangle \\ : C[X := \star] \ .$$

Case (R_Blame_C): We are given

$$M_2 = \mathsf{blame}\, p, \quad M_1 = \mathsf{blame}\, p, \quad \Sigma_1 = \Sigma \quad (\exists p) \ .$$

Because $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx M'_2 : \forall X.C$, the case (5) implies

$$\Sigma \triangleright M'_2 \longrightarrow^*_S \Sigma \triangleright \mathsf{blame}\, p \ .$$

Therefore, by (R_CtxE_S) and (R_BlameE_S),

$$\Sigma \triangleright M'_2\, B \longrightarrow^*_S \Sigma \triangleright (\mathsf{blame}\, p)\, B \\ \longrightarrow_S \Sigma \triangleright \mathsf{blame}\, p \ .$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : C[X := B]$. Lemma I.23 implies $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash C[X := B]$. Therefore, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : C[X := B]$.

Case (R_Ctx_C): We are given

$$M_1 = M_3\, B, \quad \Sigma \triangleright M_2 \longrightarrow_C \Sigma_1 \triangleright M_3 \quad (\exists M_3) \ .$$

Therefore, because $\Sigma \mid \emptyset \vdash M_2 \approx M'_2 : \forall X.C$ and $\Sigma \triangleright M_2 \longrightarrow_C \Sigma_1 \triangleright M_3$, the IH implies

$$\Sigma_1 \triangleright M_3 \longrightarrow^*_C \Sigma_2 \triangleright M_4, \quad \Sigma \triangleright M'_2 \longrightarrow^*_S \Sigma_2 \triangleright M'_4, \quad \Sigma_2 \mid \emptyset \vdash M_4 \approx M'_4 : \forall X.C \quad (\exists \Sigma_2, M_4, M'_4) \ .$$

Therefore, by applying (R_Ctx_C) repeatedly, we have

$$\Sigma \triangleright M_2\, B \longrightarrow_C \Sigma_1 \triangleright M_3\, B \\ \longrightarrow^*_C \Sigma_2 \triangleright M_4\, B \ .$$

Furthermore, by applying (R_CtxE_S), we have

$$\Sigma \triangleright M'_2\, B \longrightarrow^*_S \Sigma_2 \triangleright M'_4\, B \ .$$

Therefore, it suffices to show that $\Sigma_2 \mid \emptyset \vdash M_4\, B \approx M'_4\, B : C[X := B]$. Since $\Sigma \subseteq \Sigma_2$, by Lemma D.2, we have $\Sigma_2 \mid \emptyset \vdash B$. Hence, by (Bs_Tyapp), $\Sigma_2 \mid \emptyset \vdash M_4\, B \approx M'_4\, B : C[X := B]$.

Case (Bs_Crc): We are given

$$M = M_2\langle c\rangle, \quad M' = M_2'\langle|c|_\emptyset\rangle, \quad \Sigma \mid \emptyset \vdash M_2 \approx M_2' : B, \quad \Sigma \mid \emptyset \vdash_C c : B \rightsquigarrow A \quad (\exists B, c, M_2, M_2') .$$

By Lemma I.3, $\Sigma \mid \emptyset \vdash_S |c|_\emptyset : \Sigma(B) \rightsquigarrow \Sigma(A)$. We perform case analysis on the rule applied last to derive $\Sigma \triangleright M_2\langle c\rangle \longrightarrow_C \Sigma_1 \triangleright M_1$, which is one of the following rules.

Case (R_Id_C): We are given

$$M_2 = M_1 = V_2, \quad c = \mathsf{id}_{A'}, \quad \Sigma_1 = \Sigma \quad (\exists A', V_2) .$$

Since $\Sigma \mid \emptyset \vdash_C \mathsf{id}_{A'} : B \rightsquigarrow A$ is derived by (Ct_Id_C), we have $A' = B = A$. By (Bs_CrcId), $\Sigma \mid \emptyset \vdash V_2 \approx M_2'\langle|\mathsf{id}_A|_\emptyset\rangle : A$. Hence, by the case (3), we have

$$\Sigma \triangleright M_2'\langle|\mathsf{id}_A|_\emptyset\rangle \longrightarrow_S^* \Sigma \triangleright V_2', \quad \Sigma \mid \emptyset \vdash V_2 \approx V_2' : A \quad (\exists V_2') .$$

Therefore,

$$\begin{aligned} \Sigma \triangleright M_2'\langle|c|_\emptyset\rangle &= \Sigma \triangleright M_2'\langle|\mathsf{id}_A|_\emptyset\rangle \\ &\longrightarrow_S^* \Sigma \triangleright V_2' . \end{aligned}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash V_2 \approx V_2' : A$, which holds already.

Case (R_Fail_C): We are given

$$M_2 = V_2, \quad c = \bot^p_{B' \rightsquigarrow A'}, \quad M_1 = \mathsf{blame}\, p, \quad \Sigma_1 = \Sigma \quad (\exists p, A', B', V_2) .$$

Therefore, because $\Sigma \mid \emptyset \vdash V_2 \approx M_2' : B$, by the case (3), we have

$$\Sigma \triangleright M_2' \longrightarrow_S^* \Sigma \triangleright V_2', \quad \Sigma \mid \emptyset \vdash V_2 \approx V_2' : B \quad (\exists V_2') .$$

Furthermore, by (R_Fail_S),

$$\begin{aligned} \Sigma \triangleright V_2'\langle|\bot^p_{B' \rightsquigarrow A'}|_\emptyset\rangle &= \Sigma \triangleright V_2'\langle\bot^p\rangle \\ &\longrightarrow_S \Sigma \triangleright \mathsf{blame}\, p . \end{aligned}$$

Hence, by Lemma I.17, we have

$$\Sigma \triangleright M_2'\langle|\bot^p_{B' \rightsquigarrow A'}|_\emptyset\rangle \longrightarrow_S^* \Sigma \triangleright \mathsf{blame}\, p .$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$. By Lemma I.23, we have $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Therefore, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case (R_Collapse_C): We are given

$$M_2 = V_2\langle G!\rangle, \quad c = G?^p, \quad M_1 = V_2, \quad \Sigma_1 = \Sigma \quad (\exists p, G, V_2) .$$

Since $\Sigma \mid \emptyset \vdash_C G?^p : B \rightsquigarrow A$ is derived by (Ct_Proj_C), we have $B = \star$ and $A = G$. Therefore, since $\Sigma \mid \emptyset \vdash V_2\langle G!\rangle \approx M_2' : \star$ and $V_2\langle G_1!\rangle$ is a value, by the case (3), we have

$$\Sigma \triangleright M_2' \longrightarrow_S^* \Sigma \triangleright V_2', \quad \Sigma \mid \emptyset \vdash V_2\langle G!\rangle \approx V_2' : \star \quad (\exists V_2') .$$

Since $\Sigma(\star) = \star$, by Lemma I.22, we have $\Sigma \mid \emptyset \vdash_S V_2' : \star$. Therefore, by Lemma E.25, we have

$$V_2' = U_2'\langle h\, ;\, H!\rangle \quad (\exists H, h, U_2') .$$

Hence, $\Sigma \mid \emptyset \vdash V_2\langle G!\rangle \approx U_2'\langle h\, ;\, H!\rangle : \star$. We perform case analysis on the last rule to derive $\Sigma \mid \emptyset \vdash V_2\langle G!\rangle \approx U_2'\langle h\, ;\, H!\rangle : \star$, which is either of (Bs_CrcId), (Bs_CrcIdL), (Bs_Crc), or (Bs_CrcMore).

Case (Bs_CrcId): Contradictory because there is no identity coercion id_A such that $|\mathsf{id}_A|_\emptyset = h\, ; H!$.

Case (Bs_CrcIdL): Contradictory because $G!$ is not a no-op coercion.

Case (Bs_Crc): We are given

$$h\,;H! = |G!|_{\emptyset}, \quad \Sigma \mid \emptyset \vdash V_2 \approx U'_2 : D, \quad \Sigma \mid \emptyset \vdash_C G! : D \rightsquigarrow \star \quad (\exists D)\ .$$

Since $\Sigma \mid \emptyset \vdash_C G! : D \rightsquigarrow \star$ is derived by (Ct_Inj_C), we have

$$D = G, \quad \vdash \Sigma, \quad \Sigma \vdash \emptyset, \quad \Sigma \mid \emptyset \vdash G\ .$$

Also, G does not include any free type variables. Therefore,

$$h\,;H! = |G!|_{\emptyset} = |\mathsf{id}_G|_{\emptyset} \mathbin{;} G!, \quad |G?^p|_{\emptyset} = G?^p\,;|\mathsf{id}_G|_{\emptyset}\ .$$

Hence, $H = G$ and $h = |\mathsf{id}_G|_{\emptyset}$. Furthermore, by Lemma I.11, $|\mathsf{id}_G|_{\emptyset} \mathbin{;} |\mathsf{id}_G|_{\emptyset} = |\mathsf{id}_G|_{\emptyset}$. Therefore,

$$\begin{aligned}(h\,;H!) \mathbin{;} |G?^p|_{\emptyset} &= (|\mathsf{id}_G|_{\emptyset}\,;G!) \mathbin{;} |G?^p|_{\emptyset}\\ &= (|\mathsf{id}_G|_{\emptyset}\,;G!) \mathbin{;} (G?^p\,;|\mathsf{id}_G|_{\emptyset})\\ &= |\mathsf{id}_G|_{\emptyset} \mathbin{;} |\mathsf{id}_G|_{\emptyset}\\ &= |\mathsf{id}_G|_{\emptyset}\ .\end{aligned}$$

Therefore, by (R_Merge_S), we have

$$\begin{aligned}\Sigma \triangleright V'_2\langle |G?^p|_{\emptyset}\rangle &= \Sigma \triangleright U'_2\langle h\,;H!\rangle\langle |G?^p|_{\emptyset}\rangle\\ &\longrightarrow_S \Sigma \triangleright U'_2\langle (h\,;H!) \mathbin{;} |G?^p|_{\emptyset}\rangle\\ &= \Sigma \triangleright U'_2\langle |\mathsf{id}_G|_{\emptyset}\rangle\ .\end{aligned}$$

Also, by Lemma I.23 and $B = \star$, $\Sigma \mid \emptyset \vdash_S M'_2 : \Sigma(\star)$. By (Ct_Proj_C), we have $\Sigma \mid \emptyset \vdash_C G?^p : \star \rightsquigarrow G$. By Lemma I.3, we have $\Sigma \mid \emptyset \vdash_S |G?^p|_{\emptyset} : \Sigma(\star) \rightsquigarrow \Sigma(G)$. Therefore, by Lemma I.14,

$$\Sigma \triangleright M'_2\langle |G?^p|_{\emptyset}\rangle \longrightarrow^*_S \Sigma \triangleright U'_2\langle |\mathsf{id}_G|_{\emptyset}\rangle\ .$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash V_2 \approx U'_2\langle |\mathsf{id}_G|_{\emptyset}\rangle : G$, which holds by (Bs_CrcId).

Case (Bs_CrcMore): We are given

$$h\,;H! = s' \mathbin{;} |G!|_{\emptyset}, \quad \Sigma \mid \emptyset \vdash V_2 \approx U'_2\langle s'\rangle : D, \quad \Sigma \mid \emptyset \vdash_C G! : D \rightsquigarrow \star \quad (\exists s', D)\ .$$

Since $\Sigma \mid \emptyset \vdash_C G! : D \rightsquigarrow \star$ is derived by (Ct_Inj_C), we have $D = G$. Furthermore, by Lemma I.46, there exists an intermediate coercion j such that $s' = j$. Hence, by Lemma I.22, we have $\Sigma \mid \emptyset \vdash_S U'_2\langle j\rangle : \Sigma(G)$. Since this judgment is derived (T_Crc_S), we have $\Sigma \mid \emptyset \vdash_S j : D' \rightsquigarrow \Sigma(G)$ $(\exists D')$. Therefore, by Lemma I.11, we have $j \mathbin{;} |\mathsf{id}_G|_{\emptyset} = j$. Therefore,

$$\begin{aligned}h\,;H! &= s' \mathbin{;} |G!|_{\emptyset}\\ &= j \mathbin{;} |G!|_{\emptyset}\\ &= j \mathbin{;} |G!|_{\emptyset}\\ &= j \mathbin{;} (|\mathsf{id}_G|_{\emptyset}\,;G!)\\ &= (j \mathbin{;} |\mathsf{id}_G|_{\emptyset})\,;G!\\ &= (j \mathbin{;} |\mathsf{id}_G|_{\emptyset})\,;G!\\ &= j\,;G!\ .\end{aligned}$$

Hence, $H = G$ and $h = j$. Furthermore,

$$\begin{aligned}(h\,;G!) \mathbin{;} |G?^p|_{\emptyset} &= (h\,;G!) \mathbin{;} |G?^p|_{\emptyset}\\ &= (h\,;G!) \mathbin{;} (G?^p\,;|\mathsf{id}_G|_{\emptyset})\\ &= h \mathbin{;} |\mathsf{id}_G|_{\emptyset}\\ &= h \mathbin{;} |\mathsf{id}_G|_{\emptyset}\\ &= h\ .\end{aligned}$$

Therefore, by (R_Merge_S), we have

$$\begin{array}{rl}\Sigma \triangleright V_2'\langle |G?^p|_\emptyset\rangle & = \ \Sigma \triangleright U_2'\langle h\,;\,G!\rangle\langle |G?^p|_\emptyset\rangle \\ & = \ \Sigma \triangleright U_2'\langle h\,;\,G!\rangle\langle |G?^p|_\emptyset\rangle \\ & = \ \Sigma \triangleright U_2'\langle h\,;\,G!\rangle\langle G?^p\,;\,|\mathsf{id}_G|_\emptyset\rangle \\ & = \ \Sigma \triangleright U_2'\langle h\,;\,G!\rangle\langle G?^p\,;\,|\mathsf{id}_G|_\emptyset\rangle \\ & \longrightarrow_S \Sigma \triangleright U_2'\langle (h\,;\,G!) \fatsemi (G?^p\,;\,|\mathsf{id}_G|_\emptyset)\rangle \\ & \longrightarrow_S \Sigma \triangleright U_2'\langle h \fatsemi |\mathsf{id}_G|_\emptyset\rangle \\ & = \ \Sigma \triangleright U_2'\langle h\rangle\ .\end{array}$$

By case analysis on h.

Case $h = \mathsf{id}$: We have $\Sigma \mid \emptyset \vdash_S \mathsf{id} : D' \rightsquigarrow \Sigma(G)$. Since this judgment is derived by (Ct_Id_S), we have $D' = \Sigma(G)$. Therefore, by (R_Id_S) and what have been proven, we have

$$\begin{array}{rl}\Sigma \triangleright V_2'\langle |G?^p|_\emptyset\rangle & \longrightarrow_S \Sigma \triangleright U_2'\langle h\rangle \\ & = \ \Sigma \triangleright U_2'\langle \mathsf{id}\rangle \\ & \longrightarrow_S \Sigma \triangleright U_2'\ .\end{array}$$

Therefore, by Lemma I.18, we have

$$\begin{array}{rl}\Sigma \triangleright M_2'\langle |c|_\emptyset\rangle & = \ \Sigma \triangleright M_2'\langle |G?^p|_\emptyset\rangle \\ & \longrightarrow_S^* \Sigma \triangleright U_2'\ .\end{array}$$

It suffices to show that $\Sigma \mid \emptyset \vdash V_2 \approx U_2' : G$. Since $\Sigma \mid \emptyset \vdash V_2\langle G!\rangle \approx U_2'\langle \mathsf{id}\,;\,G!\rangle : \star$, by Lemma I.55, we have $\Sigma \mid \emptyset \vdash V_2 \approx U_2' : G$.

Otherwise: $U_2'\langle h\rangle$ is a value. Therefore, by Lemma I.18, we have

$$\begin{array}{rl}\Sigma \triangleright M_2'\langle |c|_\emptyset\rangle & = \ \Sigma \triangleright M_2'\langle |G?^p|_\emptyset\rangle \\ & \longrightarrow_S^* \Sigma \triangleright U_2'\langle h\rangle\ .\end{array}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash V_2 \approx U_2'\langle h\rangle : G$, which already holds.

Case (R_Conflict_C): We are given

$$M_2 = V_2\langle G!\rangle, \quad c = H?^p, \quad M_1 = \mathsf{blame}\, p, \quad G \neq H, \quad \Sigma_1 = \Sigma \quad (\exists p, G, H, V_2)\ .$$

Because $\Sigma \mid \emptyset \vdash_C H?^p : B \rightsquigarrow A$ is derived by (Ct_Proj_C), we have $B = \star$ and $A = H$. Therefore, $\Sigma \mid \emptyset \vdash V_2\langle G!\rangle \approx M_2' : \star$, so by the case (3), we have

$$\Sigma \triangleright M_2' \longrightarrow_S^* \Sigma \triangleright V_2', \quad \Sigma \mid \emptyset \vdash V_2\langle G!\rangle \approx V_2' : \star \quad (\exists V_2')\ .$$

By Lemma I.22 and $\Sigma(\star) = \star$, we have $\Sigma \mid \emptyset \vdash_S V_2' : \star$. Therefore, by Lemma E.25, we have

$$V_2' = U_2'\langle h_2\,;\,H_2!\rangle\ (\exists h_2, H_2, U_2')\ .$$

We perform case analysis on whether $H_2 = H$ or not.

Case $H_2 = H$: We perform case analysis on the last rule to derive $\Sigma \mid \emptyset \vdash V_2\langle G!\rangle \approx U_2'\langle h_2\,;\,H_2!\rangle : \star$, which is either of , which is either of (Bs_CrcId), (Bs_CrcIdL), (Bs_Crc), or (Bs_CrcMore).

Case (Bs_CrcId): Contradictory because there is no identity coercion id_A such that $|\mathsf{id}_A|_\emptyset = h\,;\,H!$.

Case (Bs_CrcIdL): Contradictory because $G!$ is not a no-op coercion.

Case (Bs_Crc): We are given

$$h_2\,;\,H_2! = |G!|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_2 \approx U_2' : D, \quad \Sigma \mid \emptyset \vdash_C G! : D \rightsquigarrow \star \quad (\exists D)\ .$$

Since $\Sigma \mid \emptyset \vdash_C G! : D \rightsquigarrow \star$ is derived by (Ct_Inj_C), we have $D = G$. Also, G does not include any type variables. Therefore, $|G!|_\emptyset = |\mathsf{id}_G|_\emptyset \fatsemi G!$. Hence, by Lemma I.22, we have $\Sigma \mid \emptyset \vdash_S U_2' : \Sigma(G)$. Therefore,

$$\begin{aligned} h_2 \, ; H_2! &= |G!|_\emptyset \\ &= |\mathsf{id}_G|_\emptyset \fatsemi G! \ . \end{aligned}$$

Hence, $H = H_2 = G$, which is contradictory to $G \neq H$.

Case (Bs_CrcMore): We are given

$$h_2 \, ; H_2! = s' \fatsemi |G!|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_2 \approx U_2'\langle s' \rangle : D, \quad \Sigma \mid \emptyset \vdash_C G! : D \rightsquigarrow \star \quad (\exists s', D) \ .$$

Since $\Sigma \mid \emptyset \vdash_C G! : D \rightsquigarrow \star$ is derived by (Ct_Inj_C), we have $D = G$. Furthermore, by Lemma I.46, there exists an intermediate coercion j such that $s' = j$. Hence, by Lemma I.22, we have $\Sigma \mid \emptyset \vdash_S U_2'\langle j \rangle : \Sigma(G)$. Since this judgment is derived (T_Crc_S), we have $\Sigma \mid \emptyset \vdash_S j : D' \rightsquigarrow \Sigma(G)$ $(\exists D')$. Therefore, by Lemma I.11, we have $j \fatsemi |\mathsf{id}_G|_\emptyset = j$. Therefore,

$$\begin{aligned} h_2 \, ; H_2! &= s' \fatsemi |G!|_\emptyset \\ &= j \fatsemi |G!|_\emptyset \\ &= j \fatsemi |G!|_\emptyset \\ &= j \fatsemi (|\mathsf{id}_G|_\emptyset \, ; G!) \\ &= (j \fatsemi |\mathsf{id}_G|_\emptyset) \, ; G! \\ &= (j \fatsemi |\mathsf{id}_G|_\emptyset) \, ; G! \\ &= j \, ; G! \ . \end{aligned}$$

Hence, $H = H_2 = G$, which is contradictory to $G \neq H$.

Case $H_2 \neq H$: We are given

$$\begin{aligned} (h_2 \, ; H_2!) \fatsemi |H?^p|_\emptyset &= (h_2 \, ; H_2!) \fatsemi (H?^p \, ; |\mathsf{id}_H|_\emptyset) \\ &= \bot^p \ . \end{aligned}$$

Therefore, by (R_Merge_S) and (R_Fail_S),

$$\begin{aligned} \Sigma \triangleright U_2'\langle h_2 \, ; H_2! \rangle \langle |H?^p|_\emptyset \rangle &\longrightarrow_S \Sigma \triangleright U_2'\langle \bot^p \rangle \\ &\longrightarrow_S \Sigma \triangleright \mathsf{blame}\, p \ . \end{aligned}$$

Hence,

$$\Sigma \mid \emptyset \vdash_S M_2' : \Sigma(\star), \quad \Sigma \mid \emptyset \vdash_S |H?^p|_\emptyset : \Sigma(\star) \rightsquigarrow \Sigma(G),$$
$$\Sigma \triangleright M_2' \longrightarrow_S^* \Sigma \triangleright U_2'\langle h_2 \, ; H_2! \rangle, \quad \Sigma \triangleright U_2'\langle h_2 \, ; H_2! \rangle \langle |H?^p|_\emptyset \rangle \longrightarrow_S^* \Sigma \triangleright \mathsf{blame}\, p \ .$$

Therefore, by Lemma I.19, we have

$$\begin{aligned} \Sigma \triangleright M_2'\langle |c|_\emptyset \rangle &= \Sigma \triangleright M_2'\langle |H?^p|_\emptyset \rangle \\ &\longrightarrow_S^* \Sigma \triangleright \mathsf{blame}\, p \ . \end{aligned}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$. By Lemma I.23, we have $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Therefore, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case (R_Remove_C): We are given

$$M_2 = V_2\langle \alpha^- \rangle, \quad c = \alpha^+, \quad M_1 = V_2, \quad \Sigma_1 = \Sigma \quad (\exists p, \alpha, V_2) \ .$$

Since $\Sigma \mid \emptyset \vdash_C \alpha^+ : B \rightsquigarrow A$ is derived by (Ct_Reveal_C), we have

$$B = \alpha, \quad A = \mathbb{A}, \quad \alpha := \mathbb{A} \in \Sigma \quad (\exists \mathbb{A}) \ .$$

Furthermore, we have $|\alpha^+|_\emptyset = \mathsf{id}$. By (Bs_CrcId), $\Sigma \mid \emptyset \vdash V_2\langle \alpha^- \rangle \approx M_2'\langle |\mathsf{id}_\alpha|_\emptyset \rangle : \alpha$. Since $|\mathsf{id}_\alpha|_\emptyset = \mathsf{id}$, we have $\Sigma \mid \emptyset \vdash V_2\langle \alpha^- \rangle \approx M_2'\langle \mathsf{id} \rangle : \alpha$. Therefore, since $V_2\langle \alpha^- \rangle$ is a value, by the case (3),

$$M_2'\langle |\alpha^+|_\emptyset \rangle = M_2'\langle \mathsf{id} \rangle \longrightarrow_S^* V_2', \quad \Sigma \mid \emptyset \vdash V_2\langle \alpha^- \rangle \approx V_2' : \alpha \quad (\exists V_2') .$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash V_2 \approx V_2' : \mathbb{A}$. We perform case analysis on whether V_2' is a coercion application or not.

Case $V_2' = U_2'$ $(\exists U_2')$: Since $\Sigma \mid \emptyset \vdash V_2\langle \alpha^- \rangle \approx U_2' : \alpha$ is derived by (Bs_CrcIdL), we have $\Sigma \mid \emptyset \vdash V_2 \approx U_2' : \mathbb{A}$.

Case $V_2' = U_2'\langle t \rangle$ $(\exists U_2', t)$: By Lemma I.36 and V_2 is a value, we have $V_2 = V_4\langle vc_1 \rangle \cdots \langle vc_{n-1} \rangle$ and $vc_n = \alpha^-$ for some $V_4, vc_1, \dots, vc_n$. By Lemma I.57, there exists nonnegative integer $j \leq n$ such that

$$\begin{gathered} A_n = A, \quad A_{n-1} = \mathbb{A}, \quad \Sigma \mid \emptyset \vdash V_4 \approx U_2' : A_0, \\ \Sigma \mid \emptyset \vdash vc_i : A_{i-1} \rightsquigarrow A_i, \quad (\exists A_0, \dots, A_n) , \end{gathered}$$

and

$$t = |\mathsf{id}_{A_{j+1}}|_\emptyset \fatsemi |vc_{j+1}|_\emptyset \fatsemi \cdots \fatsemi |vc_{n+1}|_\emptyset, \quad vc_i = vc_i^I \quad (\exists vc_1^I, \dots, vc_j^I) \ (1 \leq i \leq j) .$$

By Lemma I.11, we have

$$\begin{aligned} t &= |\mathsf{id}_{A_{j+1}}|_\emptyset \fatsemi |vc_{j+1}|_\emptyset \fatsemi \cdots \fatsemi |vc_n|_\emptyset \\ &= |\mathsf{id}_{A_{j+1}}|_\emptyset \fatsemi |vc_{j+1}|_\emptyset \fatsemi \cdots \fatsemi |vc_{n-1}|_\emptyset \fatsemi |\alpha^-|_\emptyset \\ &= |vc_{j+1}|_\emptyset \fatsemi \cdots \fatsemi |vc_{n-1}|_\emptyset \\ &= |\mathsf{id}_{A_0}|_\emptyset \fatsemi |vc_1^I|_\emptyset \fatsemi \cdots \fatsemi |vc_j^I|_\emptyset \fatsemi |vc_{j+1}|_\emptyset \fatsemi \cdots \fatsemi |vc_{n-1}|_\emptyset \\ &= |\mathsf{id}_{A_0}|_\emptyset \fatsemi |vc_1|_\emptyset \fatsemi \cdots \fatsemi |vc_{n-1}|_\emptyset . \end{aligned}$$

By Lemma E.9, $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A_0$. By (Ct_Id_C), $\Sigma \mid \emptyset \vdash_C \mathsf{id}_{A_0} : A_0 \rightsquigarrow A_0$. Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash V_4\langle vc_1 \rangle \cdots \langle vc_{n-1} \rangle \approx U_2'\langle |\mathsf{id}_{A_0}|_\emptyset \fatsemi |vc_1|_\emptyset \fatsemi \cdots \fatsemi |vc_{n-1}|_\emptyset \rangle : \mathbb{A}$, which is given by applying (Bs_CrcId) and (Bs_CrcMore) $n-1$ times.

Case (R_Split_C): We are given

$$M_2 = V_2, \quad c = c_1 \mathbin{;} c_2, \quad M_1 = V_2\langle c_1 \rangle \langle c_2 \rangle, \quad \Sigma_1 = \Sigma \quad (\exists c_1, c_2, V_2) .$$

Therefore, it suffices to show that

$$\Sigma \mid \emptyset \vdash V_2\langle c_1 \rangle \langle c_2 \rangle \approx M_2'\langle |c_1 \mathbin{;} c_2|_\emptyset \rangle : A .$$

Because $\Sigma \mid \emptyset \vdash_C c_1 \mathbin{;} c_2 : B \rightsquigarrow A$ is derived by (Ct_Seq_C), we have

$$\Sigma \mid \emptyset \vdash_C c_1 : B \rightsquigarrow D, \quad \Sigma \mid \emptyset \vdash_C c_2 : D \rightsquigarrow A \quad (\exists D) .$$

Therefore, because $\Sigma \mid \emptyset \vdash V_2 \approx M_2' : B$, by applying (Bs_CrcMore) twice, we have

$$\Sigma \mid \emptyset \vdash V_2\langle c_1 \rangle \langle c_2 \rangle \approx M_2'\langle (|c_1|_\emptyset) \fatsemi |c_2|_\emptyset \rangle : A .$$

Hence, it suffices to show that

$$|c_1 \mathbin{;} c_2|_\emptyset = (|c_1|_\emptyset) \fatsemi |c_2|_\emptyset .$$

By Lemma I.3, we have

$$\Sigma \mid \emptyset \vdash_S |c_1|_\emptyset : \Sigma(B) \rightsquigarrow \Sigma(D), \quad \Sigma \mid \emptyset \vdash_S |c_2|_\emptyset : \Sigma(D) \rightsquigarrow \Sigma(A) .$$

Hence, by Lemma I.7, we have

$$(|c_1|_\emptyset) \fatsemi |c_2|_\emptyset = (|c_1|_\emptyset \fatsemi |c_2|_\emptyset) .$$

Therefore,

$$\begin{aligned} |c_1 \mathbin{;} c_2|_\emptyset &= (|c_1|_\emptyset \fatsemi |c_2|_\emptyset) \\ &= (|c_1|_\emptyset) \fatsemi |c_2|_\emptyset . \end{aligned}$$

Case (R_Blame_C): We are given

$$M_2 = \mathsf{blame}\, p, \quad M_1 = \mathsf{blame}\, p \quad (\exists p) \ .$$

Therefore, because $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx M_2' : B$, by the case (5), we have

$$\Sigma \rhd M_2' \longrightarrow_S^* \Sigma \rhd \mathsf{blame}\, p \ .$$

By case analysis on the length of the evaluation sequence $\Sigma \rhd M_2' \longrightarrow_S^* \Sigma \rhd \mathsf{blame}\, p$.

Case the length is zero: We are given $M_2' = \mathsf{blame}\, p$, but it is contradictory.

Case the length is larger than zero: We are given

$$\Sigma \rhd M_2' \longrightarrow_S \Sigma' \rhd M_3', \quad \Sigma' \rhd M_3' \longrightarrow_S^* \Sigma \rhd \mathsf{blame}\, p \quad (\exists \Sigma', M_3') \ .$$

By Lemma I.21, we consider the following three cases.

Case (1) in Lemma I.21: We are given

$$\Sigma \rhd M_2'\langle| c |_\emptyset\rangle \longrightarrow_S^* \Sigma \rhd (\mathsf{blame}\, p)\langle| c |_\emptyset\rangle \ .$$

Therefore, by (R_BlameC),

$$\begin{aligned} \Sigma \rhd M_2'\langle| c |_\emptyset\rangle &\longrightarrow_S^* \Sigma \rhd (\mathsf{blame}\, p)\langle| c |_\emptyset\rangle \\ &\longrightarrow_S \Sigma \rhd \mathsf{blame}\, p \ . \end{aligned}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$. By Lemma I.23, we have $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Therefore, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case (2) in Lemma I.21: We are given

$$\begin{aligned} \Sigma \rhd M_2'\langle| c |_\emptyset\rangle &\longrightarrow_S^* \Sigma \rhd M_4'\langle (s' \,;\, s) \,;\, | c |_\emptyset\rangle, \\ \mathsf{blame}\, p = M_4'\langle s' \,;\, s\rangle, &\quad \Sigma \mid \emptyset \vdash_S s' : D \rightsquigarrow C \quad (\exists D, s', M_4') \ . \end{aligned}$$

However, $\mathsf{blame}\, p = M_4'\langle s' \,;\, s\rangle$ does not hold, so there is a contradiction.

Case (3) in Lemma I.21: We are given

$$\Sigma \rhd M_2'\langle| c |_\emptyset\rangle \longrightarrow_S^* \Sigma \rhd \mathsf{blame}\, p', \quad \mathsf{blame}\, p = \mathsf{blame}\, p' \quad (\exists p') \ .$$

Because $\mathsf{blame}\, p = \mathsf{blame}\, p'$, we have $p = p'$. Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$. By Lemma I.23, we have $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Therefore, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case (R_Ctx_C): We are given

$$M_1 = M_3\langle c\rangle, \quad \Sigma \rhd M_2 \longrightarrow_C \Sigma_1 \rhd M_3 \quad (\exists M_3) \ .$$

Therefore, by the IH, we have

$$\Sigma_1 \rhd M_3 \longrightarrow_C^* \Sigma_2 \rhd M_4, \quad \Sigma \rhd M_2' \longrightarrow_S^* \Sigma_2 \rhd M_4', \quad \Sigma_2 \mid \emptyset \vdash M_4 \approx M_4' : B \quad (\exists \Sigma_2, M_4, M_4') \ .$$

Hence, by (R_Ctx_C),

$$\begin{aligned} \Sigma \rhd M_2\langle c\rangle &\longrightarrow_C \Sigma_1 \rhd M_3\langle c\rangle \\ &\longrightarrow_C^* \Sigma_2 \rhd M_4\langle c\rangle \ . \end{aligned}$$

Furthermore, by Lemma I.22, we have $\Sigma \mid \emptyset \vdash_C M_2\langle c\rangle : A$. We perform case analysis on the length of the evaluation sequence $\Sigma \rhd M_2' \longrightarrow_S^* \Sigma_2 \rhd M_4'$.

Case the length is zero: We have $\Sigma = \Sigma_2$ and $M_2' = M_4'$. Furthermore, we have $\Sigma \mid \emptyset \vdash M_4 \approx M_2' : B$. Therefore, by (Bs_CrcMore),

$$\Sigma \mid \emptyset \vdash M_4\langle c\rangle \approx M_2'\langle| c |_\emptyset\rangle : A \ .$$

Case the length is larger than zero: We are given

$$\Sigma \triangleright M'_2 \longrightarrow_S \Sigma' \triangleright M'_5, \quad \Sigma' \triangleright M'_5 \longrightarrow^*_S \Sigma_2 \triangleright M'_4 \quad (\exists \Sigma', M'_5) \ .$$

Therefore, by Lemma I.21, we consider the following three cases.

Case (1) in Lemma I.21: We are given

$$\Sigma \triangleright M'_2 \langle |c|_\emptyset \rangle \longrightarrow^*_S \Sigma_2 \triangleright M'_4 \langle |c|_\emptyset \rangle \ .$$

Therefore, it suffices to show that

$$\Sigma_2 \mid \emptyset \vdash M_4 \langle c \rangle \approx M'_4 \langle |c|_\emptyset \rangle : A \ .$$

Now, we have $\Sigma_2 \mid \emptyset \vdash M_4 \approx M'_4 : B$. By Lemma I.23, $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash B$. By (Ct_Id_C), $\Sigma \mid \emptyset \vdash_C \mathsf{id}_B : B \rightsquigarrow B$. Therefore, by (Bs_CrcId),

$$\Sigma_2 \mid \emptyset \vdash M_4 \approx M'_4 \langle |\mathsf{id}_B|_\emptyset \rangle : B \ .$$

Hence, by (Bs_CrcMore), $\Sigma_2 \mid \emptyset \vdash M_4 \langle c \rangle \approx M'_4 \langle |\mathsf{id}_B|_\emptyset \fatsemi |c|_\emptyset \rangle : A$. Therefore, it suffices to show that $|\mathsf{id}_B|_\emptyset \fatsemi |c|_\emptyset = |c|_\emptyset$, which is given by Lemma I.11.

Case (2) in Lemma I.21: We are given

$$\Sigma \triangleright M'_2 \langle |c|_\emptyset \rangle \longrightarrow^*_S \Sigma_2 \triangleright M'_6 \langle (s' \fatsemi s) \fatsemi |c|_\emptyset \rangle, \quad M'_4 = M'_6 \langle s' \fatsemi s \rangle,$$
$$\Sigma_2 \mid \emptyset \vdash_S s' : D \rightsquigarrow C \quad (\exists D, s', M'_6) \ .$$

Therefore, it suffices to show that

$$\Sigma_2 \mid \emptyset \vdash M_4 \langle c \rangle \approx M'_6 \langle (s' \fatsemi s) \fatsemi |c|_\emptyset \rangle : A \ .$$

Because $\Sigma_2 \mid \emptyset \vdash M_4 \approx M'_6 \langle s' \fatsemi s \rangle : B$, by (Bs_CrcMore), we have

$$\Sigma_2 \mid \emptyset \vdash M_4 \langle c \rangle \approx M'_6 \langle (s' \fatsemi s) \fatsemi |c|_\emptyset \rangle : A \ .$$

Case (3) in Lemma I.21: We are given

$$\Sigma \triangleright M'_2 \langle |c|_\emptyset \rangle \longrightarrow^*_S \Sigma_2 \triangleright \mathsf{blame}\, p, \quad M'_4 = \mathsf{blame}\, p \quad (\exists p) \ .$$

Therefore, $\Sigma_2 \mid \emptyset \vdash M_4 \approx \mathsf{blame}\, p : B$. Since $\Sigma_2 \mid \emptyset \vdash M_4 \approx \mathsf{blame}\, p : B$ is derived by the combination of (Bs_CrcIdL) and (Bs_Blame), there exists $n \geq 0$ and $c^I_1, \ldots, c^I_n$ such that

$$M_4 = (\mathsf{blame}\, p) \langle c^I_1 \rangle \cdots \langle c^I_n \rangle, \quad A_{n+1} = B, \quad \Sigma_2 \mid \emptyset \vdash c^I_i : A_i \rightsquigarrow A_{i+1},$$
$$\Sigma_2 \mid \emptyset \vdash c^I_i : A_i \rightsquigarrow A_{i+1}, \quad \vdash \Sigma_2, \quad \Sigma_2 \vdash \emptyset, \quad \Sigma_2 \mid \emptyset \vdash A_1 \quad (\exists A_1, \ldots, A_{n+1}) \ .$$

Therefore, by (R_Blame_C), (R_Ctx_C) and what have been proven, we have

$$\begin{aligned}
\Sigma \triangleright M_2 \langle c \rangle &\longrightarrow^*_C \Sigma_2 \triangleright M_4 \langle c \rangle \\
&= \Sigma_2 \triangleright (\mathsf{blame}\, p) \langle c^I_1 \rangle \cdots \langle c^I_n \rangle \langle c \rangle \\
&\longrightarrow_C \Sigma_2 \triangleright (\mathsf{blame}\, p) \langle c^I_2 \rangle \cdots \langle c^I_n \rangle \langle c \rangle \\
&\longrightarrow_C \cdots \\
&\longrightarrow_C \Sigma_2 \triangleright (\mathsf{blame}\, p) \langle c \rangle \\
&\longrightarrow_C \Sigma_2 \triangleright \mathsf{blame}\, p \ .
\end{aligned}$$

Furthermore, by Lemma I.22, we have $\Sigma_2 \mid \emptyset \vdash_C (\mathsf{blame}\, p) \langle c^I_1 \rangle \cdots \langle c^I_n \rangle \langle c \rangle : A$. Therefore, by Theorem E.19, we have $B = A_i = A$ $(1 \leq i \leq n+1)$. Hence, it suffices to show that

$$\Sigma_2 \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A \ .$$

Therefore, by Lemma I.23, we have $\vdash \Sigma_2$ and $\Sigma_2 \vdash \emptyset$ and $\Sigma_2 \mid \emptyset \vdash A$. Hence, by (Bs_Blame), $\Sigma_2 \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case (Bs_CrcId): We are given

$$M' = M'_1\langle|\mathsf{id}_A|_\emptyset\rangle, \quad \Sigma \mid \emptyset \vdash M \approx M'_1 : A \quad (\exists M'_1) \ .$$

By the IH,

$$\Sigma_1 \triangleright M_1 \longrightarrow^*_C \Sigma_2 \triangleright M_2, \quad \Sigma \triangleright M'_1 \longrightarrow^*_S \Sigma_2 \triangleright M'_2, \quad \Sigma_2 \mid \emptyset \vdash M_2 \approx M'_2 : A \quad (\exists \Sigma_2, M_2, M'_2) \ .$$

Furthermore, by Lemma I.22, we have

$$\Sigma \mid \emptyset \vdash_S M'_1 : \Sigma(A) \ .$$

Moreover, by Lemma I.23, we have $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Therefore, by (Ct_Id_C), $\Sigma \mid \emptyset \vdash_C \mathsf{id}_A : A \rightsquigarrow A$. Hence, by Lemma I.3, we have

$$\Sigma \mid \emptyset \vdash_S |\mathsf{id}_A|_\emptyset : \Sigma(A) \rightsquigarrow \Sigma(A) \ .$$

Therefore, by Lemma I.20, we consider the following three cases.

Case (1) in Lemma I.20: We are given

$$\Sigma \triangleright M'_1\langle|\mathsf{id}_A|_\emptyset\rangle \longrightarrow^*_S \Sigma_2 \triangleright M'_2\langle|\mathsf{id}_A|_\emptyset\rangle \ .$$

Therefore, it suffices to show that

$$\Sigma_2 \mid \emptyset \vdash M_2 \approx M'_2\langle|\mathsf{id}_A|_\emptyset\rangle : A \ .$$

Now, we have $\Sigma_2 \mid \emptyset \vdash M_2 \approx M'_2 : A$, so, by (Bs_CrcId), we have

$$\Sigma_2 \mid \emptyset \vdash M_2 \approx M'_2\langle|\mathsf{id}_A|_\emptyset\rangle : A \ .$$

Case (2) in Lemma I.20: We are given

$$\Sigma \triangleright M'_1\langle|\mathsf{id}_A|_\emptyset\rangle \longrightarrow^*_S \Sigma_2 \triangleright M'_3\langle s \fatsemi |\mathsf{id}_A|_\emptyset\rangle, \quad M'_2 = M'_3\langle s\rangle \quad (\exists s, M'_3) \ .$$

Therefore, it suffices to show that

$$\Sigma_2 \mid \emptyset \vdash M_2 \approx M'_3\langle s \fatsemi |\mathsf{id}_A|_\emptyset\rangle : A \ .$$

Now, we have $\Sigma_2 \mid \emptyset \vdash M_2 \approx M'_3\langle s\rangle : A$, so by Lemma I.22, we have $\Sigma_2 \mid \emptyset \vdash_S M'_3\langle s\rangle : \Sigma(A)$. Because this judgment is derived by (T_Crc_S), we have $\Sigma_2 \mid \emptyset \vdash_S s : B \rightsquigarrow \Sigma(A)$ $(\exists B)$. Therefore, by Lemma Lemma I.11,

$$\begin{aligned} s \fatsemi |\mathsf{id}_A|_\emptyset &= s \fatsemi |\mathsf{id}_A|_\emptyset \\ &= s \ . \end{aligned}$$

Therefore, it suffices to show that $\Sigma_2 \mid \emptyset \vdash M_2 \approx M'_3\langle s\rangle : A$, which holds already.

Case (3) in Lemma I.20: We are given

$$\Sigma \triangleright M'_1\langle|\mathsf{id}_A|_\emptyset\rangle \longrightarrow^*_S \Sigma_2 \triangleright M'_2 \ .$$

Therefore, it suffices to show that

$$\Sigma_2 \mid \emptyset \vdash M_2 \approx M'_2 : A \ ,$$

which holds already.

Case (Bs_CrcMore): We are given

$$M = M_2\langle c\rangle, \quad M' = M_2'\langle s \mathbin{;} |c|_\emptyset\rangle,$$
$$\Sigma \mid \emptyset \vdash M_2 \approx M_2'\langle s\rangle : B, \quad \Sigma \mid \emptyset \vdash_C \ c : B \rightsquigarrow A \quad (\exists B, c, s, M_2, M_2') \ .$$

By Lemma I.22, we have

$$\Sigma \mid \emptyset \vdash_S \ M_2'\langle s\rangle : \Sigma(B) \ .$$

Since this judgment is derived by (T_Crc_S), we have

$$\Sigma \mid \emptyset \vdash_S \ s : C \rightsquigarrow \Sigma(B) \quad (\exists C) \ .$$

By Lemma I.3, we have

$$\Sigma \mid \emptyset \vdash_S \ |c|_\emptyset : \Sigma(B) \rightsquigarrow \Sigma(A) \ .$$

We perform case analysis on the rule applied last to derive $\Sigma \triangleright M_2\langle c\rangle \longrightarrow_C \Sigma_1 \triangleright M_1$, which is one of the following rules.

Case (R_Id_C): We are given

$$M_2 = V_2, \quad c = \mathsf{id}_{A'}, \quad M_1 = V_2, \quad \Sigma_1 = \Sigma \quad (\exists A', V_2) \ .$$

Therefore, $\Sigma \mid \emptyset \vdash_C \ \mathsf{id}_{A'} : B \rightsquigarrow A$. Because this judgment is derived by (Ct_Id_C), we have $A' = B = A$. Therefore, since $\Sigma \mid \emptyset \vdash_S \ s : C \rightsquigarrow \Sigma(A)$, so by Lemma I.11, we have $s \mathbin{;} |\mathsf{id}_A|_\emptyset = s$. Furthermore, since $\Sigma \mid \emptyset \vdash V_2 \approx M_2'\langle s\rangle : A$, by the case (3), we have

$$\Sigma \triangleright M_2'\langle s\rangle \longrightarrow_S^* \Sigma \triangleright V_2', \quad \Sigma \mid \emptyset \vdash V_2 \approx V_2' : A \quad (\exists V_2') \ .$$

Therefore,

$$\begin{aligned} \Sigma \triangleright M_2'\langle s \mathbin{;} |c|_\emptyset\rangle &= \Sigma \triangleright M_2'\langle s \mathbin{;} |\mathsf{id}_A|_\emptyset\rangle \\ &= \Sigma \triangleright M_2'\langle s\rangle \\ &\longrightarrow_S^* \Sigma \triangleright V_2' \ . \end{aligned}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash V_2 \approx V_2' : A$, which holds already.

Case (R_Fail_C): We are given

$$M_2 = V_2, \quad c = \bot^p_{B' \rightsquigarrow A'}, \quad M_1 = \mathsf{blame}\, p, \quad \Sigma_1 = \Sigma \quad (\exists p, A', B', V_2) \ .$$

Therefore, because $\Sigma \mid \emptyset \vdash V_2 \approx M_2'\langle s\rangle : B$, by the case (3), we have

$$\Sigma \triangleright M_2'\langle s\rangle \longrightarrow_S^* \Sigma \triangleright V_2', \quad \Sigma \mid \emptyset \vdash V_2 \approx V_2' : B \quad (\exists V_2') \ .$$

Furthermore, by (R_Fail_S),

$$\begin{aligned} \Sigma \triangleright V_2'\langle |\bot^p_{B' \rightsquigarrow A'}|_\emptyset\rangle &= \Sigma \triangleright V_2'\langle |\bot^p_{B' \rightsquigarrow A'}|_\emptyset\rangle \\ &= \Sigma \triangleright V_2'\langle \bot^p\rangle \\ &\longrightarrow_S \Sigma \triangleright \mathsf{blame}\, p \ . \end{aligned}$$

Hence, by Lemma I.19, we have

$$\Sigma \triangleright M_2'\langle s \mathbin{;} |\bot^p_{B' \rightsquigarrow A'}|_\emptyset\rangle \longrightarrow_S^* \Sigma \triangleright \mathsf{blame}\, p \ .$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$. By Lemma I.23, we have $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Therefore, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case (R_Collapse_C): We are given

$$M_2 = V_2\langle G!\rangle, \quad c = G?^p, \quad M_1 \;=\; V_2, \quad \Sigma_1 \;=\; \Sigma \quad (\exists p, G, V_2)\ .$$

Since $\Sigma \mid \emptyset \vdash_C\ G?^p : B \rightsquigarrow A$ is derived by (Ct_Proj_C), we have $B = \star$ and $A = G$. Therefore, $\Sigma \mid \emptyset \vdash V_2\langle G!\rangle \approx M_2'\langle s\rangle : \star$, so by the case (3), we have

$$\Sigma \triangleright M_2'\langle s\rangle \longrightarrow_S^* \Sigma \triangleright V_2', \quad \Sigma \mid \emptyset \vdash V_2\langle G!\rangle \approx V_2' : \star \quad (\exists V_2')\ .$$

By Lemma I.22 and $\Sigma(\star) = \star$, we have $\Sigma \mid \emptyset \vdash_S\ V_2' : \star$. Therefore, by Lemma E.25, we have

$$V_2' = U_2'\langle h\,;H!\rangle \quad (\exists H, h, U_2')\ .$$

Hence, $\Sigma \mid \emptyset \vdash V_2\langle G!\rangle \approx U_2'\langle h\,;H!\rangle : \star$. We perform case analysis on the last rule to derive $\Sigma \mid \emptyset \vdash V_2\langle G!\rangle \approx U_2'\langle h\,;H!\rangle : \star$, which is either of (Bs_CrcId), (Bs_CrcIdL), (Bs_Crc), or (Bs_CrcMore).

Case (Bs_CrcId): Contradictory because there is no identity coercion id_A such that $|\mathsf{id}_A|_\emptyset = h\,;H!$.

Case (Bs_CrcIdL): Contradictory because $G!$ is not a no-op coercion.

Case (Bs_Crc): We are given

$$h\,;H! = |G!|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_2 \approx U_2' : D, \quad \Sigma \mid \emptyset \vdash_C\ G! : D \rightsquigarrow \star \quad (\exists D)\ .$$

Since $\Sigma \mid \emptyset \vdash_C\ G! : D \rightsquigarrow \star$ is derived by (Ct_Inj_C), we have $D = G$. Also, G does not include any free type variables. Therefore,

$$\begin{aligned} h\,;H! &= |G!|_\emptyset \\ &= |\mathsf{id}_G|_\emptyset\,;G!\ . \end{aligned}$$

Hence, $H = G$ and $h = |\mathsf{id}_G|_\emptyset$. Furthermore, by Lemma I.11, $|\mathsf{id}_G|_\emptyset \fatsemi |\mathsf{id}_G|_\emptyset = |\mathsf{id}_G|_\emptyset$. Therefore,

$$\begin{aligned} (h\,;G!) \fatsemi |G?^p|_\emptyset &= (|\mathsf{id}_G|_\emptyset\,;G!) \fatsemi |G?^p|_\emptyset \\ &= (|\mathsf{id}_G|_\emptyset\,;G!) \fatsemi (G?^p\,;|\mathsf{id}_G|_\emptyset) \\ &= |\mathsf{id}_G|_\emptyset \fatsemi |\mathsf{id}_G|_\emptyset \\ &= |\mathsf{id}_G|_\emptyset\ . \end{aligned}$$

Therefore, by (R_Merge_S), we have

$$\begin{aligned} \Sigma \triangleright V_2'\langle |G?^p|_\emptyset\rangle &= \Sigma \triangleright U_2'\langle h\,;G!\rangle\langle |G?^p|_\emptyset\rangle \\ &\longrightarrow_S \Sigma \triangleright U_2'\langle (h\,;G!) \fatsemi |G?^p|_\emptyset\rangle \\ &= \Sigma \triangleright U_2'\langle |\mathsf{id}_G|_\emptyset\rangle\ . \end{aligned}$$

Hence, by Lemma I.14,

$$\Sigma \triangleright M_2'\langle |G?^p|_\emptyset\rangle \longrightarrow_S \Sigma \triangleright U_2'\langle |\mathsf{id}_G|_\emptyset\rangle\ .$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash V_2 \approx U_2'\langle |\mathsf{id}_G|_\emptyset\rangle : G$, which is given by (Bs_CrcId).

Case (Bs_CrcMore): We are given

$$h\,;H! = s' \fatsemi |G!|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_2 \approx U_2'\langle s'\rangle : D, \quad \Sigma \mid \emptyset \vdash_C\ G! : D \rightsquigarrow \star \quad (\exists s', D)\ .$$

Since $\Sigma \mid \emptyset \vdash_C\ G! : D \rightsquigarrow \star$ is derived by (Ct_Inj_C), we have $D = G$. Furthermore, by Lemma I.46, there exists an intermediate coercion j such that $s' = j$. Hence, by Lemma I.22, we have $\Sigma \mid \emptyset \vdash_S\ U_2'\langle j\rangle : \Sigma(G)$. Since this judgment is derived (T_Crc_S), we have $\Sigma \mid \emptyset \vdash_S\ j : D' \rightsquigarrow \Sigma(G)$ $(\exists D')$. Therefore, by Lemma I.11, we have $j \fatsemi |\mathsf{id}_G|_\emptyset = j$. Therefore,

$$\begin{aligned} h\,;H! &= s' \fatsemi |G!|_\emptyset \\ &= j \fatsemi |G!|_\emptyset \\ &= j \fatsemi |G!|_\emptyset \\ &= j \fatsemi (|\mathsf{id}_G|_\emptyset\,;G!) \\ &= (j \fatsemi |\mathsf{id}_G|_\emptyset)\,;G! \\ &= (j \fatsemi |\mathsf{id}_G|_\emptyset)\,;G! \\ &= j\,;G!\ . \end{aligned}$$

Hence, $H = G$ and $h = j$. Furthermore,

$$\begin{aligned}(h \,;\, G!) \fatsemi |G?^p|_\emptyset &= (h \,;\, G!) \fatsemi |G?^p|_\emptyset \\ &= (h \,;\, G!) \fatsemi (G?^p \,;\, |\mathsf{id}_G|_\emptyset) \\ &= h \fatsemi |\mathsf{id}_G|_\emptyset \\ &= h \fatsemi |\mathsf{id}_G|_\emptyset \\ &= h \ .\end{aligned}$$

Therefore, by (R_Merge_S), we have

$$\begin{array}{rcl}\Sigma \triangleright V_2'\langle |G?^p|_\emptyset\rangle &=& \Sigma \triangleright U_2'\langle h \,;\, G!\rangle\langle |G?^p|_\emptyset\rangle \\ &=& \Sigma \triangleright U_2'\langle h \,;\, G!\rangle\langle |G?^p|_\emptyset\rangle \\ &=& \Sigma \triangleright U_2'\langle h \,;\, G!\rangle\langle G?^p \,;\, |\mathsf{id}_G|_\emptyset\rangle \\ &=& \Sigma \triangleright U_2'\langle h \,;\, G!\rangle\langle G?^p \,;\, |\mathsf{id}_G|_\emptyset\rangle \\ &\longrightarrow_S& \Sigma \triangleright U_2'\langle (h \,;\, G!) \fatsemi (G?^p \,;\, |\mathsf{id}_G|_\emptyset)\rangle \\ &\longrightarrow_S& \Sigma \triangleright U_2'\langle h \fatsemi |\mathsf{id}_G|_\emptyset\rangle \\ &=& \Sigma \triangleright U_2'\langle h\rangle \ .\end{array}$$

By case analysis on h.

Case $h = \mathsf{id}$: We have $\Sigma \mid \emptyset \vdash_S \mathsf{id} : D' \rightsquigarrow \Sigma(G)$. Since this judgment is derived by (Ct_Id_S), we have $D' = \Sigma(G)$. Therefore, by (R_Id_S) and what have been proven, we have

$$\begin{array}{rcl}\Sigma \triangleright V_2'\langle |G?^p|_\emptyset\rangle &\longrightarrow_S& \Sigma \triangleright U_2'\langle h\rangle \\ &=& \Sigma \triangleright U_2'\langle \mathsf{id}\rangle \\ &\longrightarrow_S& \Sigma \triangleright U_2' \ .\end{array}$$

Therefore, by Lemma I.18, we have

$$\begin{array}{rcl}\Sigma \triangleright M_2'\langle s \fatsemi |c|_\emptyset\rangle &=& \Sigma \triangleright M_2'\langle s \fatsemi |G?^p|_\emptyset\rangle \\ &\longrightarrow_S^*& \Sigma \triangleright U_2' \ .\end{array}$$

It suffices to show that $\Sigma \mid \emptyset \vdash V_2 \approx U_2' : G$. Since $\Sigma \mid \emptyset \vdash V_2\langle G!\rangle \approx U_2'\langle \mathsf{id} \,;\, G!\rangle : \star$, by Lemma I.55, we have $\Sigma \mid \emptyset \vdash V_2 \approx U_2' : G$.

Otherwise: $U_2'\langle h\rangle$ is a value. Therefore, by Lemma I.18, we have

$$\begin{array}{rcl}\Sigma \triangleright M_2'\langle s \fatsemi |c|_\emptyset\rangle &=& \Sigma \triangleright M_2'\langle s \fatsemi |G?^p|_\emptyset\rangle \\ &\longrightarrow_S^*& \Sigma \triangleright U_2'\langle h\rangle \ .\end{array}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash V_2 \approx U_2'\langle h\rangle : G$, which already holds.

Case (R_Conflict_C): We are given

$$M_2 = V_2\langle G!\rangle, \quad c = H?^p, \quad M_1 = \mathsf{blame}\, p, \quad G \neq H, \quad \Sigma_1 = \Sigma \quad (\exists p, G, H, V_2) \ .$$

Because $\Sigma \mid \emptyset \vdash_C H?^p : B \rightsquigarrow A$ is derived by (Ct_Proj_C), we have $B = \star$ and $A = H$. Therefore, $\Sigma \mid \emptyset \vdash V_2\langle G!\rangle \approx M_2'\langle s\rangle : \star$, so by the case (3), we have

$$\Sigma \triangleright M_2'\langle s\rangle \longrightarrow_S^* \Sigma \triangleright V_2', \quad \Sigma \mid \emptyset \vdash V_2\langle G!\rangle \approx V_2' : \star \quad (\exists V_2') \ .$$

By Lemma I.22 and $\Sigma(\star) = \star$, we have $\Sigma \mid \emptyset \vdash_S V_2' : \star$. Therefore, by Lemma E.25, we have

$$V_2' = U_2'\langle h_2 \,;\, H_2!\rangle \ (\exists h_2, H_2, U_2') \ .$$

We perform case analysis on whether $H_2 = H$ or not.

Case $H_2 = H$: We perform case analysis on the last rule to derive $\Sigma \mid \emptyset \vdash V_2\langle G!\rangle \approx U_2'\langle h_2 \,;\, H_2!\rangle : \star$, which is either of , which is either of (Bs_CrcId), (Bs_CrcIdL), (Bs_Crc), or (Bs_CrcMore).

Case (Bs_CrcId): Contradictory because there is no identity coercion id_A such that $|\mathsf{id}_A|_\emptyset = h\,;H!$.

Case (Bs_CrcIdL): Contradictory because $G!$ is not a no-op coercion.

Case (Bs_Crc): We are given

$$h_2\,;H_2! = |G!|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_2 \approx U_2' : D, \quad \Sigma \mid \emptyset \vdash_C G! : D \rightsquigarrow \star \quad (\exists D)\ .$$

Since $\Sigma \mid \emptyset \vdash_C G! : D \rightsquigarrow \star$ is derived by (Ct_Inj_C), we have $D = G$. Also, G does not include any free type variables. Therefore, we have $|G!|_\emptyset = |\mathsf{id}_G|_\emptyset\,; G!$. Therefore,

$$\begin{aligned} h_2\,;H_2! &= |G!|_\emptyset \\ &= |\mathsf{id}_G|_\emptyset\,;G!\ . \end{aligned}$$

Hence, $H = H_2 = G$, which is contradictory to $G \neq H$.

Case (Bs_CrcMore): We are given

$$h_2\,;H_2! = s' \mathbin{⨟} |G!|_\emptyset, \quad \Sigma \mid \emptyset \vdash V_2 \approx U_2'\langle s'\rangle : D, \quad \Sigma \mid \emptyset \vdash_C G! : D \rightsquigarrow \star \quad (\exists s', D)\ .$$

Since $\Sigma \mid \emptyset \vdash_C G! : D \rightsquigarrow \star$ is derived by (Ct_Inj_C), we have $D = G$. Furthermore, by Lemma I.46, there exists an intermediate coercion j such that $s' = j$. Hence, by Lemma I.22, we have $\Sigma \mid \emptyset \vdash_S U_2'\langle j\rangle : \Sigma(G)$. Since this judgment is derived (T_Crc_S), we have $\Sigma \mid \emptyset \vdash_S j : D' \rightsquigarrow \Sigma(G)$ $(\exists D')$. Therefore, by Lemma I.11, we have $j \mathbin{⨟} |\mathsf{id}_G|_\emptyset = j$. Therefore,

$$\begin{aligned} h_2\,;H_2! &= s' \mathbin{⨟} |G!|_\emptyset \\ &= j \mathbin{⨟} |G!|_\emptyset \\ &= j \mathbin{⨟} |G!|_\emptyset \\ &= j \mathbin{⨟} (|\mathsf{id}_G|_\emptyset\,;G!) \\ &= (j \mathbin{⨟} |\mathsf{id}_G|_\emptyset)\,;G! \\ &= (j \mathbin{⨟} |\mathsf{id}_G|_\emptyset)\,;G! \\ &= j\,;G!\ . \end{aligned}$$

Hence, $H = H_2 = G$, which is contradictory to $G \neq H$.

Case $H_2 \neq H$: We are given

$$\begin{aligned} (h_2\,;H_2!) \mathbin{⨟} |H?^p|_\emptyset &= (h_2\,;H_2!) \mathbin{⨟} (H?^p\,;|\mathsf{id}_H|_\emptyset) \\ &= \bot^p\ . \end{aligned}$$

Therefore, by (R_Merge_S) and (R_Fail_S),

$$\begin{aligned} \Sigma \triangleright U_2'\langle h_2\,;H_2!\rangle\langle |H?^p|_\emptyset\rangle &\longrightarrow_S \Sigma \triangleright U_2'\langle \bot^p\rangle \\ &\longrightarrow_S \Sigma \triangleright \mathsf{blame}\,p\ . \end{aligned}$$

Hence,

$$\Sigma \mid \emptyset \vdash_S M_2'\langle s\rangle : \Sigma(\star), \quad \Sigma \mid \emptyset \vdash_S |H?^p|_\emptyset : \Sigma(\star) \rightsquigarrow \Sigma(G),$$
$$\Sigma \triangleright M_2'\langle s\rangle \longrightarrow_S^* \Sigma \triangleright U_2'\langle h_2\,;H_2!\rangle, \quad \Sigma \triangleright U_2'\langle h_2\,;H_2!\rangle\langle |H?^p|_\emptyset\rangle \longrightarrow_S^* \Sigma \triangleright \mathsf{blame}\,p\ .$$

Therefore, by Lemma I.19, we have

$$\begin{aligned} \Sigma \triangleright M_2'\langle s \mathbin{⨟} |c|_\emptyset\rangle &= \Sigma \triangleright M_2'\langle s \mathbin{⨟} |H?^p|_\emptyset\rangle \\ &\longrightarrow_S^* \Sigma \triangleright \mathsf{blame}\,p\ . \end{aligned}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\,p \approx \mathsf{blame}\,p : A$. By Lemma I.23, we have $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Therefore, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\,p \approx \mathsf{blame}\,p : A$.

Case (R_Remove_C): We are given

$$M_2 = V_2\langle\alpha^-\rangle, \quad c = \alpha^+, \quad M_1 \ = \ V_2, \quad \Sigma_1 \ = \ \Sigma \quad (\exists p, \alpha, V_2) \ .$$

Since $\Sigma \mid \emptyset \vdash_C \ \alpha^+ : B \rightsquigarrow A$ is derived by (Ct_Reveal_C), we have

$$B = \alpha, \quad A = \mathbb{A}, \quad \alpha := \mathbb{A} \in \Sigma \quad (\exists \mathbb{A}) \ .$$

By Lemma I.50, $s = i$ $(\exists i)$. Therefore, we have

$$\begin{aligned} s \mathbin{;} |\alpha^+|_\emptyset &= s \mathbin{;} |\alpha^+|_\emptyset \\ &= s \mathbin{;} \mathsf{id} \\ &= i \mathbin{;} \mathsf{id} \\ &= i \ . \end{aligned}$$

Here, we have $\Sigma \mid \emptyset \vdash V_2\langle\alpha^-\rangle \approx M'_2\langle i\rangle : \alpha$. Then, by the case (3),

$$M'_2\langle s \mathbin{;} |\alpha^+|_\emptyset\rangle = M'_2\langle i\rangle \longrightarrow^*_S V'_2, \quad \Sigma \mid \emptyset \vdash V_2\langle\alpha^-\rangle \approx V'_2 : \alpha \quad (\exists V'_2) \ .$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash V_2 \approx V'_2 : \mathbb{A}$. We perform case analysis on whether V'_2 is a coercion application or not.

Case $V'_2 = U'_2$ $(\exists U'_2)$: Since $\Sigma \mid \emptyset \vdash V_2\langle\alpha^-\rangle \approx U'_2 : \alpha$ is derived by (Bs_CrcIdL), we have $\Sigma \mid \emptyset \vdash V_2 \approx U'_2 : \mathbb{A}$.

Case $V'_2 = U'_2\langle t\rangle$ $(\exists U'_2, t)$: By Lemma I.36 and V_2 is a value, we have $V_2 = V_4\langle vc_1\rangle \cdots \langle vc_{n-1}\rangle$ and $vc_n = \alpha^-$ for some $V_4, vc_1, \ldots, vc_n$. By Lemma I.57, there exists nonnegative integer $j \leq n$ such that

$$\begin{gathered} A_n = A, \quad A_{n-1} = \mathbb{A}, \quad \Sigma \mid \emptyset \vdash V_4 \approx U'_2 : A_0, \\ \Sigma \mid \emptyset \vdash \ vc_i : A_{i-1} \rightsquigarrow A_i, \quad (\exists A_0, \ldots, A_n) \ , \end{gathered}$$

and

$$t = |\mathsf{id}_{A_{j+1}}|_\emptyset \mathbin{;} |vc_{j+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |vc_{n+1}|_\emptyset, \quad vc_i = vc_i^I \quad (\exists vc_1^I, \ldots, vc_j^I) \ (1 \leq i \leq j) \ .$$

By Lemma I.11, we have

$$\begin{aligned} t &= |\mathsf{id}_{A_{j+1}}|_\emptyset \mathbin{;} |vc_{j+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |vc_n|_\emptyset \\ &= |\mathsf{id}_{A_{j+1}}|_\emptyset \mathbin{;} |vc_{j+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |vc_{n-1}|_\emptyset \mathbin{;} |\alpha^-|_\emptyset \\ &= |vc_{j+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |vc_{n-1}|_\emptyset \\ &= |\mathsf{id}_{A_0}|_\emptyset \mathbin{;} |vc_1^I|_\emptyset \mathbin{;} \cdots \mathbin{;} |vc_j^I|_\emptyset \mathbin{;} |vc_{j+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |vc_{n-1}|_\emptyset \\ &= |\mathsf{id}_{A_0}|_\emptyset \mathbin{;} |vc_1|_\emptyset \mathbin{;} \cdots \mathbin{;} |vc_{n-1}|_\emptyset \ . \end{aligned}$$

By Lemma E.9, $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A_0$. By (Ct_Id_C), $\Sigma \mid \emptyset \vdash_C \ \mathsf{id}_{A_0} : A_0 \rightsquigarrow A_0$. Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash V_4\langle vc_1\rangle \cdots \langle vc_{n-1}\rangle \approx U'_2\langle|\mathsf{id}_{A_0}|_\emptyset \mathbin{;} |vc_1|_\emptyset \mathbin{;} \cdots \mathbin{;} |vc_{n-1}|_\emptyset\rangle : \mathbb{A}$, which is given by applying (Bs_CrcId) and (Bs_CrcMore) $n-1$ times.

Case (R_Split_C): We are given

$$M_2 \ = \ V_2, \quad c = c_1 \mathbin{;} c_2, \quad M_1 = V_2\langle c_1\rangle\langle c_2\rangle, \quad \Sigma_1 \ = \ \Sigma \quad (\exists c_1, c_2, V_2) \ .$$

Therefore, it suffices to show that

$$\Sigma \mid \emptyset \vdash V_2\langle c_1\rangle \, \langle c_2\rangle \approx M'_2\langle s \mathbin{;} |c_1 \mathbin{;} c_2|_\emptyset\rangle : A \ .$$

Because $\Sigma \mid \emptyset \vdash_C \ c_1 \mathbin{;} c_2 : B \rightsquigarrow A$ is derived by (Ct_Seq_C), we have

$$\Sigma \mid \emptyset \vdash_C \ c_1 : B \rightsquigarrow D, \quad \Sigma \mid \emptyset \vdash_C \ c_2 : D \rightsquigarrow A \quad (\exists D) \ .$$

Therefore, because $\Sigma \mid \emptyset \vdash V_2 \approx M_2'\langle s \rangle : B$, by applying (Bs_CrcMore) twice, we have

$$\Sigma \mid \emptyset \vdash V_2\langle c_1 \rangle \langle c_2 \rangle \approx M_2'\langle (s \mathbin{⨟} |c_1|_\emptyset) \mathbin{⨟} |c_2|_\emptyset \rangle : A \ .$$

Hence, it suffices to show that

$$s \mathbin{⨟} |c_1 \ ; \ c_2|_\emptyset = (s \mathbin{⨟} |c_1|_\emptyset) \mathbin{⨟} |c_2|_\emptyset \ .$$

By Lemma I.3, we have

$$\Sigma \mid \emptyset \vdash_S |c_1|_\emptyset : \Sigma(B) \rightsquigarrow \Sigma(D), \quad \Sigma \mid \emptyset \vdash_S |c_2|_\emptyset : \Sigma(D) \rightsquigarrow \Sigma(A) \ .$$

Hence, by Lemma I.7, we have

$$(s \mathbin{⨟} |c_1|_\emptyset) \mathbin{⨟} |c_2|_\emptyset = s \mathbin{⨟} (|c_1|_\emptyset \mathbin{⨟} |c_2|_\emptyset) \ .$$

Therefore,

$$\begin{aligned} s \mathbin{⨟} |c_1 \ ; \ c_2|_\emptyset &= s \mathbin{⨟} (|c_1|_\emptyset \mathbin{⨟} |c_2|_\emptyset) \\ &= (s \mathbin{⨟} |c_1|_\emptyset) \mathbin{⨟} |c_2|_\emptyset \ . \end{aligned}$$

Case (R_Blame_C): We are given

$$M_2 = \mathsf{blame}\, p, \quad M_1 = \mathsf{blame}\, p \quad (\exists p) \ .$$

Therefore, because $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx M_2'\langle s \rangle : B$, by the case (5), we have

$$\Sigma \rhd M_2'\langle s \rangle \longrightarrow_S^* \Sigma \rhd \mathsf{blame}\, p \ .$$

By case analysis on the length of the evaluation sequence $\Sigma \rhd M_2'\langle s \rangle \longrightarrow_S^* \Sigma \rhd \mathsf{blame}\, p$.
Case the length is zero: We are given $M_2'\langle s \rangle = \mathsf{blame}\, p$, but it is contradictory.
Case the length is larger than zero: We are given

$$\Sigma \rhd M_2'\langle s \rangle \longrightarrow_S \Sigma' \rhd M_3', \quad \Sigma' \rhd M_3' \longrightarrow_S^* \Sigma \rhd \mathsf{blame}\, p \quad (\exists \Sigma', M_3') \ .$$

By Lemma I.21, we consider the following three cases.
Case (1) in Lemma I.21: We are given

$$\Sigma \rhd M_2'\langle s \mathbin{⨟} |c|_\emptyset \rangle \longrightarrow_S^* \Sigma \rhd (\mathsf{blame}\, p)\langle |c|_\emptyset \rangle \ .$$

Therefore, by (R_BlameC),

$$\begin{aligned} \Sigma \rhd M_2'\langle s \mathbin{⨟} |c|_\emptyset \rangle &\longrightarrow_S^* \Sigma \rhd (\mathsf{blame}\, p)\langle |c|_\emptyset \rangle \\ &\longrightarrow_S \Sigma \rhd \mathsf{blame}\, p \ . \end{aligned}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$. By Lemma I.23, we have $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Therefore, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.
Case (2) in Lemma I.21: We are given

$$\begin{gathered} \Sigma \rhd M_2'\langle s \mathbin{⨟} |c|_\emptyset \rangle \longrightarrow_S^* \Sigma \rhd M_4'\langle (s' \mathbin{⨟} s) \mathbin{⨟} |c|_\emptyset \rangle, \\ \mathsf{blame}\, p = M_4'\langle s' \mathbin{⨟} s \rangle, \quad \Sigma \mid \emptyset \vdash_S s' : D \rightsquigarrow C \quad (\exists D, s', M_4') \ . \end{gathered}$$

However, $\mathsf{blame}\, p = M_4'\langle s' \mathbin{⨟} s \rangle$ does not hold, so there is a contradiction.
Case (3) in Lemma I.21: We are given

$$\Sigma \rhd M_2'\langle s \mathbin{⨟} |c|_\emptyset \rangle \longrightarrow_S^* \Sigma \rhd \mathsf{blame}\, p', \quad \mathsf{blame}\, p = \mathsf{blame}\, p' \quad (\exists p') \ .$$

Because $\mathsf{blame}\, p = \mathsf{blame}\, p'$, we have $p = p'$. Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$. By Lemma I.23, we have $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Therefore, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case (R_Ctx_C): We are given

$$M_1 = M_3\langle c\rangle, \quad \Sigma \triangleright M_2 \longrightarrow_C \Sigma_1 \triangleright M_3 \quad (\exists M_3) .$$

Therefore, by the IH, we have

$$\Sigma_1 \triangleright M_3 \longrightarrow_C^* \Sigma_2 \triangleright M_4, \quad \Sigma \triangleright M_2'\langle s\rangle \longrightarrow_S^* \Sigma_2 \triangleright M_4', \quad \Sigma_2 \mid \emptyset \vdash M_4 \approx M_4' : B \quad (\exists \Sigma_2, M_4, M_4') .$$

Hence, by (R_Ctx_C),

$$\begin{aligned} \Sigma \triangleright M_2\langle c\rangle &\longrightarrow_C \Sigma_1 \triangleright M_3\langle c\rangle \\ &\longrightarrow_C^* \Sigma_2 \triangleright M_4\langle c\rangle . \end{aligned}$$

Furthermore, by Lemma I.22, we have $\Sigma \mid \emptyset \vdash_C M_2\langle c\rangle : A$. We perform case analysis on the length of the evaluation sequence $\Sigma \triangleright M_2'\langle s\rangle \longrightarrow_S^* \Sigma_2 \triangleright M_4'$.

Case the length is zero: We have $\Sigma = \Sigma_2$ and $M_2'\langle s\rangle = M_4'$. Furthermore, we have $\Sigma \mid \emptyset \vdash M_4 \approx M_2'\langle s\rangle : B$. Therefore, by (Bs_CrcMore),

$$\Sigma \mid \emptyset \vdash M_4\langle c\rangle \approx M_2'\langle s \mathbin{⨟} |c|_\emptyset\rangle : A .$$

Case the length is larger than zero: We are given

$$\Sigma \triangleright M_2'\langle s\rangle \longrightarrow_S \Sigma' \triangleright M_5', \quad \Sigma' \triangleright M_5' \longrightarrow_S^* \Sigma_2 \triangleright M_4' \quad (\exists \Sigma', M_5') .$$

Therefore, by Lemma I.21, we consider the following three cases.

Case (1) in Lemma I.21: We are given

$$\Sigma \triangleright M_2'\langle s \mathbin{⨟} |c|_\emptyset\rangle \longrightarrow_S^* \Sigma_2 \triangleright M_4'\langle |c|_\emptyset\rangle .$$

Therefore, it suffices to show that

$$\Sigma_2 \mid \emptyset \vdash M_4\langle c\rangle \approx M_4'\langle |c|_\emptyset\rangle : A .$$

Now, we have $\Sigma_2 \mid \emptyset \vdash M_4 \approx M_4' : B$. By Lemma I.23, $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash B$. By (Ct_Id_C), $\Sigma \mid \emptyset \vdash_C \mathsf{id}_B : B \rightsquigarrow B$. Therefore, by (Bs_CrcId),

$$\Sigma_2 \mid \emptyset \vdash M_4 \approx M_4'\langle |\mathsf{id}_B|_\emptyset\rangle : B .$$

Hence, by (Bs_CrcMore), $\Sigma_2 \mid \emptyset \vdash M_4\langle c\rangle \approx M_4'\langle |\mathsf{id}_B|_\emptyset \mathbin{⨟} |c|_\emptyset\rangle : A$. Therefore, it suffices to show that $|\mathsf{id}_B|_\emptyset \mathbin{⨟} |c|_\emptyset = |c|_\emptyset$, which is given by Lemma I.11.

Case (2) in Lemma I.21: We are given

$$\begin{gathered} \Sigma \triangleright M_2'\langle s \mathbin{⨟} |c|_\emptyset\rangle \longrightarrow_S^* \Sigma_2 \triangleright M_6'\langle (s' \mathbin{⨟} s) \mathbin{⨟} |c|_\emptyset\rangle, \quad M_4' = M_6'\langle s' \mathbin{⨟} s\rangle, \\ \Sigma_2 \mid \emptyset \vdash_S s' : D \rightsquigarrow C \quad (\exists D, s', M_6') . \end{gathered}$$

Therefore, it suffices to show that

$$\Sigma_2 \mid \emptyset \vdash M_4\langle c\rangle \approx M_6'\langle (s' \mathbin{⨟} s) \mathbin{⨟} |c|_\emptyset\rangle : A .$$

Because $\Sigma_2 \mid \emptyset \vdash M_4 \approx M_6'\langle s' \mathbin{⨟} s\rangle : B$, by (Bs_CrcMore), we have

$$\Sigma_2 \mid \emptyset \vdash M_4\langle c\rangle \approx M_6'\langle (s' \mathbin{⨟} s) \mathbin{⨟} |c|_\emptyset\rangle : A .$$

Case (3) in Lemma I.21: We are given

$$\Sigma \triangleright M_2'\langle s \mathbin{⨟} |c|_\emptyset\rangle \longrightarrow_S^* \Sigma_2 \triangleright \mathsf{blame}\, p, \quad M_4' = \mathsf{blame}\, p \quad (\exists p) .$$

Therefore, $\Sigma_2 \mid \emptyset \vdash M_4 \approx \mathsf{blame}\, p : B$. Since $\Sigma_2 \mid \emptyset \vdash M_4 \approx \mathsf{blame}\, p : B$ is derived by the combination of (Bs_CrcIdL) and (Bs_Blame), there exists $n \geq 0$ and $c^I_1, \ldots, c^I_n$ such that

$$M_4 = (\mathsf{blame}\, p)\langle c^I_1 \rangle \cdots \langle c^I_n \rangle, \quad A_{n+1} = B, \quad \Sigma_2 \mid \emptyset \vdash c^I_i : A_i \rightsquigarrow A_{i+1},$$
$$\Sigma_2 \mid \emptyset \vdash c^I_i : A_i \rightsquigarrow A_{i+1}, \quad \vdash \Sigma_2, \quad \Sigma_2 \vdash \emptyset, \quad \Sigma_2 \mid \emptyset \vdash A_1 \quad (\exists A_1, \ldots, A_{n+1}) .$$

Therefore, by (R_Blame_C), (R_Ctx_C) and what have been proven, we have

$$\begin{aligned}\Sigma \triangleright M_2\langle c \rangle &\longrightarrow^*_C \Sigma_2 \triangleright M_4\langle c \rangle \\ &= \Sigma_2 \triangleright (\mathsf{blame}\, p)\langle c^I_1 \rangle \cdots \langle c^I_n \rangle \langle c \rangle \\ &\longrightarrow_C \Sigma_2 \triangleright (\mathsf{blame}\, p)\langle c^I_2 \rangle \cdots \langle c^I_n \rangle \langle c \rangle \\ &\longrightarrow_C \cdots \\ &\longrightarrow_C \Sigma_2 \triangleright (\mathsf{blame}\, p)\langle c \rangle \\ &\longrightarrow_C \Sigma_2 \triangleright \mathsf{blame}\, p \ .\end{aligned}$$

Furthermore, by Lemma I.22, we have $\Sigma_2 \mid \emptyset \vdash_C (\mathsf{blame}\, p)\langle c^I_1 \rangle \cdots \langle c^I_n \rangle \langle c \rangle : A$. Therefore, by Theorem E.19, we have $B = A_i = A$ $(1 \leq i \leq n+1)$. Hence, it suffices to show that

$$\Sigma_2 \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A \ .$$

Therefore, by Lemma I.23, we have $\vdash \Sigma_2$ and $\Sigma_2 \vdash \emptyset$ and $\Sigma_2 \mid \emptyset \vdash A$. Hence, by (Bs_Blame), $\Sigma_2 \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case (Bs_CrcIdL): We are given

$$M = M_2\langle c^I \rangle, \quad \Sigma \mid \emptyset \vdash M_2 \approx M' : B, \quad \Sigma \mid \emptyset \vdash_C c^I : B \rightsquigarrow A \quad (\exists B, c^I, M_2) .$$

We perform case analysis on the rule applied last to derive $\Sigma \triangleright M_2\langle c^I \rangle \longrightarrow_C \Sigma_1 \triangleright M_1$, which is one of the following rules.

Case (R_Id_C): We are given

$$M_2 = V_2, \quad c^I = \mathsf{id}_{A'}, \quad M_1 = V_2, \quad \Sigma_1 = \Sigma \quad (\exists A', V_2) .$$

Since $\Sigma \mid \emptyset \vdash_C \mathsf{id}_{A'} : B \rightsquigarrow A$ is derived by (Ct_Id_C), we have $A' = B = A$. Therefore, by Lemma I.56, $\Sigma \triangleright V_2\langle \mathsf{id}_{A'} \rangle \longrightarrow^*_C \Sigma \triangleright V_2$ and $\Sigma \mid \emptyset \vdash V_2 \approx M' : A$.

Case (R_Remove_C): We are given

$$M_2 = V_2\langle \alpha^- \rangle, \quad c^I = \alpha^+, \quad M_1 = V_2, \quad \Sigma_1 = \Sigma \quad (\exists \alpha, V_2) .$$

Since $\Sigma \mid \emptyset \vdash \alpha^+ : B \rightsquigarrow A$ is derived by (Ct_Reveal_C), we have

$$B = \alpha, \quad A = \mathbb{A}, \quad \alpha := \mathbb{A} \in \Sigma \quad (\exists \mathbb{A}) .$$

By Lemma I.36, there exists a non-cercion application value V_3 and coercions $c_1, \cdots, c_n$ such that $V_2\langle \alpha^- \rangle = V_3\langle c_1 \rangle \cdots \langle c_{n-1} \rangle \langle \alpha^- \rangle$ $(c_n = \alpha^-)$ and $V_2 = V_3\langle c_1 \rangle \cdots \langle c_{n-1} \rangle$. Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash V_3\langle c_1 \rangle \cdots \langle c_{n-1} \rangle \approx M' : \mathbb{A}$. We perform case analysis on M'.

Case $M' = M'_2\langle s \rangle$ $(\exists s)$: By Lemma I.39, there exists nonnegative j $(1 \leq j \leq n)$ such that

$$c_i = c^I_i \ (1 \leq i \leq j), \quad c_n = \alpha^-, \quad A_{n+1} = \alpha, \quad \Sigma \mid \emptyset \vdash V_3 \approx M'_2 : A_1, \quad \Sigma \mid \emptyset \vdash_C c_i : A_i \rightsquigarrow A_{i+1},$$
$$s = |\mathsf{id}_{A_j}|_\emptyset \mathbin{;} |c_{j+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n|_\emptyset \quad (\exists A_1, \ldots, A_{n+1}, c^I_1, \ldots, c^I_j) .$$

Since $\Sigma \mid \emptyset \vdash_C \alpha^- : A_n \rightsquigarrow \alpha$ is derived by (Ct_Conceal_C) and $\alpha := \mathbb{A} \in \Sigma$, we have $A_n = \mathbb{A}$. By applying (Bs_CrcIdL) j times, (Bs_CrcId), and (Bs_CrcMore) $n - j - 1$ times, we have

$$\Sigma \mid \emptyset \vdash V_3\langle c^I_1 \rangle, \ldots, \langle c^I_j \rangle \, \langle c_{j+1} \rangle, \ldots, \langle c_{n-1} \rangle \approx M'_2\langle |\mathsf{id}_{A_j}|_\emptyset \mathbin{;} |c_{j+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_{n-1}|_\emptyset \rangle : \mathbb{A} \ .$$

Furthermore, since $c_n = \alpha^-$ is a no-op coercion, by Lemma I.11, we have

$$\begin{aligned} s &= |\mathsf{id}_{A_j}|_\emptyset \fatsemi |c_{j+1}|_\emptyset \fatsemi \cdots \fatsemi |c_n|_\emptyset \\ &= |\mathsf{id}_{A_j}|_\emptyset \fatsemi |c_{j+1}|_\emptyset \fatsemi \cdots \fatsemi |\alpha^-|_\emptyset \\ &= |\mathsf{id}_{A_j}|_\emptyset \fatsemi |c_{j+1}|_\emptyset \fatsemi \cdots \fatsemi |c_{n-1}|_\emptyset \end{aligned}$$

Therefore, we have $\Sigma \mid \emptyset \vdash V_3\langle c_1^I\rangle, \ldots, \langle c_j^I\rangle\, \langle c_{j+1}\rangle, \ldots, \langle c_{n-1}\rangle \approx M_2'\langle s\rangle : \mathbb{A}$.

Otherwise: Since $\Sigma \mid \emptyset \vdash V_3\langle c_1\rangle, \ldots, \langle c_{n-1}\rangle\, \langle \alpha^-\rangle \approx M' : \alpha$ is derived by (Bs_CrcIdL), we have

$$\Sigma \mid \emptyset \vdash V_3\langle c_1\rangle, \ldots, \langle c_{n-1}\rangle \approx M' : C, \quad \Sigma \mid \emptyset \vdash_C \alpha^- : C \rightsquigarrow \alpha \quad (\exists C) \ .$$

Since $\Sigma \mid \emptyset \vdash_C \alpha^- : C \rightsquigarrow \alpha$ is derived by (Ct_Conceal_C) and $\alpha := \mathbb{A} \in \Sigma$, we have $C = \mathbb{A} = A$.

Case (R_Split_C): We are given

$$M_2 \ = \ V_2, \quad c^I = c_1^I \,;c_2^I, \quad M_1 = V_2\langle c_1^I\rangle\langle c_2^I\rangle, \quad \Sigma_1 \ = \ \Sigma \quad (\exists c_1^I, c_2^I, V_2) \ .$$

Furthermore, $\Sigma \mid \emptyset \vdash_C c_1^I \,; c_2^I : B \rightsquigarrow A$ is derived by (Ct_Seq_C), we have

$$\Sigma \mid \emptyset \vdash_C c_1^I : B \rightsquigarrow C, \quad \Sigma \mid \emptyset \vdash_C c_2^I : C \rightsquigarrow A \quad (\exists C) \ .$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash V_2\langle c_1^I\rangle\, \langle c_2^I\rangle \approx M' : A$, which is given by applying (Bs_CrcIdL) twice.

Case (R_Blame_C): We are given

$$M_2 \ = \ \mathsf{blame}\, p, \quad M_1 \ = \ \mathsf{blame}\, p \quad (\exists p) \ .$$

Therefore, because $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx M' : B$, by the case (5), we have

$$\Sigma \rhd M' \longrightarrow_S^* \Sigma \rhd \mathsf{blame}\, p \ .$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$. By Lemma I.23, we have $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Therefore, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case (R_Ctx_C): We are given

$$M_1 = M_3\langle c^I\rangle, \quad \Sigma \rhd M_2 \longrightarrow_C \Sigma_1 \rhd M_3 \quad (\exists M_3) \ .$$

Therefore, by the IH, we have

$$\Sigma_1 \rhd M_3 \longrightarrow_C^* \Sigma_2 \rhd M_4, \quad \Sigma \rhd M' \longrightarrow_S^* \Sigma_2 \rhd M_4', \quad \Sigma_2 \mid \emptyset \vdash M_4 \approx M_4' : B \quad (\exists \Sigma_2, M_4, M_4') \ .$$

Hence, by (R_Ctx_C),

$$\begin{aligned} \Sigma \rhd M_2\langle c^I\rangle &\longrightarrow_C \Sigma_1 \rhd M_3\langle c^I\rangle \\ &\longrightarrow_C^* \Sigma_2 \rhd M_4\langle c^I\rangle \ . \end{aligned}$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash M_4\langle c^I\rangle \approx M_4' : A$, which is given by (Bs_CrcIdL).

(4) By induction on the derivation of $\Sigma \mid \emptyset \vdash M \approx V' : A$ with case analysis on the last rule used.

Case (Bs_Const), (Bs_Abs), (Bs_Tyabs): Because M is a value, we have the conclusion by letting $V = M$.

Case (Bs_CrcId): We are given

$$V' \ = \ U_1'\langle |\mathsf{id}_A|_\emptyset\rangle, \quad \Sigma \mid \emptyset \vdash M \approx U_1' : A \quad \Sigma \mid \emptyset \vdash_C \mathsf{id}_A : A \rightsquigarrow A \quad (\exists U_1') \ .$$

By the IH,

$$\Sigma \rhd M \longrightarrow_C^* \Sigma \rhd V_1, \quad \Sigma \mid \emptyset \vdash V_1 \approx U_1' : A \quad (\exists V_1) \ .$$

By (Bs_CrcId), $\Sigma \mid \emptyset \vdash V_1 \approx U_1'\langle |\mathsf{id}_A|_\emptyset\rangle : A$.

Case (Bs_Crc): We are given

$$M = M_1\langle c\rangle, \quad V' = U_1'\langle |c|_\emptyset\rangle, \quad \Sigma \mid \emptyset \vdash M_1 \approx U_1' : B, \quad \Sigma \mid \emptyset \vdash_C c : B \rightsquigarrow A \quad (\exists B, c, M_1, U_1') .$$

By Lemma I.36, there exist some M_2 that is not a coercion application, $n > 0$, and $c_1, \dots c_n$ such that $M_1\langle c\rangle = M_2\langle c_1\rangle \cdots \langle c_n\rangle$ and $c_n = c$. Since U_1' is a value, there exists $c_1^I, \dots, c_{n-1}^I$ such that $c_i = c_i^I$ $(1 \le i \le n-1)$. Furthermore, by Lemma I.40, we have

$$M_2 = V_2, \quad A_{n+1} = A, \quad \Sigma \mid \emptyset \vdash c_i : A_i \rightsquigarrow A_{i+1}\ (1 \le i \le n), \quad \Sigma \mid \emptyset \vdash V_2 \approx U_1' : A_1 \quad (\exists A_1, ..., A_{n+1}) .$$

By Lemma I.52, we consider the following two cases.

Case (1): where

$$\Sigma \triangleright V_2\langle c_1\rangle \cdots \langle c_n\rangle \longrightarrow_C^* \Sigma \triangleright V_3, \quad \Sigma \mid \emptyset \vdash V_3 \approx U_1'\langle |c|_\emptyset\rangle : A \quad (\exists V_3) .$$

Then,

$$\begin{aligned} \Sigma \triangleright M_1\langle c\rangle &= \Sigma \triangleright V_2\langle c_1\rangle \cdots \langle c_n\rangle \\ &\longrightarrow_C^* \Sigma \triangleright V_3 . \end{aligned}$$

It suffices to show that $\Sigma \mid \emptyset \vdash V_3 \approx U_1'\langle |c|_\emptyset\rangle : A$, which holds already.

Case (2): where

$$\Sigma \triangleright V_2\langle c_1\rangle \cdots \langle c_n\rangle \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p, \quad |c|_\emptyset = \bot^p \quad (\exists p) .$$

It means that

$$\begin{aligned} V' &= U_1'\langle |c|_\emptyset\rangle \\ &= U_1'\langle \bot^p\rangle . \end{aligned}$$

However, it is contradictory because $U_1'\langle \bot^p\rangle$ is not a value.

Case (Bs_CrcMore): We are given

$$M = M_1\langle c\rangle, \quad V' = U_1'\langle s \fatsemi |c|_\emptyset\rangle, \quad \Sigma \mid \emptyset \vdash M_1 \approx U_1'\langle s\rangle : B, \quad \Sigma \mid \emptyset \vdash_C c : B \rightsquigarrow A \quad (\exists B, c, s, M_1, U_1') .$$

By Lemma I.36, there exist some M_2 that is not a coercion application, $n > 0$, and $c_1, \dots c_n$ such that $M_1\langle c\rangle = M_2\langle c_1\rangle \cdots \langle c_n\rangle$ and $c_n = c$. By Lemma I.39,

$$\Sigma \mid \emptyset \vdash M_2 \approx U_1' : A_1, \quad \Sigma \mid \emptyset \vdash c_i : A_i \rightsquigarrow A_{i+1}\ (1 \le i \le n), \quad A_{n+1} = A, \quad (\exists A_1, ..., A_{n+1})$$

and, for some $k \le n$, $c_1, \dots, c_k$ are no-op and $s = |\mathsf{id}_{A_{k+1}}|_\emptyset \fatsemi |c_{k+1}|_\emptyset \fatsemi \cdots \fatsemi |c_n|_\emptyset$. By Lemma I.40, there exists a vlaue V_2 such that $M_2 = V_2$. By Lemma I.52, we consider the following two cases.

Case (1): where

$$\Sigma \triangleright V_2\langle c_1\rangle \cdots \langle c_n\rangle \longrightarrow_C^* \Sigma \triangleright V_3, \quad \Sigma \mid \emptyset \vdash V_3 \approx U_1'\langle s \fatsemi |c|_\emptyset\rangle : A \quad (\exists V_3) .$$

Then,

$$\begin{aligned} \Sigma \triangleright M_1\langle c\rangle &= \Sigma \triangleright V_2\langle c_1\rangle \cdots \langle c_n\rangle \\ &\longrightarrow_C^* \Sigma \triangleright V_3 . \end{aligned}$$

It suffices to show that $\Sigma \mid \emptyset \vdash V_3 \approx U_1'\langle s \fatsemi |c|_\emptyset\rangle : A$, which holds already.

Case (2): where

$$\Sigma \triangleright V_2\langle c_1\rangle \cdots \langle c_n\rangle \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p, \quad s \fatsemi |c|_\emptyset = \bot^p \quad (\exists p) .$$

It means that

$$\begin{aligned} V' &= U_1'\langle s \fatsemi |c|_\emptyset\rangle \\ &= U_1'\langle \bot^p\rangle . \end{aligned}$$

However, it is contradictory because $U_1'\langle \bot^p\rangle$ is not a value.

Case (Bs_CrcIdL): We are given

$$M = M_1\langle c^I\rangle, \quad \Sigma \mid \emptyset \vdash M_1 \approx V' : B, \quad \Sigma \mid \emptyset \vdash_C c^I : B \rightsquigarrow A \quad (\exists M_1, c^I, B)\ .$$

By the IH, there exists V_1 such that $\Sigma \triangleright M_1 \longrightarrow_C^* \Sigma \triangleright V_1$ and $\Sigma \mid \emptyset \vdash V_1 \approx V' : B$. Therefore, by (R_Ctx_C), we have $\Sigma \triangleright M_1\langle c^I\rangle \longrightarrow_C^* \Sigma \triangleright V_1\langle c^I\rangle$. Moreover, by Lemma I.56, there exists a value V_2 such that $\Sigma \triangleright V_1\langle c^I\rangle \longrightarrow_C^* \Sigma \triangleright V_2$ and $\Sigma \mid \emptyset \vdash V_2 \approx V' : A$.

Case (Bs_Var), (Bs_App), (Bs_Tyapp), (Bs_Blame): Cannot happen because the RHS is a value.

(6) By induction on the derivation of $\Sigma \mid \emptyset \vdash M \approx \mathsf{blame}\, p : A$. We perform case analysis on the rule applied last to derive $\Sigma \mid \emptyset \vdash M \approx \mathsf{blame}\, p : A$, which is either of (Bs_Blame) or (Bs_CrcIdL).

Case (Bs_Blame): We have $M = \mathsf{blame}\, p$. Hence, $\Sigma \triangleright \mathsf{blame}\, p \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p$.

Case (Bs_CrcIdL): We are given

$$M = M_1\langle c^I\rangle, \quad \Sigma \mid \emptyset \vdash_C c^I : B \rightsquigarrow A, \quad \Sigma \mid \emptyset \vdash M_1 \approx \mathsf{blame}\, p : B \quad (\exists M_1, c^I, B)\ .$$

By the IH, $\Sigma \triangleright M_1 \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p$. By (R_Blame_C), $\Sigma \triangleright (\mathsf{blame}\, p)\langle c^I\rangle \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p$.

(2) By induction on the depth of the derivation of $\Sigma \mid \emptyset \vdash M \approx M' : A$. We perform case analysis on the rule applied last to derive $\Sigma \mid \emptyset \vdash M \approx M' : A$.

Case (Bs_Const), (Bs_Var), (Bs_Abs), (Bs_Tyabs), (Bs_Blame): Contradictory with $\Sigma \triangleright M' \longrightarrow_S \Sigma_1 \triangleright M_1'$.

Case (Bs_App): We are given

$$M = M_2\, M_3, \quad M' = M_2'\, M_3', \quad \Sigma \mid \emptyset \vdash M_2 \approx M_2' : B \to A, \quad \Sigma \mid \emptyset \vdash M_3 \approx M_3' : B \quad (\exists B, M_2, M_3, M_2', M_3')\ .$$

By case analysis on the rule applied last to derive $\Sigma \triangleright M_2'\, M_3' \longrightarrow_S \Sigma_1 \triangleright M_1'$, which is one of the following rules.

Case (R_Delta_S): We are given

$$M_2' = k_2, \quad M_3' = k_3, \quad M_1' = \delta(k_2, k_3), \quad \Sigma_1 = \Sigma \quad (\exists k_2, k_3)\ .$$

By the case (4), we have

$$\Sigma \triangleright M_2 \longrightarrow_C^* \Sigma \triangleright V_2, \quad \Sigma \mid \emptyset \vdash V_2 \approx k_2 : B \to A$$
$$\Sigma \triangleright M_3 \longrightarrow_C^* \Sigma \triangleright V_3, \quad \Sigma \mid \emptyset \vdash V_3 \approx k_3 : B\ .$$

We perform case analysis on whether V_2 is a coercion application or not.

Case V_2 is not a coercion application: Since $\Sigma \mid \emptyset \vdash V_2 \approx k_2 : B \to A$ is derived by (Bs_Const), we have

$$V_2 = k_2, \quad \vdash \Sigma, \quad \Sigma \vdash \emptyset, \quad ty(k_2) = B \to A\ .$$

By the definition of ty, B is a base type. Therefore, by Lemma I.58, V_3 is not a coercion application and

$$V_3 = k_3, \quad ty(k_3) = B\ .$$

Therefore, by (R_Ctx_C) and (R_Delta_C),

$$\begin{aligned} \Sigma \triangleright M_2\, M_3 &\longrightarrow_C^* \Sigma \triangleright M_2\, V_3 \\ &\longrightarrow_C^* \Sigma \triangleright V_2\, V_3 \\ &= \Sigma \triangleright k_2\, k_3 \\ &\longrightarrow_C \Sigma \triangleright \delta(k_2, k_3)\ . \end{aligned}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash \delta(k_2, k_3) \approx \delta(k_2, k_3) : A$. By (Bs_App) and Lemma I.22, we have $\Sigma \mid \emptyset \vdash_C k_2\, k_3 : A$. Therefore, by the assumption on δ, $\Sigma \mid \emptyset \vdash_C \delta(k_2, k_3) : A$. Because this judgment is derived by (T_Const_C), we have $ty(\delta(k_2, k_3)) = A$. Hence, by (Bs_Const), $\Sigma \mid \emptyset \vdash \delta(k_2, k_3) \approx \delta(k_2, k_3) : A$.

Case V_2 is a coercion application: By Lemma I.58, there eixsts V_4 that is not a coercion application such that

$$V_2 = V_4\langle c_1^I \to d_1^I\rangle \cdots \langle c_n^I \to d_n^I\rangle, \quad \Sigma \mid \emptyset \vdash V_4 \approx k_2 : B_0 \to C_0, \quad B = B_n, \quad A = C_n,$$
$$\Sigma \mid \emptyset \vdash_C c_i^I : B_i \rightsquigarrow B_{i-1}, \quad \Sigma \mid \emptyset \vdash_C d_i^I : C_{i-1} \rightsquigarrow C_i$$
$$(\exists c_1^I \dots c_n^I, d_1^I \dots d_n^I, B_0 \dots B_n, C_0 \dots C_n)\ .$$

Since $\Sigma \mid \emptyset \vdash V_4 \approx k_2 : B_0 \to C_0$ is derived by (Bs_Const), we have $V_4 = k_2$, and B_0 and C_0 are base types. Furthermore, by Lemma I.56, we have

$$\Sigma \triangleright V_3\langle c_n^I\rangle \longrightarrow_C^* \Sigma \triangleright V_3', \quad \Sigma \mid \emptyset \vdash V_3' \approx k_3 : B_{n-1} \quad (\exists V_3')\ .$$

Therefore, by applying (R_Wrap_C), (R_Ctx_C), we have

$$\begin{aligned}\Sigma \triangleright M_2\, M_3 &\longrightarrow_C^* \Sigma \triangleright (k_2\langle c_1^I \to d_1^I\rangle \cdots \langle c_n^I \to d_n^I\rangle)\ V_3\\ &\longrightarrow_C^* \Sigma \triangleright ((k_2\langle c_1^I \to d_1^I\rangle \cdots \langle c_{n-1}^I \to d_{n-1}^I\rangle)\,(V_3\langle c_n^I\rangle))\langle d_n^I\rangle\\ &\longrightarrow_C^* \Sigma \triangleright ((k_2\langle c_1^I \to d_1^I\rangle \cdots \langle c_{n-1}^I \to d_{n-1}^I\rangle)\,(V_3'))\langle d_n^I\rangle\ .\end{aligned}$$

Similarly, by Lemma I.56 and applying (R_Wrap_C), (R_Ctx_C) repeatedly, we have

$$\Sigma \mid \emptyset \vdash V_3'' \approx k_3 : B_0\ ,$$

and

$$\begin{aligned}\Sigma \triangleright M_2\, M_3 &\longrightarrow_C^* \Sigma \triangleright ((k_2\langle c_1^I \to d_1^I\rangle \cdots \langle c_{n-1}^I \to d_{n-1}^I\rangle)\,(V_3'))\langle d_n^I\rangle\\ &\longrightarrow_C^* \Sigma \triangleright (k_2\ V_3'')\langle d_n^I\rangle \cdots \langle d_1^I\rangle\ .\end{aligned}$$

Since B_0 is a base type, by Lemma I.58, V_3'' is not a coercion application. Therefore, since $\Sigma \mid \emptyset \vdash V_3'' \approx k_3 : B_0$ is derived (Bs_Const), we have $V_3'' = k_3$. Thus,

$$\begin{aligned}\Sigma \triangleright (k_2\ V_3'')\langle d_n^I\rangle \cdots \langle d_1^I\rangle &= \Sigma \triangleright (k_2\ k_3)\langle d_n^I\rangle \cdots \langle d_1^I\rangle\\ &\longrightarrow_C \Sigma \triangleright \delta(k_2, k_3)\langle d_n^I\rangle \cdots \langle d_1^I\rangle\ .\end{aligned}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash \delta(k_2, k_3)\langle d_n^I\rangle \cdots \langle d_1^I\rangle \approx \delta(k_2, k_3) : A$. By (Bs_App) and Lemma I.22, we have $\Sigma \mid \emptyset \vdash_C k_2\, k_3 : C_0$. Therefore, by the assumption on δ, $\Sigma \mid \emptyset \vdash_C \delta(k_2, k_3) : C_0$. Because this judgment is derived by (T_Const_C), we have $ty(\delta(k_2, k_3)) = C_0$. Hence, by (Bs_Const), $\Sigma \mid \emptyset \vdash \delta(k_2, k_3) \approx \delta(k_2, k_3) : C_0$. By (Bs_Const) and (Bs_CrcIdL), $\Sigma \mid \emptyset \vdash \delta(k_2, k_3)\langle d_n^I\rangle \cdots \langle d_1^I\rangle \approx \delta(k_2, k_3) : A$.

Case (R_Beta_S): We are given

$$M_2' = \lambda x\!:\!A'.M_4', \quad M_3' = V_3', \quad M_1' = M_4'[x := V_3'], \quad \Sigma_1 = \Sigma \quad (\exists A', x, M_4', V_3')\ .$$

By the case (4), we have

$$\Sigma \triangleright M_2 \longrightarrow_C^* \Sigma \triangleright V_2, \quad \Sigma \mid \emptyset \vdash V_2 \approx \lambda x\!:\!A'.M_4' : B \to A\ .$$

By Lemma I.58, there exists V_4 that is not a coercion application such that

$$V_2 = V_4\langle c_1^I \to d_1^I\rangle \cdots \langle c_n^I \to d_n^I\rangle, \quad B_n = B, \quad C_n = A, \quad \Sigma \mid \emptyset \vdash V_4 \approx \lambda x\!:\!A'.M_4' : B_0 \to C_0,$$
$$\Sigma \mid \emptyset \vdash_C c^I : B_i \rightsquigarrow B_{i-1}, \quad \Sigma \mid \emptyset \vdash_C d^I : C_{i-1} \rightsquigarrow C_i \quad (\exists c_1^I \dots c_n^I, d_1^I \dots d_n^I, B_0 \dots B_n, C_0 \dots C_n)\ .$$

Furthermore, $\Sigma \mid \emptyset \vdash V_4 \approx \lambda x\!:\!A'.M_4' : B_0 \to C_0$ is derived by (Bs_Abs), we have

$$B_0 = A', \quad V_4 = \lambda x\!:\!B_0.M_4, \quad \Sigma \mid \emptyset, x : B_0 \vdash M_4 \approx M_4' : C_0 \quad (\exists M_4)\ .$$

Similarly, by the case (4), we have

$$\Sigma \triangleright M_3 \longrightarrow_C \Sigma \triangleright V_3, \quad \Sigma \mid \emptyset \vdash V_3 \approx V_3' : B\ .$$

Furthermore, by Lemma I.56, we have

$$\Sigma \rhd V_3\langle c_n^I\rangle \longrightarrow_C^* \Sigma \rhd V_3'', \quad \Sigma \mid \emptyset \vdash V_3'' \approx k_3 : B_{n-1} \quad (\exists V_3'') \ .$$

Therefore, by (R_Ctx_C) and (R_Wrap_C),

$$\begin{aligned} \Sigma \rhd M_2\, M_3 &\longrightarrow_C^* \Sigma \rhd V_2\ V_3 \\ &= \ \Sigma \rhd ((\lambda x : B_0.M_4)\langle c_1^I \to d_1^I\rangle \cdots \langle c_n^I \to d_n^I\rangle)\ V_3 \\ &\longrightarrow_C^* \Sigma \rhd (((\lambda x : B_0.M_4)\langle c_1^I \to d_1^I\rangle \cdots \langle c_{n-1}^I \to d_{n-1}^I\rangle)\,(V_3\langle c_n^I\rangle))\langle d_n^I\rangle \\ &\longrightarrow_C^* \Sigma \rhd (((\lambda x : B_0.M_4)\langle c_1^I \to d_1^I\rangle \cdots \langle c_{n-1}^I \to d_{n-1}^I\rangle)\ V_3'')\langle d_n^I\rangle \ . \end{aligned}$$

Similarly, by Lemma I.56 and applying (R_Wrap_C), (R_Ctx_C) repeatedly, we have

$$\Sigma \mid \emptyset \vdash V_3''' \approx V_3' : B_0 \ ,$$

and

$$\begin{aligned} \Sigma \rhd M_2\, M_3 &\longrightarrow_C^* \Sigma \rhd (((\lambda x : B_0.M_4)\langle c_1^I \to d_1^I\rangle \cdots \langle c_{n-1}^I \to d_{n-1}^I\rangle)\ V_3')\langle d_n^I\rangle \\ &\longrightarrow_C^* \Sigma \rhd ((\lambda x : B_0.M_4)\ V_3''')\langle d_n^I\rangle \cdots \langle d_1^I\rangle \\ &\longrightarrow_C^* \Sigma \rhd (M_4[x := V_3'''])\langle d_n^I\rangle \cdots \langle d_1^I\rangle \ . \end{aligned}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash (M_4[x := V_3'''])\langle d_n^I\rangle \cdots \langle d_1^I\rangle \approx M_4'[x := V_3'] : A$ By Lemma I.27, $\Sigma \mid \emptyset \vdash M_4[x := V_3'''] \approx M_4'[x := V_3'] : C_0$. By applying (Bs_CrcIdL) repeatedly, we have $\Sigma \mid \emptyset \vdash M_4[x := V_3''']\langle d_n^I\rangle \cdots \langle d_1^I\rangle \approx M_4'[x := V_3'] : A$.

Case (R_Wrap_S): We are given

$$M_2' = U_2'\langle s \to t\rangle, \quad M_3' \ = \ V_3', \quad M_1' = (\,U_2'\,(\,V_3'\langle s\rangle))\langle t\rangle, \quad \Sigma_1 \ = \ \Sigma \quad (\exists s, t, U_2', V_3') \ .$$

Since $M_2'(= U_2'\langle s \to t\rangle)$ and V_3' are values, by the case (4),

$$\Sigma \rhd M_2 \ \longrightarrow_C^* \ \Sigma \rhd V_2, \quad \Sigma \mid \emptyset \vdash V_2 \approx U_2'\langle s \to t\rangle : B \to A \quad (\exists V_2) \ ,$$

and

$$\Sigma \rhd M_3 \ \longrightarrow_C^* \ \Sigma \rhd V_3, \quad \Sigma \mid \emptyset \vdash V_3 \approx V_3' : B \quad (\exists V_3) \ .$$

By applying (R_Ctx_C) repeatedly, we have

$$\begin{aligned} \Sigma \rhd M_2\, M_3 &\longrightarrow_C^* \Sigma \rhd V_2\, M_3 \\ &\longrightarrow_C^* \Sigma \rhd V_2\ V_3 \ . \end{aligned}$$

We perform case analysis on whether V_2 is a coercion application or not.

Case V_2 is not a coercion application: Since $\Sigma \mid \emptyset \vdash V_2 \approx U_2'\langle s \to t\rangle : B \to A$ is derive by (Bs_CrcId), we have

$$|\mathsf{id}_{B\to A}|_\emptyset = s \to t, \quad \Sigma \mid \emptyset \vdash_C \mathsf{id}_{B\to A} : (B \to A) \rightsquigarrow (B \to A), \quad \Sigma \mid \emptyset \vdash V_2 \approx U_2' : (B \to A) \ .$$

Furthermore, since

$$\begin{aligned} s \to t &= |\mathsf{id}_{B\to A}|_\emptyset \\ &= |\mathsf{id}_{B\to A}|_\emptyset \\ &= |\mathsf{id}_B|_\emptyset \to |\mathsf{id}_A|_\emptyset \\ &= |\mathsf{id}_B|_\emptyset \to |\mathsf{id}_A|_\emptyset \ , \end{aligned}$$

we have

$$s = |\mathsf{id}_B|_\emptyset, \quad t = |\mathsf{id}_A|_\emptyset \ .$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash V_2\ V_3 \approx (U_2'\,(V_3'\langle|\mathsf{id}_B|_\emptyset\rangle))\langle|\mathsf{id}_A|_\emptyset\rangle : A$. By Lemma I.23, we have $\vdash \Sigma$, $\Sigma \vdash \emptyset$, $\Sigma \mid \emptyset \vdash A$, $\Sigma \mid \emptyset \vdash B$. By (Ct_Id_C), we have

$$\Sigma \mid \emptyset \vdash_C \mathsf{id}_A : A \rightsquigarrow A, \quad \Sigma \mid \emptyset \vdash_C \mathsf{id}_B : B \rightsquigarrow B\ .$$

Hence, $\Sigma \mid \emptyset \vdash V_2\ V_3 \approx (U_2'\,(V_3'\langle|\mathsf{id}_B|_\emptyset\rangle))\langle|\mathsf{id}_A|_\emptyset\rangle : A$ is derived as follows:

$$\dfrac{\dfrac{\Sigma \mid \emptyset \vdash V_2 \approx U_2' : B \to A \qquad \dfrac{\Sigma \mid \emptyset \vdash V_3 \approx V_3' : B}{\Sigma \mid \emptyset \vdash V_3 \approx V_3'\langle|\mathsf{id}_B|_\emptyset\rangle : B}\ (\text{Bs\_CrcId})}{\Sigma \mid \emptyset \vdash V_2\ V_3 \approx U_2'\,(V_3'\langle|\mathsf{id}_B|_\emptyset\rangle) : A}\ (\text{Bs\_App})}{\Sigma \mid \emptyset \vdash V_2\ V_3 \approx (U_2'\,(V_3'\langle|\mathsf{id}_B|_\emptyset\rangle))\langle|\mathsf{id}_A|_\emptyset\rangle : A}\ (\text{Bs\_CrcId})$$

Case V_2 is a coercion application: By Lemma I.36 and V_2 is a value, $V_2 = V_4\langle vc_1\rangle \cdots \langle vc_n\rangle$ for some V_4, $vc_1, \ldots, vc_n$, and $n > 0$. By Lemma I.57, we have

$$B_n = B, \quad C_n = A, \quad vc_i = c_i \to d_i, \quad \Sigma \mid \emptyset \vdash V_4 \approx U_2' : B_0 \to C_0,$$
$$\Sigma \mid \emptyset \vdash_C c_i : B_i \rightsquigarrow B_{i-1}, \quad \Sigma \mid \emptyset \vdash_C d_i : C_{i-1} \rightsquigarrow C_i \quad (\exists c_1, \ldots, c_n, d_1, \ldots, d_n, B_0, \ldots, B_n, C_0, \ldots, C_n)\ ,$$

and there exists nonegative integer j such that

$$j \le n, \quad c_i \to d_i = c_i^I \to d_i^I\ (1 \le i \le j),$$
$$s \to t = |\mathsf{id}_{A_{j+1}}|_\emptyset \mathbin{;} |c_{j+1} \to d_{j+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n \to d_n|_\emptyset \quad (\exists c_1^I, \ldots, c_j^I, d_1^I, \ldots, d_j^I)\ .$$

By Lemma E.9, $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash B_0$ and $\Sigma \mid \emptyset \vdash C_0$. By (Ct_Id_C), $\Sigma \mid \emptyset \vdash_C \mathsf{id}_{B_0} : B_0 \rightsquigarrow B_0$ and $\Sigma \mid \emptyset \vdash_C \mathsf{id}_{C_0} : C_0 \rightsquigarrow C_0$. Furthermore, by Lemma I.11, we have

$$\begin{aligned}
s \to t &= |\mathsf{id}_{A_{j+1}}|_\emptyset \mathbin{;} |c_{j+1} \to d_{j+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n \to d_n|_\emptyset \\
&= |c_{j+1} \to d_{j+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n \to d_n|_\emptyset \\
&= |c_1^I \to d_1^I|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_j^I \to d_j^I|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_{j+1} \to d_{j+1}|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n \to d_n|_\emptyset \\
&= |c_1 \to d_1|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n \to d_n|_\emptyset \\
&= |c_1 \to d_1|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n \to d_n|_\emptyset \\
&= (|c_1|_\emptyset \to |d_1|_\emptyset) \mathbin{;} \cdots \mathbin{;} (|c_n|_\emptyset \to |d_n|_\emptyset) \\
&= (|c_n|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_1|_\emptyset) \to (|d_1|_\emptyset \mathbin{;} \cdots \mathbin{;} |d_n|_\emptyset) \\
&= (|c_n|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_1|_\emptyset) \to (|d_1|_\emptyset \mathbin{;} \cdots \mathbin{;} |d_n|_\emptyset)\ .
\end{aligned}$$

Therefore,

$$s = |c_n|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_1|_\emptyset, \quad t = |d_1|_\emptyset \mathbin{;} \cdots \mathbin{;} |d_n|_\emptyset\ .$$

We perform case analysis on V_3' is coercion application or not.

Case $V_3' = U_3'$ ($\exists U_3'$): By Lemma I.23, $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash B$. By (Ct_Id_C), $\Sigma \mid \emptyset \vdash_C \mathsf{id}_B : B \rightsquigarrow B$. By (Bs_CrcId), $\Sigma \mid \emptyset \vdash V_3 \approx U_3'\langle|\mathsf{id}_B|_\emptyset\rangle : B$. Now, we have

$$\Sigma \mid \emptyset \vdash V_4\langle c_1 \to d_1\rangle \cdots \langle c_n \to d_n\rangle \approx U_2'\langle(|c_n|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_1|_\emptyset) \to (|d_1|_\emptyset \mathbin{;} \cdots \mathbin{;} |d_n|_\emptyset)\rangle : B \to A,$$
$$\Sigma \mid \emptyset \vdash V_3 \approx U_3'\langle|\mathsf{id}_B|_\emptyset\rangle : B\ .$$

Therefore, by Lemma I.53, we consider the following two cases.

Case (1) in Lemma I.53: We are given

$$\begin{aligned}
\Sigma \rhd M_2\, M_3 &\longrightarrow_C^* \Sigma \rhd V_2\ V_3 \\
&= \Sigma \rhd (V_4\langle c_1 \to d_1\rangle \cdots \langle c_n \to d_n\rangle)\ V_3 \\
&\longrightarrow_C^* \Sigma \rhd (V_4\ M_5)\langle d_1\rangle \cdots \langle d_n\rangle \quad (\exists M_5)\ ,
\end{aligned}$$

and

$$\Sigma \mid \emptyset \vdash M_5 \approx U_3'\langle|\mathsf{id}_B|_\emptyset \mathbin{;} |c_n|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_1|_\emptyset\rangle : B_0\ .$$

Hence, it suffices to show that

$$\Sigma \mid \emptyset \vdash (V_4\, M_5)\langle d_1\rangle \cdots \langle d_n\rangle \approx (U_2'\,(U_3'\langle |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset\rangle))\langle |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset\rangle : A \ .$$

By (Bs_App),

$$\Sigma \mid \emptyset \vdash V_4\, M_5 \approx U_2'\,(U_3'\langle |\mathsf{id}_B|_\emptyset \fatsemi |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset\rangle) : C_0 \ .$$

By (Bs_CrcId) and applying (Bs_CrcMore) repeatedly, we have

$$\Sigma \mid \emptyset \vdash (V_4\, M_5)\langle d_1\rangle \cdots \langle d_n\rangle \approx (U_2'\,(U_3'\langle |\mathsf{id}_B|_\emptyset \fatsemi |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset\rangle))\langle |\mathsf{id}_{C_0}|_\emptyset \fatsemi |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset\rangle : A \ .$$

Therefore, by Lemma I.11, we conclude the case by

$$|\mathsf{id}_B|_\emptyset \fatsemi |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset = |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset, \quad |\mathsf{id}_{C_0}|_\emptyset \fatsemi |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset = |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset \ .$$

Case (2) in Lemma I.53: We are given

$$\begin{aligned} \Sigma \rhd M_2\, M_3 &\longrightarrow_C^* \Sigma \rhd V_2\, V_3 \\ &= \Sigma \rhd (V_4\langle c_1 \to d_1\rangle \cdots \langle c_n \to d_n\rangle)\, V_3 \\ &\longrightarrow_C^* \Sigma \rhd \mathsf{blame}\, p \quad (\exists p)\ , \end{aligned}$$

and

$$|\mathsf{id}_B|_\emptyset \fatsemi |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset = \bot^p \ .$$

By Lemma E.9, $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash B$. By (Ct_Id_C), $\Sigma \mid \emptyset \vdash_C \mathsf{id}_B : B \rightsquigarrow B$. By Lemma I.11 and Lemma I.7, $|\mathsf{id}_B|_\emptyset \fatsemi |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset = |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset$. Therefore, we have

$$|c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset = \bot^p \ .$$

Hence, by (R_Wrap_S), (R_Fail_S), (R_CtxE_S), (R_CtxC_S), (R_BlameE_S), (R_BlameC_S), we have

$$\begin{aligned} \Sigma \rhd M_2'\, M_3' &\longrightarrow_S^* \Sigma \rhd V_2'\, V_3' \\ &= \Sigma \rhd (U_2'\langle (|c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset) \to (|d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset)\rangle)\, U_3' \\ &\longrightarrow_S \Sigma \rhd (U_2'\,(U_3'\langle |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset\rangle))\langle |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset\rangle \\ &= \Sigma \rhd (U_2'\,(U_3'\langle \bot^p\rangle))\langle |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset\rangle \\ &\longrightarrow_S \Sigma \rhd (U_2'\,(\mathsf{blame}\, p))\langle |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset\rangle \\ &\longrightarrow_S \Sigma \rhd (\mathsf{blame}\, p)\langle |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset\rangle \\ &\longrightarrow_S \Sigma \rhd \mathsf{blame}\, p\ . \end{aligned}$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$. By Lemma I.23, we have $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Therefore, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case $V_3' = U_3'\langle t\rangle$ $(\exists U_3', t)$: By Lemma I.53, we consider the following two cases.

Case (1) in Lemma I.53: We are given

$$\begin{aligned} \Sigma \triangleright M_2\, M_3 \quad &= \quad \Sigma \triangleright (\, V_4 \langle c_1 \to d_1 \rangle \cdots \langle c_n \to d_n \rangle)\, V_3 \\ &\longrightarrow_C^* \Sigma \triangleright (\, V_4\, M_5) \langle d_1 \rangle \cdots \langle d_n \rangle \quad (\exists M_5)\ , \end{aligned}$$

and

$$\Sigma \mid \emptyset \vdash M_5 \approx U_3' \langle t \fatsemi |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset \rangle : B_0\ .$$

Hence, by (R_Wrap_S), (R_Merge_S), (R_CtxE_S), (R_CtxC_S), (R_BlameE_S), (R_BlameC_S), we have

$$\begin{aligned} \Sigma \triangleright M_2'\, M_3' &\longrightarrow_S^* \Sigma \triangleright V_2'\, V_3' \\ &= \quad \Sigma \triangleright (\, U_2' \langle (|c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset) \to (|d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset) \rangle)\, (U_3' \langle t \rangle) \\ &\longrightarrow_S \Sigma \triangleright (\, U_2'\, (\, U_3' \langle t \rangle \langle |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset \rangle)) \langle |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset \rangle \\ &\longrightarrow_S \Sigma \triangleright (\, U_2'\, (\, U_3' \langle t \fatsemi (|c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset) \rangle)) \langle |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset \rangle \\ &= \quad \Sigma \triangleright (\, U_2'\, (\, U_3' \langle t \fatsemi |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset \rangle)) \langle |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset \rangle\ . \end{aligned}$$

Therefore, it suffices to show that

$$\Sigma \mid \emptyset \vdash (\, V_4\, M_5) \langle d_1 \rangle \cdots \langle d_n \rangle \approx (\, U_2'\, (\, U_3' \langle t \fatsemi |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset \rangle)) \langle |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset \rangle : A\ .$$

By (Bs_App),

$$\Sigma \mid \emptyset \vdash V_4\, M_5 \approx U_2'\, (\, U_3' \langle t \fatsemi |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset \rangle) : C_0\ .$$

By Lemma I.23, $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash C_0$. By (Ct_Id_C), $\Sigma \mid \emptyset \vdash_C \mathsf{id}_{C_0} : C_0 \rightsquigarrow C_0$. By (Bs_CrcId) and (Bs_CrcMore) repeatedly,

$$\Sigma \mid \emptyset \vdash (\, V_4\, M_5) \langle d_1 \rangle \cdots \langle d_n \rangle \approx (\, U_2'\, (\, U_3' \langle t \fatsemi |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset \rangle)) \langle |\mathsf{id}_{C_0}|_\emptyset \fatsemi |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset \rangle : A\ .$$

Furthermore, by Lemma I.11 and Lemma I.7, we have $|\mathsf{id}_{C_0}|_\emptyset \fatsemi |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset = |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset$. Therefore, we finish the case.

Case (2) in Lemma I.53: We are given

$$\begin{aligned} \Sigma \triangleright M_2\, M_3 \quad &= \quad \Sigma \triangleright (\, V_4 \langle c_1 \to d_1 \rangle \cdots \langle c_n \to d_n \rangle)\, V_3 \\ &\longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p\ , \end{aligned}$$

and

$$t \fatsemi |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset = \bot^p \quad (\exists p)\ .$$

Furthermore, by (R_Wrap_S), (R_Merge_S) (R_Fail_S), (R_CtxE_S), (R_CtxC_S), (R_BlameE_S), (R_BlameC_S), we have

$$\begin{aligned} \Sigma \triangleright M_2'\, M_3' &\longrightarrow_S^* \Sigma \triangleright V_2'\, V_3' \\ &= \quad \Sigma \triangleright (\, U_2' \langle (|c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset) \to (|d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset) \rangle)\, (U_3' \langle t \rangle) \\ &\longrightarrow_S \Sigma \triangleright (\, U_2'\, (\, U_3' \langle t \rangle \langle |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset \rangle)) \langle |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset \rangle \\ &\longrightarrow_S \Sigma \triangleright (\, U_2'\, (\, U_3' \langle t \fatsemi (|c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset) \rangle)) \langle |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset \rangle \\ &= \quad \Sigma \triangleright (\, U_2'\, (\, U_3' \langle t \fatsemi |c_n|_\emptyset \fatsemi \cdots \fatsemi |c_1|_\emptyset \rangle)) \langle |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset \rangle \\ &= \quad \Sigma \triangleright (\, U_2'\, (\, U_3' \langle \bot^p \rangle)) \langle |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset \rangle \\ &\longrightarrow_S \Sigma \triangleright (\, U_2'\, (\mathsf{blame}\, p)) \langle |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset \rangle \\ &\longrightarrow_S \Sigma \triangleright (\mathsf{blame}\, p) \langle |d_1|_\emptyset \fatsemi \cdots \fatsemi |d_n|_\emptyset \rangle \\ &\longrightarrow_S \Sigma \triangleright \mathsf{blame}\, p\ . \end{aligned}$$

Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$. By Lemma I.23, we have $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Therefore, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case (R_BlameE_S): We are given

$$M'_2\, M'_3 = E[\mathsf{blame}\, p], \quad M'_1 \;=\; \mathsf{blame}\, p \quad (\exists p, E)\ .$$

Because $M'_2\, M'_3 = E[\mathsf{blame}\, p]$, we consider the following two cases.

Case $E = \Box\, M'_3$ and $M'_2 \;=\; \mathsf{blame}\, p$: Since $\Sigma \mid \emptyset \vdash M_2 \approx \mathsf{blame}\, p : B \to A$, by the case (6), we have $\Sigma \triangleright M_2 \;\longrightarrow^*_C\; \Sigma \triangleright \mathsf{blame}\, p$. By (R_Ctx_C) and (R_Blame_C),

$$\begin{aligned}\Sigma \triangleright M_2\, M_3 &\longrightarrow^*_C \Sigma \triangleright (\mathsf{blame}\, p)\, M_3 \\ &\longrightarrow_C \Sigma \triangleright \mathsf{blame}\, p\ .\end{aligned}$$

Thus, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$. By Lemma I.23, $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Therefore, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case $E = V'_2\, \Box$ and $M'_2 \;=\; V'_2$ and $M'_3 \;=\; \mathsf{blame}\, p$ $(\exists V'_2)$: Since $\Sigma \mid \emptyset \vdash M_3 \approx \mathsf{blame}\, p : B$, by the case (6), $\Sigma \triangleright M_3 \;\longrightarrow^*_C\; \Sigma \triangleright \mathsf{blame}\, p$. By the case (4), there exists a value V_2 such that

$$\Sigma \triangleright M_2 \;\longrightarrow^*_C\; \Sigma \triangleright V_2, \quad \Sigma \mid \emptyset \vdash V_2 \approx V'_2 : B \to A\ .$$

By (R_Ctx_C) and (R_Blame_C),

$$\begin{aligned}\Sigma \triangleright M_2\, M_3 &\longrightarrow^*_C \Sigma \triangleright V_2\, M_3 \\ &\longrightarrow^*_C \Sigma \triangleright V_2\, (\mathsf{blame}\, p) \\ &\longrightarrow_C \Sigma \triangleright \mathsf{blame}\, p\ .\end{aligned}$$

Thus, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$. By Lemma I.23, $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Therefore, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case (R_CtxE_S): We are given

$$M'_2\, M'_3 = E[M'_4], \quad M'_1 \;=\; E[M'_5], \quad \Sigma \triangleright M'_4 \;\longrightarrow_S\; \Sigma_1 \triangleright M'_5 \quad (\exists E, M'_4, M'_5)\ .$$

Because $M'_2\, M'_3 = E[M'_4]$, we consider the following two cases.

Case $E = \Box\, M'_3$ and $M'_2 \;=\; M'_4$: By $\Sigma \mid \emptyset \vdash M_2 \approx M'_2 : B \to A$ and $\Sigma \triangleright M'_2 \;\longrightarrow_S\; \Sigma_1 \triangleright M'_5$ and the IH,

$$\Sigma_1 \triangleright M'_5 \;\longrightarrow^*_S\; \Sigma_2 \triangleright M'_6, \quad \Sigma \triangleright M_2 \;\longrightarrow^*_C\; \Sigma_2 \triangleright M_6, \quad \Sigma_2 \mid \emptyset \vdash M_6 \approx M'_6 : B \to A \quad (\exists \Sigma_2, M_6, M'_6)\ .$$

By applying (R_CtxE_S) repeatedly, we have

$$\begin{aligned}\Sigma \triangleright M'_2\, M'_3 &\longrightarrow_S \Sigma_1 \triangleright M'_5\, M'_3 \\ &\longrightarrow^*_S \Sigma_2 \triangleright M'_6\, M'_3\ .\end{aligned}$$

Similarly, by applying (R_Ctx_C) repeatedly, we have

$$\Sigma \triangleright M_2\, M_3 \;\longrightarrow^*_C\; \Sigma_2 \triangleright M_6\, M_3\ .$$

Therefore, it suffices to show that $\Sigma_2 \mid \emptyset \vdash M_6\, M_3 \approx M'_6\, M'_3 : A$. By Lemma I.26, $\Sigma_2 \mid \emptyset \vdash M_3 \approx M'_3 : B$. Finally, by (Bs_App), $\Sigma_2 \mid \emptyset \vdash M_6\, M_3 \approx M'_6\, M'_3 : A$.

Case $E = V'_2\, \Box$ and $M'_2 \;=\; V'_2$ and $M'_3 \;=\; M'_4$ $(\exists V'_2)$: Because $\Sigma \mid \emptyset \vdash M_2 \approx V'_2 : B \to A$, by the case (4), there exists some V_2 such that

$$\Sigma \triangleright M_2 \;\longrightarrow^*_C\; \Sigma \triangleright V_2, \quad \Sigma \mid \emptyset \vdash V_2 \approx V'_2 : B \to A\ .$$

By the IH (applied to $\Sigma \mid \emptyset \vdash M_3 \approx M'_3 : B$ and $\Sigma \triangleright M'_3 \;\longrightarrow_S\; \Sigma_1 \triangleright M'_5$),

$$\Sigma_1 \triangleright M'_5 \;\longrightarrow^*_S\; \Sigma_2 \triangleright M'_6, \quad \Sigma \triangleright M_3 \;\longrightarrow^*_C\; \Sigma_2 \triangleright M_6, \quad \Sigma_2 \mid \emptyset \vdash M_6 \approx M'_6 : B \quad (\exists \Sigma_2, M_6, M'_6)\ .$$

By applying (R_CtxE_S) repeatedly, we have

$$\begin{aligned}\Sigma \triangleright M_2'\, M_3' \quad &= \quad \Sigma \triangleright V_2'\, M_3' \\ &\longrightarrow_S \Sigma_1 \triangleright V_2'\, M_5' \\ &\longrightarrow_S^* \Sigma_2 \triangleright V_2'\, M_6' \ .\end{aligned}$$

Furthermore, by applying (R_Ctx_C) repeatedly, we have

$$\begin{aligned}\Sigma \triangleright M_2\, M_3 &\longrightarrow_C^* \Sigma \triangleright V_2\, M_3 \\ &\longrightarrow_C^* \Sigma_2 \triangleright V_2\, M_6 \ .\end{aligned}$$

Therefore, it suffices to show that $\Sigma_2 \mid \emptyset \vdash V_2\, M_6 \approx V_2'\, M_6' : A$. By Lemma I.26, $\Sigma_2 \mid \emptyset \vdash V_2 \approx V_2' : B \to A$. Finally, by (Bs_App), $\Sigma_2 \mid \emptyset \vdash V_2\, M_6 \approx V_2'\, M_6' : A$.

Case (Bs_Tyapp): We are given

$$A = C[X := B], \quad M \ = \ M_2\, B, \quad M' \ = \ M_2'\, B, \quad \Sigma \mid \emptyset \vdash M_2 \approx M_2' : \forall X.C, \quad \Sigma \mid \emptyset \vdash B \quad (\exists X, B, C, M_2, M_3) \ .$$

By case analysis on the rule applied last to derive $\Sigma \triangleright M_2'\, B \ \longrightarrow_S \ \Sigma_1 \triangleright M_1'$, which is one of the following rules.

Case (R_Tybeta_S): We are given

$$M_2' \ = \ \Lambda Y.M_3', \quad B = \mathbb{B}, \quad M_1' \ = \ M_3'[Y := \alpha], \quad \Sigma' \ = \ \Sigma, \alpha := \mathbb{B} \quad (\exists\, Y, M_3', \mathbb{B}, \alpha) \ .$$

Then, since $\Lambda Y.M_3'$ is a value, by the case (4), we have

$$\Sigma \triangleright M_2 \ \longrightarrow_C^* \ \Sigma \triangleright V_2, \quad \Sigma \mid \emptyset \vdash V_2 \approx \Lambda Y.M_3' : \forall X.C \quad (\exists\, V_2) \ .$$

By Lemma I.58,

$$\begin{gathered} V_2 = \ V_3 \langle \forall X.c_1^I \rangle \cdots \langle \forall X.c_n^I \rangle, \quad C_n = C, \\ \Sigma \mid \emptyset \vdash V_3 \approx \Lambda Y.M_3' : \forall X.C_0, \quad \Sigma \mid \emptyset, X \vdash_C \ c_i^I : C_{i-1} \rightsquigarrow C_i \\ (V_3, C_0, \ldots, C_n, c_0^I, \ldots, c_j^I) \ . \end{gathered}$$

Furthermore, since $\Sigma \mid \emptyset \vdash V_3 \approx \Lambda Y.M_3' : \forall X.C$ is dericed by (Bs_Tyabs), we have

$$X \ = \ Y, \quad V_3 \ = \ \Lambda X.(M_3 : C_0), \quad \Sigma \mid \emptyset, X \vdash M_3 \approx M_3' : C_0 \quad (\exists M_3) \ .$$

Therefore, by (R_Ctx_C) and (R_Tybeta_C),

$$\begin{aligned} \Sigma \triangleright M_2\, \mathbb{B} &\longrightarrow_C^* \Sigma \triangleright V_2\, \mathbb{B} \\ &= \quad \Sigma \triangleright (\Lambda X.(M_3 : C_0)\langle \forall X.c_1^I \rangle \cdots \langle \forall X.c_n^I \rangle)\, \mathbb{B} \\ &= \quad \Sigma \triangleright (\Lambda X.(M_3 : C_0)\overline{\langle \forall X.c^I \rangle})\, \mathbb{B} \\ &\longrightarrow_C \Sigma, \alpha := \mathbb{B} \triangleright (M_3 \overline{\langle c^I \rangle})[X := \alpha]\langle \mathit{coerce}_\alpha^+(C[X := \alpha]) \rangle \\ &= \quad \Sigma, \alpha := \mathbb{B} \triangleright (M_3 \langle c_1^I \rangle \cdots \langle c_n^I \rangle)[X := \alpha]\langle \mathit{coerce}_\alpha^+(C[X := \alpha]) \rangle \end{aligned}$$

Hence, it suffices to show that

$$\begin{aligned} &\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash \\ &\qquad (M_3 \langle c_1^I \rangle \cdots \langle c_n^I \rangle)[X := \alpha]\langle \mathit{coerce}_\alpha^+(C[X := \alpha]) \rangle \\ &\qquad \approx \\ &\qquad M_3'[X := \alpha] \\ &\qquad\qquad\qquad\qquad : C[X := \mathbb{B}] \ . \end{aligned}$$

By Lemma I.23, we have $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash \forall X.C$. Therefore, by (Tew_Tyvar), $\Sigma \vdash \emptyset, X$. Furthermore, because $\Sigma \mid \emptyset \vdash \forall X.C$ is derived by (Tw_Poly), we have $\Sigma \mid \emptyset, X \vdash C$. Therefore, by Lemma E.17, we have

$$\Sigma \mid \emptyset \vdash_C \ \mathit{coerce}^+_\alpha(C[X := \alpha]) : C[X := \alpha] \rightsquigarrow C[X := \mathbb{B}] \ .$$

Moreover, by Lemma I.8, $\mathit{coerce}^+_\alpha(C[X := \alpha])$ is a no-op coercion. Now, we have $\Sigma \mid \emptyset, X \vdash M_3 \approx M'_3 : C_0$. By Lemma I.26, $\Sigma, \alpha := \mathbb{B} \mid \emptyset, X \vdash M_3 \approx M'_3 : C_0$. By Lemma I.33, $\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash M_3[X := \alpha] \approx M'_3[X := \alpha] : C_0[X := \alpha]$. By Lemma E.13 and Lemma E.6, we have $\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash_C c^I_i[X := \alpha] : C_{i-1}[X := \alpha] \rightsquigarrow C_i[X := \alpha]$. By applying (Bs_CrcIdL) repeatedly, we have

$$\begin{array}{l} \Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash \\ \qquad M_3[X := \alpha]\langle c^I_1[X := \alpha]\rangle \cdots \langle c^I_n[X := \alpha]\rangle \\ \qquad \approx \\ \qquad M'_3[X := \alpha] \\ \qquad\qquad\qquad : C[X := \alpha] \ . \end{array}$$

By (Bs_CrcIdL) again, we have

$$\begin{array}{l} \Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash \\ \qquad (M_3[X := \alpha]\langle c^I_1[X := \alpha]\rangle \cdots \langle c^I_n[X := \alpha]\rangle)\langle \mathit{coerce}^+_\alpha(C[X := \alpha])\rangle \\ \qquad \approx \\ \qquad M'_3[X := \alpha] \\ \qquad\qquad\qquad : C[X := \mathbb{B}] \ . \end{array}$$

Since $M_3[X := \alpha]\langle c^I_1[X := \alpha]\rangle \cdots \langle c^I_n[X := \alpha]\rangle \ = \ (M_3\langle c^I_1\rangle \cdots \langle c^I_n\rangle)[X := \alpha]$, we finish the case.

Case (R_TybetaDyn_S): We are given

$$M'_2 \ = \ \Lambda Y.M'_3, \quad B = \star, \quad M'_1 \ = \ M'_3[Y := \star] \quad (\exists Y, M'_3) \ .$$

Then, since $\Lambda Y.M'_3$ is a value, by the case (4), we have

$$\Sigma \rhd M_2 \ \longrightarrow^*_C \ \Sigma \rhd V_2, \quad \Sigma \mid \emptyset \vdash V_2 \approx \Lambda Y.M'_3 : \forall X.C \quad (\exists V_2) \ .$$

By Lemma I.58,

$$\begin{array}{c} V_2 \ = \ V_3\langle \forall X.c^I_1\rangle \cdots \langle \forall X.c^I_n\rangle, \quad C_n \ = \ C, \\ \Sigma \mid \emptyset \vdash V_3 \approx \Lambda Y.M'_3 : \forall X.C_0, \quad \Sigma \mid \emptyset, X \vdash_C \ c^I_i : C_{i-1} \rightsquigarrow C_i \\ (\exists V_3, C_0, \ldots, C_n, c^I_0, \ldots, c^I_j) \end{array}$$

Furthermore, since $\Sigma \mid \emptyset \vdash V_3 \approx \Lambda Y.M'_3 : \forall X.C$ is dericed by (Bs_Tyabs), we have

$$X \ = \ Y, \quad V_3 \ = \ \Lambda X.(M_3 : C_0), \quad \Sigma \mid \emptyset, X \vdash M_3 \approx M'_3 : C_0 \quad (\exists M_3) \ .$$

Therefore, by (R_Ctx_C) and (R_TybetaDyn_C),

$$\begin{array}{rl} \Sigma \rhd M_2\,\star \longrightarrow^*_C & \Sigma \rhd V_2\,\star \\ = & \Sigma \rhd (\Lambda X.(M_3 : C_0)\langle \forall X.c^I_1\rangle \cdots \langle \forall X.c^I_n\rangle)\,\star \\ = & \Sigma \rhd (\Lambda X.(M_3 : C_0)\overline{\langle \forall X.c^I\rangle})\,\star \\ \longrightarrow_C & \Sigma \rhd (M_3\overline{\langle c^I\rangle})[X := \star] \\ = & \Sigma \rhd (M_3\langle c^I_1\rangle \cdots \langle c^I_n\rangle)[X := \star] \end{array}$$

Hence, it suffices to show that

$$\Sigma \mid \emptyset \vdash (M_3\langle c_1^I\rangle \cdots \langle c_n^I\rangle)[X := \star] \approx M_3'[X := \star] : C[X := \star] \ .$$

Now, we have $\Sigma \mid \emptyset, X \vdash M_3 \approx M_3' : C_0$. By applying (Bs_CrcIdL) repeatedly, we have

$$\Sigma \mid \emptyset, X \vdash M_3\langle c_1^I\rangle \cdots \langle c_n^I\rangle \approx M_3' : C \ .$$

By Lemma I.33, we finish the case by getting

$$\Sigma \mid \emptyset \vdash (M_3\langle c_1^I\rangle \cdots \langle c_n^I\rangle)[X := \star] \approx M_3'[X := \star] : C[X := \star] \ .$$

Case (R_TybetaC_S): We are given

$$M_2' = (\Lambda Y.M_3')\langle \forall Y.s \,,, t\rangle, \quad B = \mathbb{B},$$
$$M_1' = (M_3'\langle s\rangle)[Y := \alpha], \quad \Sigma' = \Sigma, \alpha := \mathbb{B} \quad (\exists Y, M_3', s, t, \mathbb{B}, \alpha) \ .$$

Then, since $(\Lambda Y.M_3')\langle \forall Y.s \,,, t\rangle$ is a value, by the case (4), we have

$$\Sigma \triangleright M_2 \longrightarrow_C^* \Sigma \triangleright V_2, \quad \Sigma \mid \emptyset \vdash V_2 \approx (\Lambda Y.M_3')\langle \forall Y.s \,,, t\rangle : \forall X.C \quad (\exists V_2) \ .$$

By Lemma I.36, we have $V_2 = V_3\langle vc_1\rangle \cdots \langle vc_n\rangle$ $(\exists V_3)$. By Lemma I.39, we have

$$\Sigma \mid \emptyset \vdash V_3 \approx \Lambda Y.M_3' : A_0, \quad \Sigma \mid \emptyset \vdash_C vc_i : A_{i-1} \rightsquigarrow A_i, \quad A_n = \forall X.C, \quad j \leq n,$$
$$vc_i = vc_i^I \ (1 \leq i \leq j), \quad \forall Y.s \,,, t = |\mathsf{id}_{A_{j+1}}|_\emptyset \mathbin{⨟} |vc_{j+1}|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |vc_n|_\emptyset$$
$$(\exists A_0, \ldots, A_n, j, vc_0^I, \ldots, vc_j^I)$$

Furthermore, $\Sigma \mid \emptyset \vdash_C vc_n : A_{n-1} \rightsquigarrow \forall X.C$ is derived by (Ct_All_C), we have

$$A_{n-1} = \forall X.C_{n-1}, \quad C_n = C, \quad vc_n = \forall X.c_n,$$
$$vc_i = vc_i^I = \forall X.c_i^I \ (1 \leq i \leq j), \quad \Sigma \mid \emptyset, X \vdash_C c_n : C_{n-1} \rightsquigarrow C \ .$$

Similarly, $\Sigma \mid \emptyset \vdash_C vc_i : A_{i-1} \rightsquigarrow \forall X.C_i$ is derived by (Ct_All_C), we have

$$A_i = \forall X.C_i, \quad C_n = C, \quad vc_i = \forall X.c_i,$$
$$\Sigma \mid \emptyset, X \vdash_C c_i : C_{i-1} \rightsquigarrow C_i \quad (\exists C_0, \ldots, C_n, c_1, \ldots, c_n) \ (1 \leq i \leq n) \ .$$

Therefore, $\Sigma \mid \emptyset \vdash V_3 \approx \Lambda Y.M_3' : \forall X.C_0$ is derived by (Bs_Tyabs), we have

$$Y = X, \quad V_3 = \Lambda X.(M_3 : C_0), \quad \Sigma \mid \emptyset, X \vdash M_3 \approx M_3' : C_0 \quad (\exists M_3) \ .$$

Moreover, by Lemma I.11, we have

$$\begin{aligned}
\forall Y.s \,,, t &= |\mathsf{id}_{\forall X.C_{j+1}}|_\emptyset \mathbin{⨟} |\forall X.c_{j+1}|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |\forall X.c_n|_\emptyset \\
&= |\forall X.c_{j+1}|_\emptyset \mathbin{⨟} \cdots \mathbin{⨟} |\forall X.c_n|_\emptyset \\
&= (\forall X.|c_{j+1}|_{\emptyset,X} \,,, |c_{j+1}|_\emptyset) \mathbin{⨟} \cdots \mathbin{⨟} (\forall X.|c_n|_{\emptyset,X} \,,, |c_n|_\emptyset) \\
&= \forall X.(|c_{j+1}|_{\emptyset,X} \mathbin{⨟} \cdots \mathbin{⨟} |c_n|_{\emptyset,X}) \,,, (|c_{j+1}|_\emptyset \,,, |c_n|_\emptyset) \\
&= \forall X.(|c_1|_{\emptyset,X} \mathbin{⨟} \cdots \mathbin{⨟} |c_n|_{\emptyset,X}) \,,, (|c_1|_\emptyset \,,, |c_n|_\emptyset) \ .
\end{aligned}$$

Therefore, $Y = X$, $s = |c_{j+1}|_{\emptyset,X} \mathbin{⨟} \cdots \mathbin{⨟} |c_n|_{\emptyset,X}$, and $t = |c_{j+1}|_\emptyset \,,, |c_n|_\emptyset$. Furthermore, by applying (Ct_Cons_C) repeatedly, we have $\Sigma \vdash \overline{\langle \forall X.c_i\rangle} : \forall X.C_0 \rightsquigarrow \forall X.C$. Therefore, by (R_Ctx_C) and (R_Tybeta_C),

$$\begin{aligned}
\Sigma \triangleright M_2\,\mathbb{B} &\longrightarrow_C^* \Sigma \triangleright V_2\,\mathbb{B} \\
&= \quad \Sigma \triangleright (\Lambda X.(M_3 : C_0)\langle \forall X.c_1\rangle \cdots \langle \forall X.c_n\rangle)\,\mathbb{B} \\
&= \quad \Sigma \triangleright (\Lambda X.(M_3 : C_0)\overline{\langle \forall X.c\rangle})\,\mathbb{B} \\
&\longrightarrow_C \Sigma, \alpha := \mathbb{B} \triangleright (M_3\overline{\langle c\rangle})[X := \alpha]\langle coerce_\alpha^+(C[X := \alpha])\rangle \\
&= \quad \Sigma, \alpha := \mathbb{B} \triangleright (M_3\langle c_1\rangle \cdots \langle c_n\rangle)[X := \alpha]\langle coerce_\alpha^+(C[X := \alpha])\rangle
\end{aligned}$$

Hence, it suffices to show that

$$\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash \\ (M_3\langle c_1\rangle \cdots \langle c_n\rangle)[X := \alpha]\langle \mathit{coerce}^+_\alpha(C[X := \alpha])\rangle \\ \approx \\ (M'_3\langle |c_1|_{\emptyset,X} \mathbin{;} \cdots \mathbin{;} |c_n|_{\emptyset,X}\rangle)[X := \alpha] \\ : C[X := \mathbb{B}] \ .$$

By Lemma I.23, we have $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash \forall X.C$. Therefore, by (Tew_Tyvar), $\Sigma \vdash \emptyset, X$. Furthermore, because $\Sigma \mid \emptyset \vdash \forall X.C$ is derived by (Tw_Poly), we have $\Sigma \mid \emptyset, X \vdash C$. Therefore, by Lemma E.17, we have

$$\Sigma \mid \emptyset \vdash_C \ \mathit{coerce}^+_\alpha(C[X := \alpha]) : C[X := \alpha] \rightsquigarrow C[X := \mathbb{B}] \ .$$

Moreover, by Lemma I.8, $\mathit{coerce}^+_\alpha(C[X := \alpha])$ is a no-op coercion. Now, we have $\Sigma \mid \emptyset, X \vdash M_3 \approx M'_3 : C_0$. By Lemma I.26, $\Sigma, \alpha := \mathbb{B} \mid \emptyset, X \vdash M_3 \approx M'_3 : C_0$. By Lemma I.33, $\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash M_3[X := \alpha] \approx M'_3[X := \alpha] : C_0[X := \alpha]$. By (Bs_CrcId), $\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash M_3[X := \alpha] \approx M'_3[X := \alpha]\langle |\mathsf{id}_{C_0[X:=\alpha]}|_\emptyset\rangle : C_0[X := \alpha]$. By Lemma E.13 and Lemma E.6, we have $\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash_C c_i[X := \alpha] : C_{i-1}[X := \alpha] \rightsquigarrow C_i[X := \alpha]$. By applying (Bs_CrcMore) repeatedly, we have

$$\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash \\ M_3[X := \alpha]\langle c_1[X := \alpha]\rangle \cdots \langle c_n[X := \alpha]\rangle \\ \approx \\ M'_3[X := \alpha]\langle |\mathsf{id}_{C_0[X:=\alpha]}|_\emptyset \mathbin{;} |c_1[X := \alpha]|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n[X := \alpha]|_\emptyset\rangle \\ : C[X := \alpha] \ .$$

By (Bs_CrcIdL), we have

$$\Sigma, \alpha := \mathbb{B} \mid \emptyset \vdash \\ (M_3[X := \alpha]\langle c_1[X := \alpha]\rangle \cdots \langle c_n[X := \alpha]\rangle)\langle \mathit{coerce}^+_\alpha(C[X := \alpha])\rangle \\ \approx \\ M'_3[X := \alpha]\langle |\mathsf{id}_{C_0[X:=\alpha]}|_\emptyset \mathbin{;} |c_1[X := \alpha]|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n[X := \alpha]|_\emptyset\rangle \\ : C[X := \mathbb{B}] \ .$$

By Lemma I.32 and Lemma I.28, we have

$$\begin{aligned} & |\mathsf{id}_{C_0[X:=\alpha]}|_\emptyset \mathbin{;} |c_1[X := \alpha]|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n[X := \alpha]|_\emptyset \\ = {} & |\mathsf{id}_{C_0}[X := \alpha]|_\emptyset \mathbin{;} |c_1[X := \alpha]|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n[X := \alpha]|_\emptyset \\ = {} & |\mathsf{id}_{C_0}|_{\emptyset,X}[X := \alpha] \mathbin{;} |c_1|_{\emptyset,X}[X := \alpha] \mathbin{;} \cdots \mathbin{;} |c_n|_{\emptyset,X}[X := \alpha] \\ = {} & (|\mathsf{id}_{C_0}|_{\emptyset,X} \mathbin{;} |c_1|_{\emptyset,X} \mathbin{;} \cdots \mathbin{;} |c_n|_{\emptyset,X})[X := \alpha] \ . \end{aligned}$$

Therefore, since

$$\begin{aligned} & (M'_3\langle |\mathsf{id}_{C_0}|_{\emptyset,X} \mathbin{;} |c_1|_{\emptyset,X} \mathbin{;} \cdots \mathbin{;} |c_n|_{\emptyset,X}\rangle)[X := \alpha] \\ = {} & M'_3[X := \alpha]\langle (|\mathsf{id}_{C_0}|_{\emptyset,X} \mathbin{;} |c_1|_{\emptyset,X} \mathbin{;} \cdots \mathbin{;} |c_n|_{\emptyset,X})[X := \alpha]\rangle \\ = {} & M'_3[X := \alpha]\langle |\mathsf{id}_{C_0[X:=\alpha]}|_\emptyset \mathbin{;} |c_1[X := \alpha]|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n[X := \alpha]|_\emptyset\rangle \ , \end{aligned}$$

and $M_3[X := \alpha]\langle c_1[X := \alpha]\rangle \cdots \langle c_n[X := \alpha]\rangle = (M_3\langle c_1\rangle \cdots \langle c_n\rangle)[X := \alpha]$, we finish the case.

Case (R_TybetaDynC_S): We are given

$$M'_2 = (\Lambda Y.M'_3)\langle \forall Y.s \,,, t\rangle, \quad B = \star, \quad M'_1 = M'_3[Y := \star]\langle t\rangle \quad (\exists Y, M'_3, s, t) \ .$$

Then, since $(\Lambda Y.M'_3)\langle \forall Y.s \,,, t\rangle$ is a value, by the case (4), we have

$$\Sigma \triangleright M_2 \longrightarrow^*_C \Sigma \triangleright V_2, \quad \Sigma \mid \emptyset \vdash V_2 \approx (\Lambda Y.M'_3)\langle \forall Y.s \,,, t\rangle : \forall X.C \quad (\exists V_2) \ .$$

By Lemma I.36, we have $V_2 = V_3\langle vc_1\rangle \cdots \langle vc_n\rangle$ $(\exists V_3)$. By Lemma I.39, we have

$$\Sigma \mid \emptyset \vdash V_3 \approx \Lambda Y.M_3' : A_0, \quad \Sigma \mid \emptyset \vdash_C vc_i : A_{i-1} \rightsquigarrow A_i, \quad A_n = \forall X.C, \quad j \leq n,$$
$$vc_i = vc_i^I \ (1 \leq i \leq j), \quad \forall Y.s \,,, t = |\mathsf{id}_{A_{j+1}}|_\emptyset \,;\, |vc_{j+1}|_\emptyset \,;\, \cdots \,;\, |vc_n|_\emptyset$$
$$(\exists A_0, \ldots, A_n, j, vc_0^I, \ldots, vc_j^I)$$

Furthermore, $\Sigma \mid \emptyset \vdash_C vc_n : A_{n-1} \rightsquigarrow \forall X.C$ is derived by (Ct_All_C), we have

$$A_{n-1} = \forall X.C_{n-1}, \quad C_n = C, \quad vc_n = \forall X.c_n,$$
$$vc_i = vc_i^I = \forall X.c_i^I \ (1 \leq i \leq j), \quad \Sigma \mid \emptyset, X \vdash_C c_n : C_{n-1} \rightsquigarrow C \ .$$

Similarly, $\Sigma \mid \emptyset \vdash_C vc_i : A_{i-1} \rightsquigarrow \forall X.C_i$ is derived by (Ct_All_C), we have

$$A_i = \forall X.C_i, \quad C_n = C, \quad vc_i = \forall X.c_i,$$
$$\Sigma \mid \emptyset, X \vdash_C c_i : C_{i-1} \rightsquigarrow C_i \quad (\exists C_0, \ldots, C_n, c_1, \ldots, c_n) \ (1 \leq i \leq n) \ .$$

Therefore, $\Sigma \mid \emptyset \vdash V_3 \approx \Lambda Y.M_3' : \forall X.C_0$ is derived by (Bs_Tyabs), we have

$$Y = X, \quad V_3 = \Lambda X.(M_3 : C_0), \quad \Sigma \mid \emptyset, X \vdash M_3 \approx M_3' : C_0 \quad (\exists M_3) \ .$$

Moreover, by Lemma I.11, we have

$$\begin{aligned}
\forall Y.s \,,, t &= |\mathsf{id}_{\forall X.C_{j+1}}|_\emptyset \,;\, |\forall X.c_{j+1}|_\emptyset \,;\, \cdots \,;\, |\forall X.c_n|_\emptyset \\
&= |\forall X.c_{j+1}|_\emptyset \,;\, \cdots \,;\, |\forall X.c_n|_\emptyset \\
&= (\forall X.|c_{j+1}|_{\emptyset,X} \,,, |c_{j+1}|_\emptyset) \,;\, \cdots \,;\, (\forall X.|c_n|_{\emptyset,X} \,,, |c_n|_\emptyset) \\
&= \forall X.(|c_{j+1}|_{\emptyset,X} \,;\, \cdots \,;\, |c_n|_{\emptyset,X}) \,,, (|c_{j+1}|_\emptyset \,,, |c_n|_\emptyset) \\
&= \forall X.(|c_1|_{\emptyset,X} \,;\, \cdots \,;\, |c_n|_{\emptyset,X}) \,,, (|c_1|_\emptyset \,,, |c_n|_\emptyset) \ .
\end{aligned}$$

Therefore, $Y = X$, $s = |c_{j+1}|_{\emptyset,X} \,;\, \cdots \,;\, |c_n|_{\emptyset,X}$, and $t = |c_{j+1}|_\emptyset \,,, |c_n|_\emptyset$. Therefore, by (R_Ctx_C) and (R_TybetaDyn_C),

$$\begin{aligned}
\Sigma \rhd M_2 \star \longrightarrow_C^* \ & \Sigma \rhd V_2 \star \\
= \ & \Sigma \rhd (\Lambda X.(M_3 : C_0)\langle \forall X.c_1\rangle \cdots \langle \forall X.c_n\rangle) \star \\
= \ & \Sigma \rhd (\Lambda X.(M_3 : C_0)\overline{\langle \forall X.c\rangle}) \star \\
\longrightarrow_C \ & \Sigma \rhd (M_3\overline{\langle c\rangle})[X := \star] \\
= \ & \Sigma \rhd (M_3\langle c_1\rangle \cdots \langle c_n\rangle)[X := \star]
\end{aligned}$$

Hence, it suffices to show that

$$\begin{aligned}
\Sigma \mid \emptyset \vdash & \\
& (M_3\langle c_1\rangle \cdots \langle c_n\rangle)[X := \star] \\
& \approx \\
& M_3'[X := \star]\langle |c_1|_\emptyset \,;\, \cdots \,;\, |c_n|_\emptyset\rangle \\
& \qquad\qquad\qquad : C[X := \star] \ .
\end{aligned}$$

Now, we have $\Sigma \mid \emptyset, X \vdash M_3 \approx M_3' : C_0$. By Lemma I.33, $\Sigma \mid \emptyset \vdash M_3[X := \star] \approx M_3'[X := \star] : C_0[X := \star]$. By (Bs_CrcId), $\Sigma \mid \emptyset \vdash M_3[X := \star] \approx M_3'[X := \star]\langle |\mathsf{id}_{C_0[X:=\star]}|_\emptyset\rangle : C_0[X := \star]$. By Lemma E.15, we have $\Sigma \mid \emptyset \vdash_C c_i[X := \star] : C_{i-1}[X := \star] \rightsquigarrow C_i[X := \star]$. By applying (Bs_CrcMore) repeatedly, we have

$$\begin{aligned}
\Sigma, \alpha := \star \mid \emptyset \vdash & \\
& M_3[X := \star]\langle c_1[X := \star]\rangle \cdots \langle c_n[X := \star]\rangle \\
& \approx \\
& M_3'[X := \star]\langle |\mathsf{id}_{C_0[X:=\star]}|_\emptyset \,;\, |c_1[X := \star]|_\emptyset \,;\, \cdots \,;\, |c_n[X := \star]|_\emptyset\rangle \\
& \qquad\qquad\qquad\qquad\qquad : C[X := \star] \ .
\end{aligned}$$

By Lemma I.30, we have

$$\begin{aligned} & |\mathsf{id}_{C_0[X:=\star]}|_\emptyset \mathbin{;} |c_1[X := \star]|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n[X := \star]|_\emptyset \\ = & |\mathsf{id}_{C_0}[X := \star]|_\emptyset \mathbin{;} |c_1[X := \star]|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n[X := \star]|_\emptyset \\ = & |\mathsf{id}_{C_0}|_\emptyset \mathbin{;} |c_1|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n|_\emptyset \ . \end{aligned}$$

Therefore, since

$$\begin{aligned} & M_3'[X := \star]\langle |c_1|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n|_\emptyset \rangle \\ = & M_3'[X := \star]\langle |\mathsf{id}_{C_0}|_\emptyset \mathbin{;} |c_1|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n|_\emptyset \rangle \\ = & M_3'[X := \star]\langle |\mathsf{id}_{C_0[X:=\star]}|_\emptyset \mathbin{;} |c_1[X := \star]|_\emptyset \mathbin{;} \cdots \mathbin{;} |c_n[X := \star]|_\emptyset \rangle \ , \end{aligned}$$

and $M_3[X := \star]\langle c_1[X := \star]\rangle \cdots \langle c_n[X := \star]\rangle = (M_3\langle c_1\rangle \cdots \langle c_n\rangle)[X := \star]$, we finish the case.

Case (R_BlameE_S): We are given

$$M_2' = \mathsf{blame}\, p, \quad M_1' = \mathsf{blame}\, p, \quad \Sigma_1 = \Sigma \quad (\exists p) \ .$$

Because $\Sigma \mid \emptyset \vdash M_2 \approx \mathsf{blame}\, p : \forall X.C$, by the case (6), we have

$$M_2 \longrightarrow_C^* \mathsf{blame}\, p \ .$$

Therefore, by (R_Blame_C),

$$\begin{aligned} \Sigma \triangleright M_2\, B \quad &= \quad \Sigma \triangleright (\mathsf{blame}\, p)\, B \\ &\longrightarrow_C \Sigma \triangleright \mathsf{blame}\, p \ . \end{aligned}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : C[X := B]$. Lemma I.23 implies $\vdash \Sigma$, $\Sigma \vdash \emptyset$, and $\Sigma \mid \emptyset \vdash C[X := B]$. Therefore, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : C[X := B]$.

Case (R_CtxE_S): We are given

$$M_1' = M_3'\, B, \quad \Sigma \triangleright M_2' \longrightarrow_S \Sigma_1 \triangleright M_3' \quad (\exists M_3') \ .$$

Therefore, by the IH, we have

$$\Sigma_1 \triangleright M_3' \longrightarrow_S^* \Sigma_2 \triangleright M_4', \quad \Sigma \triangleright M_2 \longrightarrow_C^* \Sigma_2 \triangleright M_4, \quad \Sigma_2 \mid \emptyset \vdash M_4 \approx M_4' : \forall X.C \quad (\exists \Sigma_2, M_4, M_4') \ .$$

Hence, by applying (R_CtxE_S) repeatedly, we have

$$\begin{aligned} \Sigma \triangleright M_2'\, B &\longrightarrow_S \Sigma_1 \triangleright M_3'\, B \\ &\longrightarrow_S^* \Sigma_2 \triangleright M_4'\, B \ . \end{aligned}$$

Furthermore, by applying (R_Ctx_C) repeatedly, we have

$$\Sigma \triangleright M_2\, B \longrightarrow_C^* \Sigma_2 \triangleright M_4\, B \ .$$

Therefore, it suffices to show that $\Sigma_2 \mid \emptyset \vdash M_4\, B \approx M_4'\, B : C[X := B]$. By Lemma E.8, we have $\Sigma_2 \mid \emptyset \vdash B$. Hence, by (Bs_Tyapp), $\Sigma_2 \mid \emptyset \vdash M_4\, B \approx M_4'\, B : C[X := B]$.

Case (Bs_CrcId): We are given

$$M' = M_2'\langle |\mathsf{id}_A|_\emptyset \rangle, \quad \Sigma \mid \emptyset \vdash M \approx M_2' : A \quad \Sigma \mid \emptyset \vdash_C \mathsf{id}_A : A \rightsquigarrow A \quad (\exists M_2') \ .$$

By case analysis on the rule applied last to derive $\Sigma \triangleright M_2'\langle |\mathsf{id}_A|_\emptyset \rangle \longrightarrow_S \Sigma_1 \triangleright M_1'$, which is one of the following rules.

Case (R_Id_S): We are given

$$M_2' = U_2', \quad |\mathsf{id}_A|_\emptyset = \mathsf{id}, \quad M_1' = U_2', \quad \Sigma_1 = \Sigma \quad (\exists U_2', A') \ .$$

By the case (4),

$$\Sigma \triangleright M \longrightarrow_C^* \Sigma \triangleright V, \quad \Sigma \mid \emptyset \vdash V \approx U_2' : A \quad (\exists V) \ .$$

finishing the case.

Case (R_FAIL_S): Cannot happen since $|\mathsf{id}_A|_\emptyset$ cannot be a failure coercion $\bot^p$.

Case (R_MERGE_S): We are given

$$M_2' = M_3'\langle s\rangle, \quad M_1' = M_3'\langle s \fatsemi |\mathsf{id}_A|_\emptyset\rangle, \quad \Sigma_1 \ = \ \Sigma \quad (\exists s, M_3') \ .$$

By Lemma I.22, we have $\Sigma \mid \emptyset \vdash_S \ M_3'\langle s\rangle : \Sigma(A)$. This judgment must be derived by (T_CRC_S) and we have $\Sigma \mid \emptyset \vdash_S \ s : B \rightsquigarrow \Sigma(A)$ for some B. By Lemma I.11, we have $s \fatsemi |\mathsf{id}_A|_\emptyset = s$. Hence, $M_1' = M_3'\langle s\rangle$. Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash M \approx M_3'\langle s\rangle : A$, which holds already.

Case (R_BLAMEC_S): We are given

$$M_2' \ = \ \mathsf{blame}\, p, \quad M_1' \ = \ \mathsf{blame}\, p, \quad \Sigma_1 \ = \ \Sigma \quad (\exists p) \ .$$

Since $\Sigma \mid \emptyset \vdash M \approx \mathsf{blame}\, p : A$, by the case (6), $\Sigma \triangleright M \ \longrightarrow_C^* \ \Sigma \triangleright \mathsf{blame}\, p$. Hence, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$. By Lemma I.23 implies $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Therefore, by (BS_BLAME), $\Sigma_2 \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case (R_CTXC_S): We are given

$$M_1' = M_3'\langle |\mathsf{id}_A|_\emptyset\rangle, \quad \Sigma \triangleright M_2' \ \longrightarrow_S \ \Sigma_1 \triangleright M_3' \quad (\exists M_3') \ .$$

By the IH,

$$\Sigma_1 \triangleright M_3' \ \longrightarrow_S^* \ \Sigma_2 \triangleright M_4', \quad \Sigma \triangleright M \ \longrightarrow_C^* \ \Sigma_2 \triangleright M_4, \quad \Sigma_2 \mid \emptyset \vdash M_4 \approx M_4' : A \quad (\exists \Sigma_2, M_4, M_4') \ .$$

By Lemma I.22, $\Sigma \mid \emptyset \vdash_S \ M_2' : \Sigma(A)$. By Theorem E.45, $\Sigma_1 \mid \emptyset \vdash_S \ M_3' : \Sigma(A)$. By Lemma I.3, $\Sigma_1 \mid \emptyset \vdash_S \ |\mathsf{id}_A|_\emptyset : \Sigma(A) \rightsquigarrow \Sigma(A)$. By Lemma I.20, we consider the following three cases.

Case $\Sigma_1 \triangleright M_3'\langle |\mathsf{id}_A|_\emptyset\rangle \ \longrightarrow_S^* \ \Sigma_2 \triangleright M_4'\langle |\mathsf{id}_A|_\emptyset\rangle$: We have

$$\begin{aligned}\Sigma \triangleright M_2'\langle |\mathsf{id}_A|_\emptyset\rangle &\longrightarrow_S \Sigma_1 \triangleright M_3'\langle |\mathsf{id}_A|_\emptyset\rangle \\ &\longrightarrow_S^* \Sigma_2 \triangleright M_4'\langle |\mathsf{id}_A|_\emptyset\rangle \ .\end{aligned}$$

Thus, it suffices to show that

$$\Sigma_2 \mid \emptyset \vdash M_4 \approx M_4'\langle |\mathsf{id}_A|_\emptyset\rangle : A \ .$$

The rule (BS_CRCID) finishes the case.

Case $\Sigma \triangleright M_3'\langle |\mathsf{id}_A|_\emptyset\rangle \ \longrightarrow_S^* \ \Sigma_2 \triangleright M_5'\langle s \fatsemi |\mathsf{id}_A|_\emptyset\rangle, \quad M_4' = M_5'\langle s\rangle \quad (\exists B, s, M_5')$: We have

$$\begin{aligned}\Sigma \triangleright M_2'\langle |\mathsf{id}_A|_\emptyset\rangle &\longrightarrow_S \Sigma_1 \triangleright M_3'\langle |\mathsf{id}_A|_\emptyset\rangle \\ &\longrightarrow_S^* \Sigma_2 \triangleright M_5'\langle s \fatsemi |\mathsf{id}_A|_\emptyset\rangle \ .\end{aligned}$$

Thus, it suffices to show that

$$\Sigma_2 \mid \emptyset \vdash M_4 \approx M_5'\langle s \fatsemi |\mathsf{id}_A|_\emptyset\rangle : A \ .$$

The rule (BS_CRCMORE) finishes the case.

Case $\Sigma \triangleright M_3'\langle |\mathsf{id}_A|_\emptyset\rangle \ \longrightarrow_S^* \ \Sigma_2 \triangleright M_4', \quad M_4' \ = \ \mathsf{blame}\, p \quad (\exists p)$: We have

$$\begin{aligned}\Sigma \triangleright M_2'\langle |\mathsf{id}_A|_\emptyset\rangle &\longrightarrow_S \Sigma_1 \triangleright M_3'\langle |\mathsf{id}_A|_\emptyset\rangle \\ &\longrightarrow_S^* \Sigma_2 \triangleright \mathsf{blame}\, p \ .\end{aligned}$$

It suffices to show that $\Sigma_2 \mid \emptyset \vdash M_4 \approx \mathsf{blame}\, p : A$, which holds already because $M_4' \ = \ \mathsf{blame}\, p$.

Case (BS_CRC): We are given

$$M = M_2\langle c\rangle, \quad M' = M_2'\langle |c|_\emptyset\rangle, \quad \Sigma \mid \emptyset \vdash M_2 \approx M_2' : B, \quad \Sigma \mid \emptyset \vdash_C \ c : B \rightsquigarrow A \quad (\exists B, c, M_2, M_2') \ .$$

By case analysis on the rule applied last to derive $\Sigma \triangleright M_2'\langle |c|_\emptyset\rangle \ \longrightarrow_S \ \Sigma_1 \triangleright M_1'$, which is one of the following rules.

Case (R_Id_S): We are given

$$M_2' = U_2', \quad |c|_\emptyset = \mathsf{id}, \quad M_1' = U_2', \quad \Sigma_1 = \Sigma \quad (\exists U_2') .$$

By Lemma I.36, there exist some M_3 that is not a coercion application, $n > 0$, and $c_1, \ldots, c_n$ such that $M_2\langle c\rangle = M_3\langle c_1\rangle \cdots \langle c_n\rangle$ and $c_n = c$. Hence, by Lemma I.40, there exists some V_3 such that $M_3 = V_3$. By Lemma I.52, since we have

$$\Sigma \mid \emptyset \vdash V_3\langle c_1\rangle \cdots \langle c_n\rangle \approx U_2'\langle \mathsf{id}\rangle : A ,$$

we consider the following two cases.

Case (1) in Lemma I.52: We are given

$$\Sigma \triangleright V_3\langle c_1\rangle \cdots \langle c_n\rangle \longrightarrow_C^* \Sigma \triangleright V_4, \quad \Sigma \mid \emptyset \vdash V_4 \approx U_2'\langle \mathsf{id}\rangle : A \quad (\exists V_4) .$$

Therefore,

$$\begin{aligned} \Sigma \triangleright M_2\langle c\rangle &= \Sigma \triangleright V_3\langle c_1\rangle \cdots \langle c_n\rangle \\ &\longrightarrow_C^* \Sigma \triangleright V_4 . \end{aligned}$$

Hence, it suffices to show that

$$\Sigma \mid \emptyset \vdash V_4 \approx U_2' : A .$$

By case analysis on V_4.

Case $V_4 = V_5\langle c'\rangle$ ($\exists c'$, V_5): By Lemma I.36, there exist some V_6 that is not a coercion application, $m > 0$, and $c_1', \ldots, c_m'$ such that $V_5\langle c'\rangle = V_6\langle c_1'\rangle \cdots \langle c_m'\rangle$ and $c_m' = c'$. Because $V_6\langle c_1'\rangle \cdots \langle c_m'\rangle$ is a value, we have

$$vc_i = c_i' \quad (\exists vc_1, \ldots, vc_m)(1 \leq i \leq m) .$$

Therefore,

$$\Sigma \mid \emptyset \vdash V_6\langle vc_1\rangle \cdots \langle vc_m\rangle \approx U_2'\langle \mathsf{id}\rangle : A .$$

Then, by Lemma I.39, there exists $j \leq m$ such that $\mathsf{id} = |\mathsf{id}_{A_j}|_\emptyset \mathbin{⨾} |vc_j|_\emptyset \mathbin{⨾} \cdots \mathbin{⨾} |vc_m|_\emptyset$ and

$$A_m = A, \quad \Sigma \mid \emptyset \vdash_C vc_i : A_{i-1} \rightsquigarrow A_i \quad (\exists A_0, \ldots A_m)(1 \leq \forall i \leq m) .$$

Therefore, Lemma I.43 implies

$$vs_i = |vc_i|_\emptyset, \quad \Sigma \mid \emptyset \vdash_S vs_i : \Sigma(A_{i-1}) \rightsquigarrow \Sigma(A_i) \quad (\exists vs_1, \ldots, vs_m)(1 \leq \forall i \leq m) .$$

Hence, by Lemma I.44, there exists some vs such that $vs = vs_j \mathbin{⨾} \cdots \mathbin{⨾} vs_m$ and $\Sigma \mid \emptyset \vdash_S vs : \Sigma(A_0) \rightsquigarrow \Sigma(A)$. Therefore, by Lemma I.11, we have

$$\begin{aligned} \mathsf{id} &= |\mathsf{id}_{A_j}|_\emptyset \mathbin{⨾} |vc_j|_\emptyset \mathbin{⨾} \cdots \mathbin{⨾} |vc_m|_\emptyset \\ &= |vc_j|_\emptyset \mathbin{⨾} \cdots \mathbin{⨾} |vc_m|_\emptyset \\ &= vs_j \mathbin{⨾} \cdots \mathbin{⨾} vs_m \\ &= vs . \end{aligned}$$

Therefore, by Lemma I.54, we have $\Sigma \mid \emptyset \vdash V_6\langle vc_1'\rangle \cdots \langle vc_m'\rangle \approx U_2 : \alpha$.

Otherwise: Since V_4 is not a coercion application, $\Sigma \mid \emptyset \vdash V_4 \approx U_2'\langle |\mathsf{id}_A|_\emptyset\rangle : A$ is derived by (Bs_CrcId). Therefore, we have $\Sigma \mid \emptyset \vdash V_4 \approx U_2' : A$.

Case (2) in Lemma I.52: We are given

$$\Sigma \triangleright V_3\langle c_1\rangle \cdots \langle c_n\rangle \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p, \quad \mathsf{id} = \perp^p \quad (\exists p) .$$

Since $\mathsf{id} \neq \perp^p$, there is a contradiction.

Case (R_Fail_S): We are given

$$M_2' = U_2', \quad |c|_\emptyset = \bot^p, \quad M_1' = \mathsf{blame}\,p, \quad \Sigma_1 = \Sigma \quad (\exists p, U_2') .$$

By Lemma I.36, there exist some M_3 that is not a coercion application, $n > 0$, and $c_1, \ldots, c_n$ such that $M_2\langle c\rangle = M_3\langle c_1\rangle \cdots \langle c_n\rangle$ and $c_n = c$. Hence, by Lemma I.40, there exists some V_3 such that $M_3 = V_3$ and $\Sigma \mid \emptyset \vdash V_3 \approx U_2' : C$. By Lemma I.52, we consider the following two cases.

Case (1) in Lemma I.52: We are given

$$\Sigma \rhd V_3\langle c_1\rangle \cdots \langle c_n\rangle \longrightarrow_C^* \Sigma \rhd V_4, \quad \Sigma \mid \emptyset \vdash V_4 \approx U_2'\langle \bot^p\rangle : A \quad (\exists V_4) .$$

By Lemma I.50, there exists some i such that $\bot^p = i$. However, this is contradictory because $\bot^p$ is not an intermediate coercion.

Case (2) in Lemma I.52: We are given

$$\Sigma \rhd V_3\langle c_1\rangle \cdots \langle c_n\rangle \longrightarrow_C^* \Sigma \rhd \mathsf{blame}\,p \quad (\exists p) .$$

Therefore,

$$\begin{aligned} \Sigma \rhd M_2\langle c\rangle &= \Sigma \rhd V_3\langle c_1\rangle \cdots \langle c_n\rangle \\ &\longrightarrow_C^* \Sigma \rhd \mathsf{blame}\,p . \end{aligned}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\,p \approx \mathsf{blame}\,p : A$. Lemma I.23 implies $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Hence, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\,p \approx \mathsf{blame}\,p : A$.

Case (R_Merge_S): We are given

$$M_2' = M_3'\langle s'\rangle, \quad M_1' = M_3'\langle s' \,;\, (|c|_\emptyset)\rangle, \quad \Sigma_1 = \Sigma \quad (\exists s', M_3') .$$

Then, it suffices to show that

$$\Sigma \mid \emptyset \vdash M_2\langle c\rangle \approx M_3'\langle s' \,;\, (|c|_\emptyset)\rangle : A ,$$

which is given by (Bs_CrcMore).

Case (R_BlameC_S): We are given

$$M_2' = \mathsf{blame}\,p, \quad M_1' = \mathsf{blame}\,p, \quad \Sigma_1 = \Sigma \quad (\exists p) .$$

By Lemma I.36, there exist some M_3 that is not a coercion application, $n > 0$, and $c_1, \ldots, c_n$ such that $M_2\langle c\rangle = M_3\langle c_1\rangle \cdots \langle c_n\rangle$ and $c_n = c$. Therefore, Lemma I.39 implies $\Sigma \mid \emptyset \vdash M_3 \approx \mathsf{blame}\,p : C$ $(\exists C)$. Since M_3 is not a coercion application, $\Sigma \mid \emptyset \vdash M_3 \approx \mathsf{blame}\,p : C$ is derived by (Bs_Blame), so we have $M_3 = \mathsf{blame}\,p$. Therefore, by applying (R_Blame_C) repeatedly, we have

$$\begin{aligned} \Sigma \rhd M_2\langle c\rangle &= \Sigma \rhd M_3\langle c_1\rangle \cdots \langle c_n\rangle \\ &= \Sigma \rhd (\mathsf{blame}\,p)\langle c_1\rangle \cdots \langle c_n\rangle \\ &\longrightarrow_C^* \Sigma \rhd \mathsf{blame}\,p . \end{aligned}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\,p \approx \mathsf{blame}\,p : A$. Lemma I.23 implies $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Therefore, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\,p \approx \mathsf{blame}\,p : A$.

Case (R_CtxC_S): We are given

$$M_1' = M_3'\langle |c|_\emptyset\rangle, \quad \Sigma \rhd M_2' \xrightarrow{e}_S \Sigma_1 \rhd M_3', \quad \Sigma_1 = \Sigma \quad (\exists M_3') .$$

By Lemma I.36, there exist some M_3 that is not a coercion application, $n > 0$, and $c_1, \ldots, c_n$ such that $M_2\langle c\rangle = M_3\langle c_1\rangle \cdots \langle c_n\rangle$ and $c_n = c$. Furthermore, because $\Sigma \rhd M_2' \xrightarrow{e}_S \Sigma_1 \rhd M_3'$, M_2' is not a coercion application. Therefore, because $\Sigma \mid \emptyset \vdash M_3\langle c_1\rangle \cdots \langle c_n\rangle \approx M_2'\langle |c|_\emptyset\rangle : A$, by Lemma I.39, there exists $j \leq n$ such that

$$|c|_\emptyset = |\mathsf{id}_{A_{j+1}}|_\emptyset \,;\, |c_{j+1}|_\emptyset \,;\, \cdots \,;\, |c_n|_\emptyset, \quad \Sigma \mid \emptyset \vdash M_3 \approx M_2' : C \quad (\exists C) .$$

Moreover, the derivation $\Sigma \mid \emptyset \vdash M_3 \approx M'_2 : C$ is a sub derivation of the derivation $\Sigma \mid \emptyset \vdash M_3\langle c_1\rangle \cdots \langle c_n\rangle \approx M'_2\langle |c|_\emptyset\rangle : A$. Therefore, by the IH,

$$\Sigma \triangleright M_3 \longrightarrow^*_C \Sigma_2 \triangleright M_4, \quad \Sigma_1 \triangleright M'_3 \longrightarrow^*_S \Sigma_2 \triangleright M'_4, \quad \Sigma_2 \mid \emptyset \vdash M_4 \approx M'_4 : C \quad (\exists \Sigma_2, M_4, M'_4) .$$

Hence, by applying (R_Ctx_C) repeatedly, we have

$$\begin{aligned}\Sigma \triangleright M_2\langle c\rangle \quad &= \quad \Sigma \triangleright M_3\langle c_1\rangle \cdots \langle c_n\rangle \\ &\longrightarrow^*_C \Sigma_2 \triangleright M_4\langle c_1\rangle \cdots \langle c_n\rangle .\end{aligned}$$

Also, $\Sigma \triangleright M'_2 \longrightarrow^*_S \Sigma_2 \triangleright M'_4$. Furthermore, Lemma I.22 implies $\Sigma \mid \emptyset \vdash_S M'_2\langle |c|_\emptyset\rangle : \Sigma(A)$. Because this judgment is derived by (T_Crc_S), we have $\Sigma \mid \emptyset \vdash_S M'_2 : D$ and $\Sigma \mid \emptyset \vdash_S |c|_\emptyset : D \rightsquigarrow A$ $(\exists D)$. Therefore, Lemma I.20, we consider the following three cases.

Case (1) in Lemma I.20: We are given

$$\Sigma \triangleright M'_2\langle |c|_\emptyset\rangle \longrightarrow^*_S \Sigma_2 \triangleright M'_4\langle |c|_\emptyset\rangle .$$

Therefore, it suffices to show that

$$\Sigma_2 \mid \emptyset \vdash M_4\langle c_1\rangle \cdots \langle c_n\rangle \approx M'_4\langle |c|_\emptyset\rangle : A .$$

By Lemma E.44 and Lemma I.26, $\Sigma_2 \mid \emptyset \vdash M_3\langle c_1\rangle \cdots \langle c_n\rangle \approx M'_2\langle |c|_\emptyset\rangle : A$. Hence, by Lemma I.45, $\Sigma_2 \mid \emptyset \vdash M_4\langle c_1\rangle \cdots \langle c_n\rangle \approx M'_4\langle |c|_\emptyset\rangle : A$.

Case (2) in Lemma I.20: We are given

$$\Sigma \triangleright M'_2\langle |c|_\emptyset\rangle \longrightarrow^*_S \Sigma_2 \triangleright M'_5\langle t \fatsemi |c|_\emptyset\rangle, \quad M'_4 = M'_5\langle t\rangle \quad (\exists t, M'_5) .$$

Therefore, it suffices to show that

$$\Sigma_2 \mid \emptyset \vdash M_4\langle c_1\rangle \cdots \langle c_n\rangle \approx M'_5\langle t \fatsemi |c|_\emptyset\rangle : A .$$

By Lemma E.44 and Lemma I.26, $\Sigma_2 \mid \emptyset \vdash M_3\langle c_1\rangle \cdots \langle c_n\rangle \approx M'_2\langle |c|_\emptyset\rangle : A$. Here, we have $\Sigma_2 \mid \emptyset \vdash M_4 \approx M'_5\langle t\rangle : C$. By Lemma I.45, $\Sigma_2 \mid \emptyset \vdash M_4\langle c_1\rangle \cdots \langle c_n\rangle \approx M'_5\langle t\rangle\langle |c|_\emptyset\rangle : A$. Therefore, Lemma I.34 implies $\Sigma_2 \mid \emptyset \vdash M_4\langle c_1\rangle \cdots \langle c_n\rangle \approx M'_5\langle t \fatsemi |c|_\emptyset\rangle : A$.

Case (3) in Lemma I.20: We are given

$$\Sigma \triangleright M'_2\langle |c|_\emptyset\rangle \longrightarrow^*_S \Sigma_2 \triangleright M'_4, \quad M'_4 = \mathsf{blame}\, p \quad (\exists p) .$$

Therefore, $\Sigma_2 \mid \emptyset \vdash M_4 \approx \mathsf{blame}\, p : C$, so by the case (6), we have

$$\Sigma \triangleright M_4 \longrightarrow^*_C \Sigma \triangleright \mathsf{blame}\, p .$$

Hence, by applying (R_Ctx_C) and (R_Blame_C), we have

$$\begin{aligned}\Sigma \triangleright M_2\langle c\rangle &\longrightarrow^*_C \Sigma_2 \triangleright M_4\langle c_1\rangle \cdots \langle c_n\rangle \\ &\longrightarrow^*_C \Sigma_2 \triangleright (\mathsf{blame}\, p)\langle c_1\rangle \cdots \langle c_n\rangle \\ &\longrightarrow^*_C \Sigma_2 \triangleright \mathsf{blame}\, p .\end{aligned}$$

Therefore, it suffices to show that $\Sigma_2 \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$. By Lemma E.44 and Lemma I.26, $\Sigma_2 \mid \emptyset \vdash M_3\langle c_1\rangle \cdots \langle c_n\rangle \approx M'_2\langle |c|_\emptyset\rangle : A$. By Lemma I.23, $\vdash \Sigma_2$ and $\Sigma_2 \vdash \emptyset$ and $\Sigma_2 \mid \emptyset \vdash A$. Hence, by (Bs_Blame), $\Sigma_2 \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case (Bs_CrcMore): We are given

$$M = M_2\langle c\rangle, \quad M' = M'_2\langle s \fatsemi |c|_\emptyset\rangle, \quad \Sigma \mid \emptyset \vdash M_2 \approx M'_2\langle s\rangle : B,$$
$$\Sigma \mid \emptyset \vdash_C c : B \rightsquigarrow A \quad (\exists B, c, s, M_2, M'_2) .$$

By case analysis on the rule applied last to derive $\Sigma \triangleright M'_2\langle s \fatsemi |c|_\emptyset\rangle \longrightarrow_S \Sigma_1 \triangleright M'_1$, which is one of the following rules.

Case (R_ID_S): We are given

$$M_2' = U_2', \quad s \mathbin{;} |c|_\emptyset = \mathsf{id}, \quad M_1' = U_2', \quad \Sigma_1 = \Sigma \quad (\exists U_2') .$$

By Lemma I.36, there exist some M_3 that is not a coercion application, $n > 0$, and $c_1, \ldots, c_n$ such that $M_2\langle c\rangle = M_3\langle c_1\rangle \cdots \langle c_n\rangle$ and $c_n = c$. Hence, by Lemma I.40, there exists some V_3 such that $M_3 = V_3$ and $\Sigma \mid \emptyset \vdash V_3 \approx U_2' : C$ $(\exists C)$. Therefore,

$$\Sigma \mid \emptyset \vdash V_3\langle c_1\rangle \cdots \langle c_n\rangle \approx U_2'\langle \mathsf{id}\rangle : A .$$

Hence, by Lemma I.52, we consider the following two cases.

Case (1) in Lemma I.52: We are given

$$\Sigma \triangleright V_3\langle c_1\rangle \cdots \langle c_n\rangle \longrightarrow_C^* \Sigma \triangleright V_4, \quad \Sigma \mid \emptyset \vdash V_4 \approx U_2'\langle \mathsf{id}\rangle : A \quad (\exists V_4) .$$

Therefore,

$$\begin{aligned} \Sigma \triangleright M_2\langle c\rangle &= \Sigma \triangleright M_3\langle c_1\rangle \cdots \langle c_n\rangle \\ &= \Sigma \triangleright V_3\langle c_1\rangle \cdots \langle c_n\rangle \\ &\longrightarrow_C^* \Sigma \triangleright V_4 . \end{aligned}$$

Hence, it suffices to show that

$$\Sigma \mid \emptyset \vdash V_4 \approx U_2' : A .$$

By case analysis on V_4.

Case $V_4 = V_5\langle c'\rangle$ $(\exists c', V_5)$: By Lemma I.36, there exist some V_6 that is not a coercion application, $m > 0$, and $c_1', \ldots, c_m'$ such that $V_5\langle c'\rangle = V_6\langle c_1'\rangle \cdots \langle c_m'\rangle$ and $c_m' = c'$. Because $V_6\langle c_1'\rangle \cdots \langle c_m'\rangle$ is a value, we have

$$vc_i = c_i' \quad (\exists vc_1, \ldots, vc_m)(1 \leq \forall i \leq m) .$$

Therefore,

$$\Sigma \mid \emptyset \vdash V_6\langle vc_1\rangle \cdots \langle vc_m\rangle \approx U_2'\langle \mathsf{id}\rangle : A .$$

Then, by Lemma I.39, there exists $j \leq m$ such that $\mathsf{id} = |\mathsf{id}_{A_j}|_\emptyset \mathbin{;} |vc_j|_\emptyset \mathbin{;} \cdots \mathbin{;} |vc_m|_\emptyset$ and

$$A_m = A, \quad \Sigma \mid \emptyset \vdash_C vc_i : A_{i-1} \rightsquigarrow A_i \quad (\exists A_0, \ldots A_m)(1 \leq \forall i \leq m) .$$

Therefore, Lemma I.43 implies

$$vs_i = |vc_i|_\emptyset, \quad \Sigma \mid \emptyset \vdash_S vs_i : \Sigma(A_{i-1}) \rightsquigarrow \Sigma(A_i) \quad (\exists vs_1, \ldots, vs_m)(1 \leq \forall i \leq m) .$$

Hence, by Lemma I.44, there exists some vs such that $vs = vs_j \mathbin{;} \cdots \mathbin{;} vs_m$ and $\Sigma \mid \emptyset \vdash_S vs : \Sigma(A_0) \rightsquigarrow \Sigma(A)$. Therefore, by Lemma I.11, we have

$$\begin{aligned} \mathsf{id} &= |\mathsf{id}_{A_j}|_\emptyset \mathbin{;} |vc_j|_\emptyset \mathbin{;} \cdots \mathbin{;} |vc_m|_\emptyset \\ &= |vc_j|_\emptyset \mathbin{;} \cdots \mathbin{;} |vc_m|_\emptyset \\ &= vs_j \mathbin{;} \cdots \mathbin{;} vs_m \\ &= vs . \end{aligned}$$

Hence, by Lemma I.54, we have $\Sigma \mid \emptyset \vdash V_6\langle vc_1'\rangle \cdots \langle vc_m'\rangle \approx U_2 : \alpha$.

Otherwise: Because $\Sigma \mid \emptyset \vdash V_4 \approx U_2'\langle |\mathsf{id}_A|_\emptyset\rangle : A$ is derived by (BS_CRCID), we have $\Sigma \mid \emptyset \vdash V_4 \approx U_2' : A$.

Case (2) in Lemma I.52: We are given

$$\Sigma \triangleright V_3\langle c_1\rangle \cdots \langle c_n\rangle \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p, \quad \mathsf{id} = \bot^p \quad (\exists p) .$$

Because $\mathsf{id} \neq \bot^p$, there is a contradiction.

Case (R_Fail_S): We are given

$$M_2' = U_2', \quad s \mathbin{;} |c|_\emptyset = \bot^p, \quad M_1' = \mathsf{blame}\, p, \quad \Sigma_1 = \Sigma \quad (\exists p, U_2') .$$

By Lemma I.36, there exist some M_3 that is not a coercion application, $n > 0$, and $c_1, \ldots, c_n$ such that $M_2\langle c\rangle = M_3\langle c_1\rangle \cdots \langle c_n\rangle$ and $c_n = c$. Hence, by Lemma I.40, there exists some V_3 such that $M_3 = V_3$ and $\Sigma \mid \emptyset \vdash V_3 \approx U_2' : C$. By Lemma I.52, we consider the following two cases.

Case (1) in Lemma I.52: We are given

$$\Sigma \triangleright V_3\langle c_1\rangle \cdots \langle c_n\rangle \longrightarrow_C^* \Sigma \triangleright V_4, \quad \Sigma \mid \emptyset \vdash V_4 \approx U_2'\langle \bot^p\rangle : A \quad (\exists V_4) .$$

By Lemma I.50, there exists some i such that $\bot^p = i$. However, this is contradictory because $\bot^p$ is not an intermediate coercion.

Case (2) in Lemma I.52: We are given

$$\Sigma \triangleright V_3\langle c_1\rangle \cdots \langle c_n\rangle \longrightarrow_C^* \Sigma \triangleright \mathsf{blame}\, p \quad (\exists p) .$$

Therefore,

$$\begin{array}{rcl} \Sigma \triangleright M_2\langle c\rangle & = & \Sigma \triangleright M_3\langle c_1\rangle \cdots \langle c_n\rangle \\ & = & \Sigma \triangleright V_3\langle c_1\rangle \cdots \langle c_n\rangle \\ & \longrightarrow_C^* & \Sigma \triangleright \mathsf{blame}\, p . \end{array}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$. Lemma I.23 implies $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Hence, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case (R_Merge_S): We are given

$$M_2' = M_3'\langle s'\rangle, \quad M_1' = M_3'\langle s' \mathbin{;} (s \mathbin{;} |c|_\emptyset)\rangle, \quad \Sigma_1 = \Sigma \quad (\exists s', M_3') .$$

Then, it suffices to show that

$$\Sigma \mid \emptyset \vdash M_2\langle c\rangle \approx M_3'\langle s' \mathbin{;} (s \mathbin{;} |c|_\emptyset)\rangle : A .$$

Now, we have $\Sigma \mid \emptyset \vdash M_2 \approx M_3'\langle s'\rangle\langle s\rangle : B$. Therefore, Lemma I.34 implies $\Sigma \mid \emptyset \vdash M_2 \approx M_3'\langle s' \mathbin{;} s\rangle : B$. Hence, by (Bs_CrcMore),

$$\Sigma \mid \emptyset \vdash M_2\langle c\rangle \approx M_3'\langle (s' \mathbin{;} s) \mathbin{;} |c|_\emptyset\rangle : A .$$

Therefore, it suffices to show that

$$(s' \mathbin{;} s) \mathbin{;} |c|_\emptyset = s' \mathbin{;} (s \mathbin{;} |c|_\emptyset) .$$

Lemma I.22 implies $\Sigma \mid \emptyset \vdash_S M_3'\langle s'\rangle\langle s\rangle : \Sigma(B)$. By inversion of its derivation, we have $\Sigma \mid \emptyset \vdash_S s' : D \rightsquigarrow C$ and $\Sigma \mid \emptyset \vdash_S s : C \rightsquigarrow B$ $(\exists D, C)$. Furthermore, Lemma I.3 implies $\Sigma \mid \emptyset \vdash_S |c|_\emptyset : \Sigma(B) \rightsquigarrow \Sigma(A)$. Therefore, Lemma I.7 implies $(s' \mathbin{;} s) \mathbin{;} |c|_\emptyset = s' \mathbin{;} (s \mathbin{;} |c|_\emptyset)$.

Case (R_BlameC_S): We are given

$$M_2' = \mathsf{blame}\, p, \quad M_1' = \mathsf{blame}\, p, \quad \Sigma_1 = \Sigma \quad (\exists p) .$$

By Lemma I.36, there exist some M_3 that is not a coercion application, $n > 0$, and $c_1, \ldots, c_n$ such that $M_2\langle c\rangle = M_3\langle c_1\rangle \cdots \langle c_n\rangle$ and $c_n = c$. Therefore, Lemma I.39 implies $\Sigma \mid \emptyset \vdash M_3 \approx \mathsf{blame}\, p : C$ $(\exists C)$. Since M_3 is not a coercion application, $\Sigma \mid \emptyset \vdash M_3 \approx \mathsf{blame}\, p : C$ is derived by (Bs_Blame), so we have $M_3 = \mathsf{blame}\, p$. Therefore, by applying (R_Blame_C) repeatedly, we have

$$\begin{array}{rcl} \Sigma \triangleright M_2\langle c\rangle & = & \Sigma \triangleright M_3\langle c_1\rangle \cdots \langle c_n\rangle \\ & = & \Sigma \triangleright (\mathsf{blame}\, p)\langle c_1\rangle \cdots \langle c_n\rangle \\ & \longrightarrow_C^* & \Sigma \triangleright \mathsf{blame}\, p . \end{array}$$

Hence, it suffices to show that $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$. Lemma I.23 implies $\vdash \Sigma$ and $\Sigma \vdash \emptyset$ and $\Sigma \mid \emptyset \vdash A$. Therefore, by (Bs_Blame), $\Sigma \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case (R_CtxC_S): We are given

$$M_1' = M_3'\langle s \fatsemi |c|_\emptyset \rangle, \quad \Sigma \rhd M_2' \xrightarrow{e}_S \Sigma_1 \rhd M_3', \quad \Sigma_1 = \Sigma \quad (\exists M_3') .$$

By Lemma I.36, there exist some M_3 that is not a coercion application, $n > 0$, and $c_1, \ldots, c_n$ such that $M_2\langle c \rangle = M_3\langle c_1 \rangle \cdots \langle c_n \rangle$ and $c_n = c$. Furthermore, because $\Sigma \rhd M_2' \xrightarrow{e}_S \Sigma_1 \rhd M_3'$, M_2' is not a coercion application. Therefore, because $\Sigma \mid \emptyset \vdash M_3\langle c_1 \rangle \cdots \langle c_n \rangle \approx M_2'\langle s \fatsemi |c|_\emptyset \rangle : A$, by Lemma I.39, there exists $j \leq n$ such that

$$s \fatsemi |c|_\emptyset = |\mathsf{id}_{A_{j+1}}|_\emptyset \fatsemi |c_{j+1}|_\emptyset \fatsemi \cdots \fatsemi |c_n|_\emptyset, \quad \Sigma \mid \emptyset \vdash M_3 \approx M_2' : C \quad (\exists C) .$$

Moreover, the derivation $\Sigma \mid \emptyset \vdash M_3 \approx M_2' : C$ is a sub derivation of the derivation $\Sigma \mid \emptyset \vdash M_3\langle c_1 \rangle \cdots \langle c_n \rangle \approx M_2'\langle s \fatsemi |c|_\emptyset \rangle : A$. Therefore, by the IH,

$$\Sigma_1 \rhd M_3' \longrightarrow_S^* \Sigma_2 \rhd M_4', \quad \Sigma \rhd M_3 \longrightarrow_C^* \Sigma_2 \rhd M_4, \quad \Sigma_2 \mid \emptyset \vdash M_4 \approx M_4' : C \quad (\exists \Sigma_2, M_4, M_4') .$$

Hence, by applying (R_Ctx_C) repeatedly, we have

$$\begin{aligned} \Sigma \rhd M_2\langle c \rangle &= \Sigma \rhd M_3\langle c_1 \rangle \cdots \langle c_n \rangle \\ &\longrightarrow_C^* \Sigma_2 \rhd M_4\langle c_1 \rangle \cdots \langle c_n \rangle . \end{aligned}$$

Also, $\Sigma \rhd M_2' \longrightarrow_S^* \Sigma_2 \rhd M_4'$. Furthermore, Lemma I.22 implies $\Sigma \mid \emptyset \vdash_S M_2'\langle s \fatsemi |c|_\emptyset \rangle : \Sigma(A)$. Because this judgment is derived by (T_Crc_S), we have $\Sigma \mid \emptyset \vdash_S M_2' : D$ and $\Sigma \mid \emptyset \vdash_S s \fatsemi |c|_\emptyset : D \rightsquigarrow A$ $(\exists D)$. Therefore, Lemma I.20, we consider the following three cases.

Case (1) in Lemma I.20: We are given

$$\Sigma \rhd M_2'\langle s \fatsemi |c|_\emptyset \rangle \longrightarrow_S^* \Sigma_2 \rhd M_4'\langle s \fatsemi |c|_\emptyset \rangle .$$

Therefore, it suffices to show that

$$\Sigma_2 \mid \emptyset \vdash M_4\langle c_1 \rangle \cdots \langle c_n \rangle \approx M_4'\langle s \fatsemi |c|_\emptyset \rangle : A .$$

By Lemma E.44 and Lemma I.26, $\Sigma_2 \mid \emptyset \vdash M_3\langle c_1 \rangle \cdots \langle c_n \rangle \approx M_2'\langle s \fatsemi |c|_\emptyset \rangle : A$. Hence, by Lemma I.45, $\Sigma_2 \mid \emptyset \vdash M_4\langle c_1 \rangle \cdots \langle c_n \rangle \approx M_4'\langle s \fatsemi |c|_\emptyset \rangle : A$.

Case (2) in Lemma I.20: We are given

$$\Sigma \rhd M_2'\langle s \fatsemi |c|_\emptyset \rangle \longrightarrow_S^* \Sigma_2 \rhd M_5'\langle t \fatsemi (s \fatsemi |c|_\emptyset) \rangle, \quad M_4' = M_5'\langle t \rangle \quad (\exists t, M_5') .$$

Therefore, it suffices to show that

$$\Sigma_2 \mid \emptyset \vdash M_4\langle c_1 \rangle \cdots \langle c_n \rangle \approx M_5'\langle t \fatsemi (s \fatsemi |c|_\emptyset) \rangle : A .$$

By Lemma E.44 and Lemma I.26, $\Sigma_2 \mid \emptyset \vdash M_3\langle c_1 \rangle \cdots \langle c_n \rangle \approx M_2'\langle s \fatsemi |c|_\emptyset \rangle : A$. Furthermore, $\Sigma_2 \mid \emptyset \vdash M_4 \approx M_5'\langle t \rangle : C$. Hence, Lemma I.45 implies $\Sigma_2 \mid \emptyset \vdash M_4\langle c_1 \rangle \cdots \langle c_n \rangle \approx M_5'\langle t \rangle\langle s \fatsemi |c|_\emptyset \rangle : A$. Therefore, Lemma I.34 implies $\Sigma_2 \mid \emptyset \vdash M_4\langle c_1 \rangle \cdots \langle c_n \rangle \approx M_5'\langle t \fatsemi (s \fatsemi |c|_\emptyset) \rangle : A$.

Case (3) in Lemma I.20: We are given

$$\Sigma \rhd M_2'\langle s \fatsemi |c|_\emptyset \rangle \longrightarrow_S^* \Sigma_2 \rhd M_4', \quad M_4' = \mathsf{blame}\, p \quad (\exists p) .$$

Therefore, $\Sigma_2 \mid \emptyset \vdash M_4 \approx \mathsf{blame}\, p : C$, so by the case (6), we have

$$\Sigma \rhd M_4 \longrightarrow_C^* \Sigma \rhd \mathsf{blame}\, p .$$

Hence, by applying (R_Ctx_C) and (R_Blame_C), we have

$$\begin{aligned} \Sigma \rhd M_2\langle c \rangle &\longrightarrow_C^* \Sigma_2 \rhd M_4\langle c_1 \rangle \cdots \langle c_n \rangle \\ &\longrightarrow_C^* \Sigma_2 \rhd (\mathsf{blame}\, p)\langle c_1 \rangle \cdots \langle c_n \rangle \\ &\longrightarrow_C^* \Sigma_2 \rhd \mathsf{blame}\, p . \end{aligned}$$

Therefore, it suffices to show that $\Sigma_2 \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$. By Lemma E.44 and Lemma I.26, $\Sigma_2 \mid \emptyset \vdash M_3\langle c_1 \rangle \cdots \langle c_n \rangle \approx M_2'\langle s \fatsemi |c|_\emptyset \rangle : A$. Hence, Lemma I.23 implies $\vdash \Sigma_2$ and $\Sigma_2 \vdash \emptyset$ and $\Sigma_2 \mid \emptyset \vdash A$. Therefore, by (Bs_Blame), $\Sigma_2 \mid \emptyset \vdash \mathsf{blame}\, p \approx \mathsf{blame}\, p : A$.

Case (Bs_CrcIdL): We are given

$$M = M_2\langle c^I \rangle, \quad \Sigma \mid \emptyset \vdash M_2 \approx M' : B, \quad \Sigma \mid \emptyset \vdash_C c^I : B \rightsquigarrow A \quad (\exists M_2, c^I, B) .$$

By the IH, we have

$$\Sigma \triangleright M'_1 \longrightarrow^*_S \Sigma_3 \triangleright M'_3, \quad \Sigma \triangleright M_2 \longrightarrow^*_C \Sigma_3 \triangleright M_3, \quad \Sigma_3 \mid \emptyset \vdash M_3 \approx M'_3 : B \quad (\exists M_3, M'_3, \Sigma_3) .$$

By (R_Ctx_C), we have $\Sigma \triangleright M_2\langle c^I \rangle \longrightarrow^*_C \Sigma_3 \triangleright M_3\langle c^I \rangle$. Therefore, it suffices to show that $\Sigma \mid \emptyset \vdash M_3\langle c^I \rangle \approx M'_3 : A$, which is given by (Bs_CrcIdL).

□

Lemma I.60 (Bisimulation Preserves Behavior)**.** Suppose that $\Sigma \mid \emptyset \vdash M \approx M' : A$.

(1) If $\Sigma \triangleright M \longrightarrow^*_C \Sigma' \triangleright V$, then there exists some V' such that $\Sigma \triangleright M' \longrightarrow^*_S \Sigma' \triangleright V'$. Furthermore, if $A = \iota$, then there exists some k such that $V = V' = k$.

(2) If $\Sigma \triangleright M' \longrightarrow^*_S \Sigma' \triangleright V'$, then there exists some V such that $\Sigma \triangleright M \longrightarrow^*_C \Sigma' \triangleright V$. Furthermore, if $A = \iota$, then there exists some k such that $V = V' = k$.

(3) $\Sigma \triangleright M \longrightarrow^*_C \Sigma' \triangleright \mathsf{blame}\, p$ iff $\Sigma \triangleright M' \longrightarrow^*_S \Sigma' \triangleright \mathsf{blame}\, p$.

(4) $\Sigma \triangleright M \Uparrow$ iff $\Sigma \triangleright M' \Uparrow$.

Proof.

(1) By induction on the length of the evaluation sequence $\Sigma \triangleright M \longrightarrow^*_C \Sigma' \triangleright V$. By case analysis on the length.

Case The length is zero: We have $\Sigma = \Sigma'$ and $M = V$. Therefore, Lemma I.59 implies

$$\Sigma \triangleright M' \longrightarrow^*_S \Sigma \triangleright V', \quad \Sigma \mid \emptyset \vdash V \approx V' : A \quad (\exists V') .$$

Furthermore, assume $A = \iota$. Then, Lemma I.22 implies $\Sigma \mid \emptyset \vdash_S V' : \Sigma(\iota)$. Hence, Lemma E.25 implies that there exists some k such that $V' = k$. By Lemma I.58, V is not a coercion application. Therefore, since $\Sigma \mid \emptyset \vdash V \approx k : \iota$ is derived by (Bs_Const), we have $V = k$.

Case The length is larger than zero: We are given

$$\Sigma \triangleright M \longrightarrow_C \Sigma_1 \triangleright M_1, \quad \Sigma_1 \triangleright M_1 \longrightarrow^*_C \Sigma' \triangleright V \quad (\exists \Sigma_1, M_1) .$$

Lemma I.59 implies

$$\Sigma_1 \triangleright M_1 \longrightarrow^*_C \Sigma_2 \triangleright M_2, \quad \Sigma \triangleright M' \longrightarrow^*_S \Sigma_2 \triangleright M'_2, \quad \Sigma_2 \mid \emptyset \vdash M_2 \approx M'_2 : A \quad (\exists \Sigma_2, M_2, M'_2) .$$

Therefore, by applying Lemma I.12 repeatedly, we have

$$\Sigma_1 \triangleright M_1 \longrightarrow^*_C \Sigma_2 \triangleright M_2 \longrightarrow^*_C \Sigma' \triangleright V .$$

Then, by the IH, there exists some V' such that $\Sigma_2 \triangleright M'_2 \longrightarrow^*_S \Sigma' \triangleright V'$. Therefore,

$$\Sigma \triangleright M' \longrightarrow^*_S \Sigma_2 \triangleright M'_2 \longrightarrow^*_S \Sigma' \triangleright V' .$$

Furthermore, if $A = \iota$, then the IH implies that there exists some k such that $V = V' = k$.

(2) Provable similarly to the case (1).

(3) Each direction is provable similarly to the case (1).

(4) Consider each direction.

- ($\Longrightarrow$)
  Lemma I.22 implies $\Sigma \mid \emptyset \vdash_S M' : \Sigma(A)$. Therefore, by Theorem E.47, we consider the following three cases.

  Case There exist some Σ', V' such that $\Sigma \triangleright M' \longrightarrow_S^* \Sigma' \triangleright V'$: By the case (2), there exists some V such that $\Sigma \triangleright M \longrightarrow_C^* \Sigma' \triangleright V$. However, by Corollary F.2, it is contradictory with $\Sigma \triangleright M \Uparrow$.

  Case There exist some Σ', p such that $\Sigma \triangleright M' \longrightarrow_S^* \Sigma' \triangleright \mathsf{blame}\, p$: By the case (3), $\Sigma \triangleright M \longrightarrow_C^* \Sigma' \triangleright \mathsf{blame}\, p$. However, by Corollary F.2, it is contradictory with $\Sigma \triangleright M \Uparrow$.

  Case $\Sigma \triangleright M' \Uparrow$: Obvious.
- ($\Longleftarrow$)
  Provable similarly to the case of ($\Longrightarrow$).

□

Lemma I.61 (Relating Terms in $\lambda C^{\forall}_{mp}$ and $\lambda S^{\forall}_{mp}$ via Translation (Lemma 4.5 of the paper))**.** If $\Sigma \mid \Gamma \vdash_C M : A$, then $\Sigma \mid \Gamma \vdash M \approx |M|_\Gamma : A$.

Proof. By induction on the derivation of $\Sigma \mid \Gamma \vdash_C M : A$. We perform case analysis on the rule applied last to derive $\Sigma \mid \Gamma \vdash_C M : A$.

Case (T_Const_C): We are given

$$M = k, \quad \vdash \Sigma, \quad \Sigma \vdash \Gamma, \quad ty(k) = A \quad (\exists k) \; .$$

We have $|k|_\Gamma = k$. Therefore, by (Bs_Const), $\Sigma \mid \Gamma \vdash k \approx k : A$.

Case (T_Var_C), (T_Blame_C): Provable similarly to the case of (T_Const_C).

Case (T_Abs_C): We are given

$$A = A_1 \to A_2, \quad M = \lambda x\!:\!A_1.M_2, \quad \Sigma \mid \Gamma, x : A_1 \vdash_C M_2 : A_2 \quad (\exists A_1, A_2, x, M_2) \; .$$

We have $|\lambda x\!:\!A_1.M_2|_\Gamma = \lambda x\!:\!A_1.|M_2|_{\Gamma,x:A_1}$. By the IH, $\Sigma \mid \Gamma, x : A_1 \vdash M_2 \approx |M_2|_{\Gamma,x:A_1} : A_2$. Therefore, by (Bs_Abs), $\Sigma \mid \Gamma \vdash \lambda x\!:\!A_1.M_2 \approx \lambda x\!:\!A_1.|M_2|_{\Gamma,x:A_1} : A_1 \to A_2$.

Case (T_App_C), (T_Tyabs_C), (T_Tyapp_C): By the IH(s), similarly to the case of (T_Abs_C).

Case (T_Crc_C): We are given

$$M = M_1\langle c\rangle, \quad \Sigma \mid \Gamma \vdash_C M_1 : B, \quad \Sigma \mid \Gamma \vdash_C c : B \rightsquigarrow A \quad (\exists c, M_1, B) \; .$$

We have $|M_1\langle c\rangle|_\Gamma = |M_1|_\Gamma\langle |c|_\Gamma\rangle$. By the IH, $\Sigma \mid \Gamma \vdash M_1 \approx |M_1|_\Gamma : B$. Then, we have the following derivation.

$$\frac{\Sigma \mid \Gamma \vdash M_1 \approx |M_1|_\Gamma : B \qquad \Sigma \mid \Gamma \vdash_C c : B \rightsquigarrow A}{\Sigma \mid \Gamma \vdash M_1\langle c\rangle \approx |M_1|_\Gamma\langle |c|_\Gamma\rangle : A} \text{(Bs\_Crc)}$$

□

Theorem I.62 (Correctness of Translation (Theorem 4.4 of the paper))**.** Suppose $\Sigma \mid \emptyset \vdash_C M : A$.

(1) If $\Sigma \triangleright M \longrightarrow_C^* \Sigma' \triangleright V$, then there exists some V' such that $\Sigma \triangleright |M|_\emptyset \longrightarrow_S^* \Sigma' \triangleright V'$. Furthermore, if $A = \iota$, then there exists some k such that $V = V' = k$.

(2) If $\Sigma \triangleright |M|_\emptyset \longrightarrow_S^* \Sigma' \triangleright V'$, then there exists some V such that $\Sigma \triangleright M \longrightarrow_C^* \Sigma' \triangleright V$. Furthermore, if $A = \iota$, then there exists some k such that $V = V' = k$.

(3) $\Sigma \triangleright M \longrightarrow_C^* \Sigma' \triangleright \mathsf{blame}\, p$ iff $\Sigma \triangleright |M|_\emptyset \longrightarrow_S^* \Sigma' \triangleright \mathsf{blame}\, p$.

(4) $\Sigma \triangleright M \Uparrow$ iff $\Sigma \triangleright |M|_\emptyset \Uparrow$.

Proof. Follows from Lemma I.61 and Lemma I.60. □

J A counterexample to Proposition 19 (Bisimulation, λC to λS) in "Blame and coercion: Together again for the first time" [Siek et al., JFP'21]

The syntax, semantics, and type system mentioned in this section are those of λC and λS in "Blame and coercion: Together again for the first time" [Siek et al., JFP'21]. The relation $\approx$ found in this section is also defined there.

Let

$$\begin{aligned}
f &\stackrel{\text{def}}{=} \lambda x : \mathsf{Int}.x \ , \\
M &\stackrel{\text{def}}{=} ((f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle))\langle \mathsf{id}_{\mathsf{Int}}\rangle \ , \\
M_0 &\stackrel{\text{def}}{=} ((f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(\mathsf{blame}\,p))\langle \mathsf{id}_{\mathsf{Int}}\rangle \ , \\
M' &\stackrel{\text{def}}{=} (f\langle \bot^{\mathsf{Int}p\mathsf{Bool}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,0 \ .
\end{aligned}$$

We show that: (1) $M \approx M'$; (2) $M \longrightarrow_C M_0$; and (3) for any M_0' such that $M' \longrightarrow_S^* M_0'$, $M_0 \not\approx M_0'$. Note that it is easy to check that the closed terms M and M' are of Int. Below, the rules (i), (ii), and (iii) are for the bisimulation relation $\approx$ in [Siek et al., JFP'21].

(1) We show that $M \approx M'$. First, we can prove $f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle \approx f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle$ and $0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle \approx 0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle$ as follows.

$$\dfrac{\dfrac{\dfrac{\vdots}{f \approx f} \quad \dfrac{\vdots}{\vdash f : \mathsf{Int} \to \mathsf{Int}} \quad |\mathsf{id}_{\mathsf{Int}\to\mathsf{Int}}|^{\mathsf{CS}} = \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}}{f \approx f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle}\,(\mathrm{i}) \quad |\mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}|^{\mathsf{CS}} = \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}}{f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle \approx f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle}\,(\mathrm{ii})$$

$$\dfrac{\dfrac{0 \approx 0 \quad \vdash 0 : \mathsf{Int} \quad |\mathsf{id}_{\mathsf{Int}}|^{\mathsf{CS}} = \mathsf{id}_{\mathsf{Int}}}{0 \approx 0\langle \mathsf{id}_{\mathsf{Int}}\rangle}\,(\mathrm{i}) \quad |\bot^{\mathsf{Int}p\mathsf{Bool}}|^{\mathsf{CS}} = \bot^{\mathsf{Int}p\mathsf{Bool}}}{0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle \approx 0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle}\,(\mathrm{ii})$$

Then, we can derive $M \approx M'$, i.e., $((f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle))\langle \mathsf{id}_{\mathsf{Int}}\rangle \approx (f\langle \bot^{\mathsf{Int}p\mathsf{Bool}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,0$.

$$\dfrac{\dfrac{\dfrac{\vdots}{f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle \approx f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle} \quad \dfrac{\vdots}{0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle \approx 0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle}}{(f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle) \approx (f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle)} \quad |\mathsf{id}_{\mathsf{Int}}|^{\mathsf{CS}} = \mathsf{id}_{\mathsf{Int}}}{((f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle))\langle \mathsf{id}_{\mathsf{Int}}\rangle \approx (f\langle \bot^{\mathsf{Int}p\mathsf{Bool}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,0}\,(\mathrm{iii})$$

Note $(\mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}) \,⨟\, (\bot^{\mathsf{Int}p\mathsf{Bool}} \to \mathsf{id}_{\mathsf{Int}}) = \bot^{\mathsf{Int}p\mathsf{Bool}} \to \mathsf{id}_{\mathsf{Int}}$ in the above application of (iii).

(2) $M \longrightarrow_C M_0$, i.e., $((f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle))\langle \mathsf{id}_{\mathsf{Int}}\rangle \longrightarrow_C ((f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(\mathsf{blame}\,p))\langle \mathsf{id}_{\mathsf{Int}}\rangle$ by the following evaluation derivation.

$$\dfrac{\dfrac{0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle \longrightarrow_C \mathsf{blame}\,p}{(f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle) \longrightarrow_C ((f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(\mathsf{blame}\,p))}}{((f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle))\langle \mathsf{id}_{\mathsf{Int}}\rangle \longrightarrow_C ((f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(\mathsf{blame}\,p))\langle \mathsf{id}_{\mathsf{Int}}\rangle}$$

(3) We show that, for any M_0' such that $M' \longrightarrow_S^* M_0'$, $M_0 \not\approx M_0'$. Because

$$\begin{aligned}
M' = (f\langle \bot^{\mathsf{Int}p\mathsf{Bool}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,0 &\longrightarrow_S (f\ (0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle))\langle \mathsf{id}_{\mathsf{Int}}\rangle \\
&\longrightarrow_S (f\ (\mathsf{blame}\,p))\langle \mathsf{id}_{\mathsf{Int}}\rangle \\
&\longrightarrow_S (\mathsf{blame}\,p)\langle \mathsf{id}_{\mathsf{Int}}\rangle \\
&\longrightarrow_S \mathsf{blame}\,p \ ,
\end{aligned}$$

we show that, for any

$$M'_0 \in \left\{ \begin{array}{l} (f\langle \bot^{\mathsf{Int}p\mathsf{Bool}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,0, \\ (f\,(0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle))\langle \mathsf{id}_{\mathsf{Int}}\rangle, \\ (f\,(\mathsf{blame}\,p))\langle \mathsf{id}_{\mathsf{Int}}\rangle, \\ (\mathsf{blame}\,p)\langle \mathsf{id}_{\mathsf{Int}}\rangle, \\ \mathsf{blame}\,p \end{array} \right\},$$

$M_0 \not\approx M'_0$. Note that $M_0 = ((f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(\mathsf{blame}\,p))\langle \mathsf{id}_{\mathsf{Int}}\rangle$.

Case I. $M'_0 = (f\langle \bot^{\mathsf{Int}p\mathsf{Bool}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,0$**:** We show that

$$((f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(\mathsf{blame}\,p))\langle \mathsf{id}_{\mathsf{Int}}\rangle \not\approx (f\langle \bot^{\mathsf{Int}p\mathsf{Bool}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,0\ .$$

Assume $((f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(\mathsf{blame}\,p))\langle \mathsf{id}_{\mathsf{Int}}\rangle \approx (f\langle \bot^{\mathsf{Int}p\mathsf{Bool}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,0$. It can be derived only by the rule (iii). Then, by inversion,

$$(f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(\mathsf{blame}\,p) \approx (f\langle t\rangle)\,(0\langle s\rangle)$$

for some coercions t and s such that $t \mathbin{⨾} (s \to \mathsf{id}_{\mathsf{Int}}) = \bot^{\mathsf{Int}p\mathsf{Bool}} \to \mathsf{id}_{\mathsf{Int}}$. Furthermore, by inversion, it implies $\mathsf{blame}\,p \approx 0\langle s\rangle$. It must be the case that $\mathsf{blame}\,p \approx 0\langle s\rangle$ is derived by the rule (i) because only the term on the right-hand side is a coercion application. Then, its inversion implies $\mathsf{blame}\,p \approx 0$, but it does not hold. Therefore, there is a contradiction.

Case II. $M'_0 = (f\,(0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle))\langle \mathsf{id}_{\mathsf{Int}}\rangle$**:** We show that

$$((f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(\mathsf{blame}\,p))\langle \mathsf{id}_{\mathsf{Int}}\rangle \not\approx (f\,(0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle))\langle \mathsf{id}_{\mathsf{Int}}\rangle\ .$$

Assume $((f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(\mathsf{blame}\,p))\langle \mathsf{id}_{\mathsf{Int}}\rangle \approx (f\,(0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle))\langle \mathsf{id}_{\mathsf{Int}}\rangle$. By case analysis on which rule is applied to derive the assumed bisimilarity.

The rule (i) is applied: We are given the following derivation.

$$\frac{((f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(\mathsf{blame}\,p))\langle \mathsf{id}_{\mathsf{Int}}\rangle \approx f\,(0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle)}{((f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(\mathsf{blame}\,p))\langle \mathsf{id}_{\mathsf{Int}}\rangle \approx (f\,(0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle))\langle \mathsf{id}_{\mathsf{Int}}\rangle}$$

However, the premise judgment cannot be derived in $\approx$ (the rule (iii) cannot be applied because it requires the function part on the right-hand side be a coercion application).

The rule (ii) is applied: We are given the following derivation for some coercion s and t.

$$\frac{(f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(\mathsf{blame}\,p) \approx (f\,(0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle))\langle s\rangle \qquad |\mathsf{id}_{\mathsf{Int}}|^{\mathsf{CS}} = t \qquad s \mathbin{⨾} t = \mathsf{id}_{\mathsf{Int}}}{((f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(\mathsf{blame}\,p))\langle \mathsf{id}_{\mathsf{Int}}\rangle \approx (f\,(0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle))\langle \mathsf{id}_{\mathsf{Int}}\rangle}$$

The premise judgment $(f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(\mathsf{blame}\,p) \approx (f\,(0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle))\langle s\rangle$ can be derived only by the rule (i) because only the term on the right-hand side is a coercion application. Then, its inversion implies

$$(f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(\mathsf{blame}\,p) \approx f\,(0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle)$$

and then

$$\mathsf{blame}\,p \approx 0\langle \bot^{\mathsf{Int}p\mathsf{Bool}}\rangle\ .$$

However, this judgment cannot be derived as discussed in Case I.

The rule (iii) is applied: This case needn't be considered because the term on the right-hand side is not a function application.

Case III. $M'_0 = (f\,(\mathsf{blame}\,p))\langle \mathsf{id}_{\mathsf{Int}}\rangle$**:** We show that

$$((f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(\mathsf{blame}\,p))\langle \mathsf{id}_{\mathsf{Int}}\rangle \not\approx (f\,(\mathsf{blame}\,p))\langle \mathsf{id}_{\mathsf{Int}}\rangle\ .$$

Assume $((f\langle \mathsf{id}_{\mathsf{Int}} \to \mathsf{id}_{\mathsf{Int}}\rangle)\,(\mathsf{blame}\,p))\langle \mathsf{id}_{\mathsf{Int}}\rangle \approx (f\,(\mathsf{blame}\,p))\langle \mathsf{id}_{\mathsf{Int}}\rangle$. By case analysis on which rule is applied to derive the assumed bisimilarity.

The rule (i) is applied: This case needn't be considered for a reason similar to Case II.

The rule (ii) is applied: By a reasoning similar to Case II/(ii), $f\langle \mathsf{id_{Int}} \to \mathsf{id_{Int}}\rangle \approx f$, but it cannot be derived.

The rule (iii) is applied: This case needn't be considered because the term on the right-hand side is not a function application.

Case IV. $M_0' = (\mathsf{blame}\, p)\langle \mathsf{id_{Int}}\rangle$**:** We show that

$$((f\langle \mathsf{id_{Int}} \to \mathsf{id_{Int}}\rangle)\,(\mathsf{blame}\, p))\langle \mathsf{id_{Int}}\rangle \not\approx (\mathsf{blame}\, p)\langle \mathsf{id_{Int}}\rangle \ .$$

Assume $((f\langle \mathsf{id_{Int}} \to \mathsf{id_{Int}}\rangle)\,(\mathsf{blame}\, p))\langle \mathsf{id_{Int}}\rangle \approx (\mathsf{blame}\, p)\langle \mathsf{id_{Int}}\rangle$. By case analysis on which rule is applied to derive the assumed bisimilarity.

The rule (i) is applied: We are given the following derivation.

$$\frac{((f\langle \mathsf{id_{Int}} \to \mathsf{id_{Int}}\rangle)\,(\mathsf{blame}\, p))\langle \mathsf{id_{Int}}\rangle \approx \mathsf{blame}\, p}{((f\langle \mathsf{id_{Int}} \to \mathsf{id_{Int}}\rangle)\,(\mathsf{blame}\, p))\langle \mathsf{id_{Int}}\rangle \approx (\mathsf{blame}\, p)\langle \mathsf{id_{Int}}\rangle}$$

However, the premise judgment cannot be derived because, for any M_1, if $M_1 \approx \mathsf{blame}\, p$, then $M_1 = \mathsf{blame}\, p$.

The rule (ii) is applied: We are given the following derivation for some coercion s and t.

$$\frac{(f\langle \mathsf{id_{Int}} \to \mathsf{id_{Int}}\rangle)\,(\mathsf{blame}\, p) \approx (\mathsf{blame}\, p)\langle s\rangle \qquad |\mathsf{id_{Int}}|^{\mathsf{CS}} = t \qquad s \,;\, t = \mathsf{id_{Int}}}{((f\langle \mathsf{id_{Int}} \to \mathsf{id_{Int}}\rangle)\,(\mathsf{blame}\, p))\langle \mathsf{id_{Int}}\rangle \approx (\mathsf{blame}\, p)\langle \mathsf{id_{Int}}\rangle}$$

The premise judgment $(f\langle \mathsf{id_{Int}} \to \mathsf{id_{Int}}\rangle)\,(\mathsf{blame}\, p) \approx (\mathsf{blame}\, p)\langle s\rangle$ can be derived only by the rule (i). Then, its inversion implies

$$(f\langle \mathsf{id_{Int}} \to \mathsf{id_{Int}}\rangle)\,(\mathsf{blame}\, p) \approx \mathsf{blame}\, p \ .$$

However, it does not hold.

The rule (iii) is applied: This case needn't be considered because the term on the right-hand side is not a function application.

Case V. $M_0' = \mathsf{blame}\, p$**:** $M_0 \not\approx \mathsf{blame}\, p$ clearly.