Supplementary Material for “CPS Transformation with Affine Types for Call-By-Value Implicit Polymorphism”

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1 Organization

This material provides the full definitions, auxiliary lemmas, and proofs that are omitted in our paper “CPS Transformation with Affine Types for Call-By-Value Implicit Polymorphism” at ICFP 2021. This is organized as follows.

Section 2 presents the definitions. Section 2.1 defines our CPS target language Λ^{open} and contextual equivalence of it. Section 2.2 defines Curry-style CBV System F, referred to as $\lambda_v^{\forall}$ in the paper. This section also provides a family of CBV full reductions and parallel reduction. Note that the CBV full reduction $\Longmapsto_{\beta\eta_v}$ in the paper is described as $\Longmapsto_{\beta_v\eta_v}$ in this material. Section 2.3 presents the following transformations. Section 2.3.1 provides our CPS transformation from $\lambda_v^{\forall}$ to Λ^{open} (shown in Section 5 in the paper). Section 2.3.2 provides the full definition of a variant $(\!|\cdot|\!)$ of Plotkin's CBV CPS transformation for the untyped λ-calculus [1]. Section 2.3.3 defines type erasure erase from Λ^{open} to (untyped) $\lambda_v^{\forall}$. Section 2.4 defines the logical relation (shown in Section 6 in the paper).

Section 3 provides the full proofs of the properties shown in the paper. Section 3.1 proves type soundness of Λ^{open} (Theorem 1) with progress (Lemma 36) and subject reduction (Lemma 40). Section 3.2 proves the properties concerning the reduction relations for $\lambda_v^{\forall}$. Note that Lemma 6 in the paper is shown as a corollary of Lemmas 53, 54, 63, and 64 in this section. Section 3.3 proves the properties concerning type erasure erase. Theorem 2 shows its meaning preservation property. Section 3.4 proves the type and meaning preservation properties of our CPS transformation from $\lambda_v^{\forall}$ to Λ^{open} in Theorems 3 and 4, respectively. Section 3.5 proves the Fundamental Property (or, parametricity) of the logical relation (Theorem 5) and soundness of the logical relation with respect to contextual equivalence (Theorem 6). Section 3.6 addresses a few partial free theorems.

2 Definition

2.1 Λ^{open}: CPS Target Language

2.1.1 Syntax

Variables x, y, z, f, k	**Type variables** α, β, γ
Base types ι	$::= \mathsf{bool} \mid \mathsf{int} \mid \ldots$
Types A, B, C, D	$::= \alpha \mid \iota \mid A \multimap B \mid \forall\alpha.A \mid !A$
Constants c	$::= \mathsf{true} \mid \mathsf{false} \mid 0 \mid + \mid \ldots$
Terms M	$::= x \mid c \mid \lambda x.M \mid M_1\, M_2 \mid !M \mid \mathsf{let}\, !x = M_1 \,\mathsf{in}\, M_2 \mid$ $\nu\alpha.\, M \mid \Lambda^\circ\langle\, \alpha, M\,\rangle \mid \Lambda\alpha.M \mid M\, A$
Values V	$::= c \mid \lambda x.M \mid !R \mid \Lambda\alpha.M$
Results R	$::= V \mid \nu\alpha.\, R$
Extrusion contexts $\mathbb{E}$	$::= [\,]\, R_2 \mid \Lambda^\circ\langle\, \alpha, [\,]\,\rangle \mid [\,]\, A$
Evaluation contexts E	$::= [\,]\, M_2 \mid R_1\, [\,] \mid \mathsf{let}\, !x = [\,] \,\mathsf{in}\, M_2 \mid \Lambda^\circ\langle\, \alpha, [\,]\,\rangle \mid [\,]\, A \mid \nu\alpha.\, [\,] \mid ![\,]$
Program contexts $\mathbb{C}$	$::= [\,] \mid \lambda x.\mathbb{C} \mid \mathbb{C}\, M_2 \mid M_1\, \mathbb{C} \mid !\mathbb{C} \mid \mathsf{let}\, !x = \mathbb{C} \,\mathsf{in}\, M_2 \mid \mathsf{let}\, !x = M_1 \,\mathsf{in}\, \mathbb{C} \mid$ $\nu\alpha.\, \mathbb{C} \mid \Lambda^\circ\langle\, \alpha, \mathbb{C}\,\rangle \mid \Lambda\alpha.\mathbb{C} \mid \mathbb{C}\, A$
Uses π	$::= \mathbf{0} \mid \mathbf{1} \mid \omega$
Typing contexts Γ	$::= \emptyset \mid \Gamma, x :^\pi A \mid \Gamma, \alpha^\pi$

Convention 1. *We write Γ_1, Γ_2 for the concatenation of Γ_1 and Γ_2. We use metavariable Δ for denoting typing contexts that consist only of α^π.*

Convention 2. *We write $\nu\overline{\alpha}.\, M$ for $\nu\alpha_1.\, \cdots \nu\alpha_n.\, M$ when $\overline{\alpha} = \alpha_1, \cdots, \alpha_n$.*

Definition 1 (Free variables and substitution)**.** *The sets $ftv(A)$, $ftv(M)$, and $ftv(\mathbb{E})$ of free type variables in a type A, a term M, and an evaluation context $\mathbb{E}$ are defined in a standard manner, respectively. The set $fv\,(M)$ of free variables in a term $[M]$ is also defined ordinarily.*

Type substitution $B[A/\alpha]$ of A for α in B and term substitution $M[M'/x]$ of M' for free variable x in M are defined in a capture-avoiding manner as usual. The notable point of type substitution is that $(\Lambda^\circ\langle\, \beta, M\,\rangle)[A/\alpha]$ is defined if and only if $\beta[A/\alpha] = \gamma$ for some type variable γ (i.e., β is mapped to γ by $[A/\alpha]$ or $\beta \neq \alpha$) and, then, $\Lambda^\circ\langle\, \beta, M\,\rangle[A/\alpha] \stackrel{\text{def}}{=} \Lambda^\circ\langle\, \gamma, M[A/\alpha]\,\rangle$.

Definition 2. *The set of uses $\{\mathbf{0}, \mathbf{1}, \omega\}$ forms a commutative monoid equipped with an binary operation $+$ such that:*

- $\mathbf{0} + \pi = \pi + \mathbf{0} = \pi$ *for any π;*
- $\omega + \pi = \pi + \omega = \omega$ *for any π; and*
- $\mathbf{1} + \mathbf{1} = \omega$.

We write $\pi_1 \leq \pi_2$ and $\pi_2 \geq \pi_1$ if and only if $\pi_1 + \pi = \pi_2$ for some π.

We also define the predicate $\Gamma_1 \leq \Gamma_2$ as the smallest relation satisfying the following rules.

$$\frac{}{\emptyset \leq \emptyset} \qquad \frac{\Gamma_1 \leq \Gamma_2 \quad \pi_1 \leq \pi_2}{\Gamma_1, x :^{\pi_1} A \leq \Gamma_2, x :^{\pi_2} A} \qquad \frac{\Gamma_1 \leq \Gamma_2 \quad \pi_1 \leq \pi_2 \quad \pi_2 \neq \omega}{\Gamma_1, \alpha^{\pi_1} \leq \Gamma_2, \alpha^{\pi_2}}$$

Definition 3 (Adding uses)**.** *Given a sequence of type variables $\overline{\alpha}$, $\mathbf{1}\overline{\alpha}$ is a typing context obtained by adding the use $\mathbf{1}$ to each type variable in $\overline{\alpha}$. Formally, it is defined by induction on $\overline{\alpha}$, as follows.*

$$\begin{array}{lcl} \mathbf{1}\langle\rangle & \stackrel{\text{def}}{=} & \emptyset \\ \mathbf{1}(\overline{\alpha}, \beta) & \stackrel{\text{def}}{=} & \mathbf{1}\overline{\alpha}, \beta^{\mathbf{1}} \end{array}$$

Assumption 1. *We suppose that each constant c is assigned a closed first-order type $ty(c)$ of the form $\iota_1 \multimap \ldots \multimap \iota_n \multimap \iota_{n+1}$. We also suppose that, for any ι, there is the set $\mathbb{K}_\iota$ of constants of ι. For any constant c, $ty(c) = \iota$ if and only if $c \in \mathbb{K}_\iota$. The function ζ gives a denotation to pairs of constants. In particular, for any constants c_1 and c_2: (1) $\zeta(c_1, c_2)$ is defined if and only if $ty(c_1) = \iota_0 \multimap A$ and $ty(c_2) = \iota_0$ for some ι_0 and A; and (2) if $\zeta(c_1, c_2)$ is defined, $\zeta(c_1, c_2)$ is a constant and $ty(\zeta(c_1, c_2)) = A$ where $ty(c_1) = \iota_0 \multimap A$.*

Reduction rules $\boxed{M_1 \rightsquigarrow M_2}$

$$
\begin{array}{rcll}
c_1\,(\nu\overline{\alpha}.\,c_2) & \rightsquigarrow & \nu\overline{\alpha}.\,\zeta(c_1, c_2) & (\textsc{R\_Const}) \\
(\lambda x.M)\,R & \rightsquigarrow & M[R/x] & (\textsc{R\_Beta}) \\
\mathsf{let}\,!x = \nu\overline{\alpha}.\,!R\,\mathsf{in}\,M & \rightsquigarrow & M[\nu\overline{\alpha}.\,R/x] & (\textsc{R\_Bang}) \\
\Lambda^{\circ}\langle\,\alpha, !R\,\rangle & \rightsquigarrow & !\Lambda\alpha.R & (\textsc{R\_Closing}) \\
(\Lambda\alpha.M)\,A & \rightsquigarrow & M[A/\alpha] & (\textsc{R\_TBeta})
\end{array}
$$

Evaluation rules $\boxed{M_1 \longrightarrow M_2}$

$$
\frac{M_1 \rightsquigarrow M_2}{M_1 \longrightarrow M_2}\ \textsc{E\_Red}
\qquad
\frac{M_1 \longrightarrow M_2}{E[M_1] \longrightarrow E[M_2]}\ \textsc{E\_Eval}
\qquad
\frac{\alpha \notin \mathit{ftv}(\mathbb{E})}{\mathbb{E}[\nu\alpha.\,R] \longrightarrow \nu\alpha.\,\mathbb{E}[R]}\ \textsc{E\_Extr}
$$

Figure 1: Semantics.

2.1.2 Semantics

Definition 4. *Relations $M_1 \longrightarrow M_2$ and $M_1 \rightsquigarrow M_2$ are the smallest relations satisfying the rules in Figure 1.*

Definition 5 (Multi-step evaluation)**.** *Binary relation $\longrightarrow^*$ over terms is the reflexive and transitive closure of $\longrightarrow$.*

Definition 6 (Nonreducible terms)**.** *We write $M \not\longrightarrow$ if and only if there is no M' such that $M \longrightarrow M'$.*

2.1.3 Type System

Definition 7. *Given a typing context Γ, $\omega\Gamma$ is a typing context obtained by induction on Γ as follows.*

$$
\begin{array}{lcl}
\omega\emptyset & \stackrel{\text{def}}{=} & \emptyset \\
\omega(\Gamma, x :^{\omega} A) & \stackrel{\text{def}}{=} & \omega\Gamma, x :^{\omega} A \\
\omega(\Gamma, x :^{\pi} A) & \stackrel{\text{def}}{=} & \omega\Gamma, x :^{\mathbf{0}} A \quad (\textit{if } \pi \neq \omega) \\
\omega(\Gamma, \alpha^{\pi}) & \stackrel{\text{def}}{=} & \omega\Gamma, \alpha^{\mathbf{0}}
\end{array}
$$

Definition 8. *Given a typing context Γ, its domain $\mathit{dom}(\Gamma)$ is defined by induction on Γ as follows.*

$$
\begin{array}{lcl}
\mathit{dom}(\emptyset) & \stackrel{\text{def}}{=} & \emptyset \\
\mathit{dom}(\Gamma, x :^{\pi} A) & \stackrel{\text{def}}{=} & \{x\} \cup \mathit{dom}(\Gamma) \\
\mathit{dom}(\Gamma, \alpha^{\pi}) & \stackrel{\text{def}}{=} & \{\alpha\} \cup \mathit{dom}(\Gamma)
\end{array}
$$

Definition 9. *Given typing contexts Γ_1 and Γ_2, their merging typing context $\Gamma_1 + \Gamma_2$ is defined as follows.*

$$
\begin{array}{lcl}
\emptyset + \emptyset & \stackrel{\text{def}}{=} & \emptyset \\
(\Gamma_1, x :^{\pi_1} A) + (\Gamma_2, x :^{\pi_2} A) & \stackrel{\text{def}}{=} & (\Gamma_1 + \Gamma_2), x :^{\pi_1 + \pi_2} A \\
(\Gamma_1, \alpha^{\pi_1}) + (\Gamma_2, \alpha^{\pi_2}) & \stackrel{\text{def}}{=} & (\Gamma_1 + \Gamma_2), \alpha^{\pi_1 + \pi_2} \quad (\textit{if } \pi_1 + \pi_2 \neq \omega)
\end{array}
$$

Definition 10. *We view Γ as a function that maps a variable to a type. $\Gamma(x) = A$ if and only if $x :^{\pi} A \in \Gamma$ for some $\pi \neq \mathbf{0}$.*

Definition 11. *Well-formedness of typing contexts $\vdash \Gamma$ is the smallest relation satisfying the rules at the top of Figure 2. Well-formedness of types under typing contexts $\Gamma \vdash A$ holds if and only $\mathit{ftv}(A) \subseteq \mathit{dom}(\Gamma)$. Typing judgment $\Gamma \vdash M : A$ is the smallest relation satisfying the rules at the bottom of Figure 2.*

Well-formedness rules $\boxed{\vdash \Gamma}$

$$\frac{}{\vdash \emptyset}\ \text{WF\_Empty} \qquad \frac{\vdash \Gamma \quad \Gamma \vdash A \quad x \notin dom(\Gamma)}{\vdash \Gamma, x :^{\pi} A}\ \text{WF\_Var} \qquad \frac{\vdash \Gamma \quad \alpha \notin dom(\Gamma) \quad \pi \neq \omega}{\vdash \Gamma, \alpha^{\pi}}\ \text{WF\_TyVar}$$

Typing rules $\boxed{\Gamma \vdash M : A}$

$$\frac{\vdash \Gamma}{\Gamma \vdash x : \Gamma(x)}\ \text{T\_Var} \qquad \frac{\vdash \Gamma}{\Gamma \vdash c : ty(c)}\ \text{T\_Const}$$

$$\frac{\Gamma, x :^{\mathbf{1}} A \vdash M : B}{\Gamma \vdash \lambda x.M : A \multimap B}\ \text{T\_Abs} \qquad \frac{\Gamma_1 \vdash M_1 : A \multimap B \quad \Gamma_2 \vdash M_2 : A}{\Gamma_1 + \Gamma_2 \vdash M_1\, M_2 : B}\ \text{T\_App}$$

$$\frac{\vdash \Gamma \quad \omega\Gamma \vdash M : A}{\Gamma \vdash !M : !A}\ \text{T\_Bang} \qquad \frac{\Gamma_1 \vdash M_1 : !B \quad \Gamma_2, x :^{\omega} B \vdash M_2 : A}{\Gamma_1 + \Gamma_2 \vdash \mathsf{let}\, !x = M_1\, \mathsf{in}\, M_2 : A}\ \text{T\_LetBang}$$

$$\frac{\Gamma, \alpha^{\mathbf{1}} \vdash M : A \quad \Gamma \vdash A}{\Gamma \vdash \nu\alpha.\, M : A}\ \text{T\_Nu} \qquad \frac{\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2 \vdash M : !A}{\Gamma_1, \alpha^{\mathbf{1}}, \Gamma_2 \vdash \Lambda^{\circ}\langle\, \alpha, M\, \rangle : !\forall\alpha.A}\ \text{T\_Gen}$$

$$\frac{\vdash \Gamma \quad \omega\Gamma, \alpha^{\mathbf{0}} \vdash M : A}{\Gamma \vdash \Lambda\alpha.M : \forall\alpha.A}\ \text{T\_TAbs} \qquad \frac{\Gamma \vdash M : \forall\alpha.B \quad \Gamma \vdash A}{\Gamma \vdash M\, A : B[A/\alpha]}\ \text{T\_TApp}$$

Figure 2: Type system.

2.1.4 Contextual Equivalence

Definition 12. *A context typing judgment* $\mathbb{C} : (\Gamma \vdash A) \rightsquigarrow (\Gamma' \vdash A')$ *is the smallest relation satisfying the inference rules in Figure 3.*

Definition 13 (Contextual Equivalence)**.** *Contextual equivalence* $\Gamma \vdash M_1 \approx_{ctx} M_2 : A$ *is the formula that states that (1)* $\Gamma \vdash M_1 : A$*, (2)* $\Gamma \vdash M_2 : A$*, and (3) for any base type* ι*, constant* c *of* ι*, program context* $\mathbb{C}$ *such that* $\mathbb{C} : (\Gamma \vdash A) \rightsquigarrow (\emptyset \vdash \iota)$*,* $\mathbb{C}[M_1] \longrightarrow^{*} \nu\overline{\alpha_1}.\, c$ *for some* $\overline{\alpha_1}$ *if and only if* $\mathbb{C}[M_2] \longrightarrow^{*} \nu\overline{\alpha_2}.\, c$ *for some* $\overline{\alpha_2}$*.*

Context typing rules $\boxed{\mathbb{C} : (\Gamma \vdash A) \rightsquigarrow (\Gamma' \vdash A')}$

$$\frac{}{[\,] : (\Gamma \vdash A) \rightsquigarrow (\Gamma \vdash A)}\ \text{CT\_HOLE} \qquad \frac{\mathbb{C} : (\Gamma \vdash A) \rightsquigarrow (\Gamma', x :^{\mathbf{1}} A' \vdash B')}{\lambda x.\mathbb{C} : (\Gamma \vdash A) \rightsquigarrow (\Gamma' \vdash A' \multimap B')}\ \text{CT\_ABS}$$

$$\frac{\mathbb{C} : (\Gamma \vdash A) \rightsquigarrow (\Gamma'_1 \vdash A' \multimap B') \quad \Gamma'_2 \vdash M_2 : A'}{\mathbb{C}\, M_2 : (\Gamma \vdash A) \rightsquigarrow (\Gamma'_1 + \Gamma'_2 \vdash B')}\ \text{CT\_APP1}$$

$$\frac{\Gamma'_1 \vdash M_1 : A' \multimap B' \quad \mathbb{C} : (\Gamma \vdash A) \rightsquigarrow (\Gamma'_2 \vdash A')}{M_1\, \mathbb{C} : (\Gamma \vdash A) \rightsquigarrow (\Gamma'_1 + \Gamma'_2 \vdash B')}\ \text{CT\_APP2} \qquad \frac{\vdash \Gamma' \quad \mathbb{C} : (\Gamma \vdash A) \rightsquigarrow (\omega\Gamma' \vdash A')}{!\mathbb{C} : (\Gamma \vdash A) \rightsquigarrow (\Gamma' \vdash\, !A')}\ \text{CT\_BANG}$$

$$\frac{\mathbb{C} : (\Gamma \vdash A) \rightsquigarrow (\Gamma'_1 \vdash\, !B') \quad \Gamma'_2, x :^{\omega} B' \vdash M_2 : A'}{\mathsf{let}\, !x = \mathbb{C}\, \mathsf{in}\, M_2 : (\Gamma \vdash A) \rightsquigarrow (\Gamma'_1 + \Gamma'_2 \vdash A')}\ \text{CT\_LETBANG1}$$

$$\frac{\Gamma'_1 \vdash M_1 : !B' \quad \mathbb{C} : (\Gamma \vdash A) \rightsquigarrow (\Gamma'_2, x :^{\omega} B' \vdash A')}{\mathsf{let}\, !x = M_1\, \mathsf{in}\, \mathbb{C} : (\Gamma \vdash A) \rightsquigarrow (\Gamma'_1 + \Gamma'_2 \vdash A')}\ \text{CT\_LETBANG2}$$

$$\frac{\mathbb{C} : (\Gamma \vdash A) \rightsquigarrow (\Gamma', \alpha^{\mathbf{1}} \vdash A') \quad \Gamma' \vdash A'}{\nu\alpha.\, \mathbb{C} : (\Gamma \vdash A) \rightsquigarrow (\Gamma' \vdash A')}\ \text{CT\_NU} \qquad \frac{\mathbb{C} : (\Gamma \vdash A) \rightsquigarrow (\Gamma'_1, \alpha^{\mathbf{0}}, \Gamma'_2 \vdash\, !A')}{\Lambda^{\circ}\langle\, \alpha, \mathbb{C}\,\rangle : (\Gamma \vdash A) \rightsquigarrow (\Gamma'_1, \alpha^{\mathbf{1}}, \Gamma'_2 \vdash\, !\forall\alpha.A')}\ \text{CT\_GEN}$$

$$\frac{\vdash \Gamma' \quad \mathbb{C} : (\Gamma \vdash A) \rightsquigarrow (\omega\Gamma', \alpha^{\mathbf{0}} \vdash A')}{\Lambda\alpha.\mathbb{C} : (\Gamma \vdash A) \rightsquigarrow (\Gamma' \vdash \forall\alpha.A')}\ \text{CT\_TABS} \qquad \frac{\mathbb{C} : (\Gamma \vdash A) \rightsquigarrow (\Gamma' \vdash \forall\alpha.B') \quad \Gamma' \vdash A'}{\mathbb{C}\, A' : (\Gamma \vdash A) \rightsquigarrow (\Gamma' \vdash B'[A'/\alpha])}\ \text{CT\_TAPP}$$

Figure 3: Typing of contexts.

Reduction rules $\boxed{e_1 \rightsquigarrow_{\aleph} e_2}$

$$c_1\ c_2 \rightsquigarrow_{\delta} \zeta(c_1, c_2) \qquad (\lambda x.e)\ w \rightsquigarrow_{\beta_v} e[w/x] \qquad (\lambda x.w\ x) \rightsquigarrow_{\eta_v} w \quad (x \notin \mathit{fv}\,(w))$$

Evaluation rules $\boxed{e_1 \longrightarrow_F e_2}$

$$\frac{e_1 \rightsquigarrow_{\aleph} e_2 \quad \aleph \in \{\delta, \beta_v\}}{e_1 \longrightarrow_F e_2} \qquad \frac{e_1 \longrightarrow_F e_1'}{e_1\ e_2 \longrightarrow_F e_1'\ e_2} \qquad \frac{e_2 \longrightarrow_F e_2'}{w_1\ e_2 \longrightarrow_F w_1\ e_2'}$$

Parallel reduction rules $\boxed{e_1 \rightrightarrows_{\overline{\aleph}} e_2}$

$$\frac{}{e \rightrightarrows_{\overline{\aleph}} e}\ \textsc{P\_Refl} \qquad \frac{e_1 \rightrightarrows_{\overline{\aleph}} e_2 \quad w_1 \rightrightarrows_{\overline{\aleph}} w_2 \quad \beta_v \in \{\overline{\aleph}\}}{(\lambda x.e_1)\ w_1 \rightrightarrows_{\overline{\aleph}} e_2[w_2/x]}\ \textsc{P\_Beta}$$

$$\frac{w_1 \rightrightarrows_{\overline{\aleph}} w_2 \quad x \notin \mathit{fv}\,(w_1) \quad \eta_v \in \{\overline{\aleph}\}}{\lambda x.w_1\ x \rightrightarrows_{\overline{\aleph}} w_2}\ \textsc{P\_Eta} \qquad \frac{\delta \in \{\overline{\aleph}\}}{c_1\ c_2 \rightsquigarrow_{\delta} \zeta(c_1, c_2)}\ \textsc{P\_Delta}$$

$$\frac{e_1 \rightrightarrows_{\overline{\aleph}} e_2}{\lambda x.e_1 \rightrightarrows_{\overline{\aleph}} \lambda x.e_2}\ \textsc{P\_Abs} \qquad \frac{e_{11} \rightrightarrows_{\overline{\aleph}} e_{21} \quad e_{12} \rightrightarrows_{\overline{\aleph}} e_{22}}{e_{11}\ e_{12} \rightrightarrows_{\overline{\aleph}} e_{21}\ e_{22}}\ \textsc{P\_App}$$

Figure 4: Semantics.

2.2 $\lambda_v^\forall$: Curry-style CBV System F

2.2.1 Syntax

Types	τ	$::= \alpha \mid \iota \mid \tau_1 \to \tau_2 \mid \forall\alpha.\tau$
Terms	e	$::= x \mid c \mid \lambda x.e \mid e_1\ e_2$
Values	w	$::= x \mid c \mid \lambda x.e$
Contexts	$\mathcal{C}$	$::= [\,] \mid \lambda x.\mathcal{C} \mid \mathcal{C}\ e_2 \mid e_1\,\mathcal{C}$
Typing contexts	Θ	$::= \emptyset \mid \Theta, x:\tau \mid \Theta, \alpha$

Definition 14 (Free variables and substitution)**.** *Free type variables in a type and free variables in a term are defined in a standard manner. We write $\mathit{ftv}(\tau)$ for the set of free type variables in a type τ and $\mathit{fv}\,(e)$ for the set of free variables in a term e. Type substitution $\tau_1[\tau_2/\alpha]$ of τ_2 for free type variable α in τ_1 and term substitution $e_1[e_2/x]$ of e_2 for free variable x in e_1 are defined in a capture-avoiding manner as usual.*

2.2.2 Semantics

Definition 15 (Reduction symbol)**.** *The metavariable $\aleph$ ranges over reduction symbols of β_v, η_v, and δ. We write $\aleph_1 \cdots \aleph_n$ for the sequence of the symbols $\aleph_1, \cdots, \aleph_n$ and abbreviate it to $\overline{\aleph}$ simply. We also write $\{\overline{\aleph}\}$ for viewing the sequence $\overline{\aleph}$ as a set by ignoring the order.*

Definition 16 (Reduction)**.** *The reduction relation $\rightsquigarrow_{\aleph}$, indexed by the reduction symbol $\aleph$, is a binary relation over terms in $\lambda_v^\forall$ defined by the rules at the top of Figure 4.*

Definition 17 (Evaluation)**.** *The evaluation relation $\longrightarrow_F$ is a binary relation over terms in $\lambda_v^\forall$ and defined as the smallest relation that satisfies the rules at the middle of Figure 4. We write: $e_1 \longrightarrow_F{}^{0,1} e_2$ if and only if $e_1 = e_2$ or $e_1 \longrightarrow_F e_2$; $e_1 \longrightarrow_F{}^{\leq 2} e_2$ if and only if (1) $e_1 = e_2$, (2) $e_1 \longrightarrow_F e_2$, or (3) $e_1 \longrightarrow_F e$ and $e \longrightarrow_F e_2$ for some e; and $e \not\longrightarrow_F$ if and only if there exists no term e' such that $e \longrightarrow_F e'$. We write $\longrightarrow_F^*$ for the reflexive, transitive closures of $\longrightarrow_F$.*

A term e gets stuck if and only if there exists some e' such that: (1) $e \longrightarrow_F^* e'$, (2) $e' \not\longrightarrow_F$, and (3) e' is not a value.

Well-formedness rules $\boxed{\vdash \Theta}$

$$\frac{}{\vdash \emptyset} \qquad \frac{\vdash \Theta \quad \Theta \vdash \tau \quad x \notin dom(\Theta)}{\vdash \Theta, x:\tau} \qquad \frac{\vdash \Theta \quad \alpha \notin dom(\Theta)}{\vdash \Theta, \alpha}$$

Typing rules $\boxed{\Theta \vdash e : \tau}$

$$\frac{\vdash \Theta}{\Theta \vdash x : \Theta(x)} \qquad \frac{\vdash \Theta}{\Theta \vdash c : ty^{\rightarrow}(c)} \qquad \frac{\Theta, x:\tau_1 \vdash e : \tau_2}{\Theta \vdash \lambda x.e : \tau_1 \rightarrow \tau_2}$$

$$\frac{\Theta \vdash e_1 : \tau_1 \rightarrow \tau_2 \quad \Theta \vdash e_2 : \tau_1}{\Theta \vdash e_1\, e_2 : \tau_2} \qquad \frac{\Theta, \alpha \vdash e : \tau}{\Theta \vdash e : \forall \alpha.\tau} \qquad \frac{\Theta \vdash e : \forall \alpha.\tau_2 \quad \Theta \vdash \tau_1}{\Theta \vdash e : \tau_2[\tau_1/\alpha]}$$

Figure 5: Type system.

Definition 18 (Full reduction)**.** *We define full reduction $\Longmapsto_{\aleph}$ indexed by $\aleph$, which is a binary relation over terms in $\lambda_v^{\forall}$, by: $e_1 \Longmapsto_{\aleph} e_2$ if and only if there exist some $\mathcal{C}$, e_1', and e_2' such that $e_1 = \mathcal{C}[e_1']$, $e_2 = \mathcal{C}[e_2']$, and $e_1' \rightsquigarrow_{\aleph} e_2'$. We write $\Longmapsto_{\overline{\aleph}}$ for the union of $\{\Longmapsto_{\aleph'} \mid \aleph' \in \{\overline{\aleph}\}\}$. We write $\Longmapsto_{\overline{\aleph}}^{*}$ for the reflexive, transitive closures of $\Longmapsto_{\overline{\aleph}}$.*

Definition 19 (Parallel reduction)**.** *We define parallel reduction $\rightrightarrows_{\overline{\aleph}}$ indexed by $\overline{\aleph}$, which is a binary relation over terms in $\lambda_v^{\forall}$, as the smallest relation that satisfies the rules at the bottom of Figure 4. We write $\rightrightarrows_{\overline{\aleph}}^{*}$ for the reflexive, transitive closures of $\rightrightarrows_{\overline{\aleph}}$.*

2.2.3 Type System

Definition 20. *Given a typing context Θ, its domain $dom(\Theta)$ is defined by induction on Θ as follows.*

$$\begin{array}{lcl} dom(\emptyset) & \stackrel{\text{def}}{=} & \emptyset \\ dom(\Theta, x:\tau) & \stackrel{\text{def}}{=} & \{x\} \cup dom(\Theta) \\ dom(\Theta, \alpha) & \stackrel{\text{def}}{=} & \{\alpha\} \cup dom(\Theta) \end{array}$$

Definition 21. *We view Θ as a function that maps a variable to a type. $\Theta(x) = \tau$ if and only if $x:\tau \in \Theta$.*

Definition 22. *We give each constant c a first-order closed type $ty^{\rightarrow}(c)$, which is the same as $ty(c)$ given in Assumption 1 except that type constructor $\multimap$ is replaced by $\rightarrow$.*

Definition 23. *Well-formedness of typing contexts $\vdash \Theta$ is the smallest relation that satisfies the rules at the top of Figure 5. Well-formedness of types under typing contexts $\Theta \vdash \tau$ holds if and only if $ftv(\tau) \subseteq dom(\Theta)$. Typing judgment $\Theta \vdash e : \tau$ is the smallest relation that satisfies the rules at the bottom of Figure 5.*

CPS transformation $\boxed{[\![\Theta \vdash e \,:\, \tau]\!] \Rightarrow R}$

$$\frac{\vdash \Theta \quad x : \tau \in \Theta}{[\![\Theta \vdash x \,:\, \tau]\!] \Rightarrow \Lambda\alpha.\lambda k.k\,!x}\ \text{C\_Var} \qquad \frac{\vdash \Theta}{[\![\Theta \vdash c \,:\, ty^{\rightarrow}(c)]\!] \Rightarrow \Lambda\alpha.\lambda k.k\,[\![c : ty^{\rightarrow}(c)]\!]}\ \text{C\_Const}$$

$$\frac{[\![\Theta, x : \tau_1 \vdash e \,:\, \tau_2]\!] \Rightarrow R \quad y \text{ is fresh}}{[\![\Theta \vdash \lambda x.e \,:\, \tau_1 \to \tau_2]\!] \Rightarrow \Lambda\alpha.\lambda k.k\,!(\lambda y.\mathsf{let}\,!x = y\,\mathsf{in}\,R)}\ \text{C\_Abs}$$

$$\frac{[\![\Theta \vdash e_1 \,:\, \tau_1 \to \tau_2]\!] \Rightarrow R_1 \quad [\![\Theta \vdash e_2 \,:\, \tau_1]\!] \Rightarrow R_2 \quad x \text{ is fresh}}{[\![\Theta \vdash e_1\, e_2 \,:\, \tau_2]\!] \Rightarrow \Lambda\alpha.\lambda k.R_1\,\alpha\,(\lambda x.R_2\,\alpha\,(\lambda y.\mathsf{let}\,!z = x\,\mathsf{in}\,z\,y\,\alpha\,k))}\ \text{C\_App}$$

$$\frac{[\![\Theta, \beta \vdash e \,:\, \tau]\!] \Rightarrow R}{[\![\Theta \vdash e \,:\, \forall\beta.\tau]\!] \Rightarrow \Lambda\alpha.\lambda k.\nu\beta.\,R\,\alpha\,(\lambda x.k\,\Lambda^{\circ}\langle\,\beta, x\,\rangle)}\ \text{C\_TAbs}$$

$$\frac{[\![\Theta \vdash e \,:\, \forall\beta.\tau_2]\!] \Rightarrow R \quad \Theta \vdash \tau_1}{[\![\Theta \vdash e \,:\, \tau_2[\tau_1/\beta]]\!] \Rightarrow \Lambda\alpha.\lambda k.R\,\alpha\,(\lambda x.\mathsf{let}\,!y = x\,\mathsf{in}\,k\,!(y\,[\![\tau_1]\!]_{\mathsf{v}}))}\ \text{C\_TApp}$$

Figure 6: CPS transformation.

2.3 Translation

Convention 3. *We use a metavariable χ for denoting variables or constants.*

2.3.1 CPS Transformation: from $\lambda_v^{\forall}$ to $\Lambda^{\mathbf{open}}$

Definition 24. *CPS transformations $[\![\tau]\!]$ of a type τ of terms and $[\![\tau]\!]_{\mathsf{v}}$ of a type τ of values are defined by induction on τ, as follows.*

$$\begin{array}{rcl}
[\![\tau]\!] & \stackrel{\text{def}}{=} & \forall\alpha.(![\![\tau]\!]_{\mathsf{v}} \multimap \alpha) \multimap \alpha \quad (\alpha \notin ftv(\tau)) \\
[\![\alpha]\!]_{\mathsf{v}} & \stackrel{\text{def}}{=} & \alpha \\
[\![\iota]\!]_{\mathsf{v}} & \stackrel{\text{def}}{=} & \iota \\
[\![\tau_1 \to \tau_2]\!]_{\mathsf{v}} & \stackrel{\text{def}}{=} & ![\![\tau_1]\!]_{\mathsf{v}} \multimap [\![\tau_2]\!] \\
[\![\forall\alpha.\tau]\!]_{\mathsf{v}} & \stackrel{\text{def}}{=} & \forall\alpha.[\![\tau]\!]_{\mathsf{v}}
\end{array}$$

CPS transformation $[\![\Theta]\!]$ of a typing context Θ is defined by induction on Θ, as follows.

$$\begin{array}{rcl}
[\![\emptyset]\!] & \stackrel{\text{def}}{=} & \emptyset \\
[\![\Theta, x : \tau]\!] & \stackrel{\text{def}}{=} & [\![\Theta]\!], x :^{\omega} [\![\tau]\!]_{\mathsf{v}} \\
[\![\Theta, \alpha]\!] & \stackrel{\text{def}}{=} & [\![\Theta]\!], \alpha^{\mathbf{0}}
\end{array}$$

CPS transformation $[\![\chi : \tau]\!]$ of χ of a type τ is defined by induction on τ, as follows.

$$\begin{array}{rcl}
[\![\chi : \iota]\!] & \stackrel{\text{def}}{=} & !\chi \\
[\![\chi : \iota \to \tau]\!] & \stackrel{\text{def}}{=} & !(\lambda x.\mathsf{let}\,!y = x\,\mathsf{in}\,\mathsf{let}\,!z = !(\chi\,y)\,\mathsf{in}\,\Lambda\alpha.\lambda k.k\,[\![z : \tau]\!]) \quad \textit{(where } k, x, y, z \notin fv\,(\chi)\textit{)}
\end{array}$$

Definition 25. *CPS transformation $[\![\Theta \vdash e \,:\, \tau]\!] \Rightarrow V$ of a typing judgment $\Theta \vdash e : \tau$ is the smallest relation satisfying the rules in Figure 6. In the rules of Figure 6, we assume that k and α are fresh, that is, $k, \alpha \notin dom(\Theta)$, $\alpha \notin ftv(\tau)$, and k and α do not occur in e as a free nor bound variable, respectively.*

2.3.2 CPS Transformation: from $\lambda_v^{\forall}$ to itself

Definition 26. *CPS transformation $(\!|\, \chi : \tau \,|\!)$ of χ of a type τ is defined by induction on τ, as follows.*

$$\begin{array}{rcl}
(\!|\, \chi : \iota \,|\!) & \stackrel{\text{def}}{=} & \chi \\
(\!|\, \chi : \iota \to \tau \,|\!) & \stackrel{\text{def}}{=} & \lambda x.(\lambda y.\lambda k.k\,(\!|\, y : \tau \,|\!))\,(\chi\,x) \quad \textit{(where } k, x, y \notin fv\,(\chi)\textit{)}
\end{array}$$

CPS transformation $(\!| e |\!)$ *of a term* e *in* $\lambda_v^\forall$ *is defined by induction on* e*, as follows:*

$$
\begin{array}{lcl}
(\!| x |\!) & \stackrel{\text{def}}{=} & \lambda k.k\, x \\
(\!| c |\!) & \stackrel{\text{def}}{=} & \lambda k.k\, (\!| c : ty^{\rightarrow}(c) |\!) \\
(\!| \lambda x.e |\!) & \stackrel{\text{def}}{=} & \lambda k.k\, \lambda x.(\!| e |\!) \\
(\!| e_1\, e_2 |\!) & \stackrel{\text{def}}{=} & \lambda k.(\!| e_1 |\!)\, (\lambda x.(\!| e_2 |\!)\, (\lambda y.x\, y\, k))
\end{array}
$$

where k *is a variable that does not occur in* e *as a free variable nor a bound variable.*

2.3.3 Type Erasure: from $\Lambda^{\mathbf{open}}$ to $\lambda_v^\forall$

Definition 27. *Type erasure* erase *is a function that translates terms in* Λ^{open} *to untyped terms in* $\lambda_v^\forall$*, defined by induction on* M *as follows.*

$$
\begin{array}{lcl}
\mathsf{erase}(x) & \stackrel{\text{def}}{=} & x \\
\mathsf{erase}(c) & \stackrel{\text{def}}{=} & c \\
\mathsf{erase}(\lambda x.M) & \stackrel{\text{def}}{=} & \lambda x.\mathsf{erase}(M) \\
\mathsf{erase}(M_1\, M_2) & \stackrel{\text{def}}{=} & \mathsf{erase}(M_1)\, \mathsf{erase}(M_2) \\
\mathsf{erase}(!M) & \stackrel{\text{def}}{=} & \mathsf{erase}(M) \\
\mathsf{erase}(\mathsf{let}\, !x = M_1\, \mathsf{in}\, M_2) & \stackrel{\text{def}}{=} & (\lambda x.\mathsf{erase}(M_2))\, \mathsf{erase}(M_1) \\
\mathsf{erase}(\nu\alpha.\, M) & \stackrel{\text{def}}{=} & \mathsf{erase}(M) \\
\mathsf{erase}(\Lambda^\circ \langle\, \alpha, M\, \rangle) & \stackrel{\text{def}}{=} & \mathsf{erase}(M) \\
\mathsf{erase}(\Lambda\alpha.M) & \stackrel{\text{def}}{=} & \mathsf{erase}(M) \\
\mathsf{erase}(M\, A) & \stackrel{\text{def}}{=} & \mathsf{erase}(M)
\end{array}
$$

Definition 28. *A term* M *is erasable if and only if, for any subterm* $\Lambda\alpha.M'$ *in* M*,* $M' = R$ *for some* R*.*

2.4 Logical Relation

Convention 4. *We employ the following conventions.*

- *For sets S_1 and S_2, we write $S_1 \# S_2$ to state that they are disjoint.*
- *The metavariable ρ ranges over interpretations, which are mappings that map type variables to triples of the form (A_1, A_2, r),*
- *The metavariable r ranges over relational interpretations, which are mappings that map worlds to sets of pairs of terms.*
- *The metavariable ς ranges over relational result substitutions, which are mappings that map variables to pairs of results.*
- *We write $dom(\rho)$ (resp. $dom(\varsigma)$) for the set of free type variables (resp. free variables) mapped by ρ (resp. ς).*
- *We write $dom(r)$ for the set of worlds mapped by r.*
- *When $W = (n, \Delta, \rho)$, we write $W.n$ for n, $W.\Delta$ for Δ, and $W.\rho$ for ρ.*
- *For ρ_1 and ρ_2 such that $dom(\rho_1) \# dom(\rho_2)$, we write $\rho_1 \uplus \rho_2$ for the mapping that maps a type variable $\alpha \in dom(\rho_1)$ to $\rho_1(\alpha)$ and a type variable $\alpha \in dom(\rho_2)$ to $\rho_2(\alpha)$.*
- *We write ρ_{fst} and ρ_{snd} for capture-avoiding type substitutions that map a type variable α in $dom(\rho)$ to A_1 and A_2 when $\rho(\alpha) = (A_1, A_2, r)$, respectively.*
- *When $\rho(\alpha) = (A_1, A_2, r)$, we write $\rho[\alpha]$ for the relational interpretation r.*
- *We write ρ^A_W for an interpretation $\{\alpha \Mapsto (W.\rho_{fst}(A), W.\rho_{snd}(A), r)\}$ for some r.*
- *We write $\Delta_1 \bot \Delta_2$ if and only if $\Delta_1 + \Delta_2$ is well defined.*
- *We identify typing contexts Δ_1 and Δ_2 up to permutation (i.e., $\Delta, \alpha^{\pi_1}, \beta^{\pi_2}, \Delta'$ is identical with $\Delta, \beta^{\pi_2}, \alpha^{\pi_1}, \Delta'$) for simplifying the technical development. Because Δ contains only type variables, this identification does not change typability of terms.*
- *We write $\longrightarrow^n$ for the n step evaluation.*
- *$\sum_{x \in I} \Delta_x$ stands for the typing context $\Delta_{x_1} + \cdots + \Delta_{x_n}$ given a family of typing contexts $\Delta_{x_1}, \cdots, \Delta_{x_n}$ with a finite index set of variables $I = \{x_1, \cdots, x_n\}$. We also write $\exists \prod_{x \in I} \Delta_x$ to existentially quantify $\Delta_{x_1}, \cdots, \Delta_{x_n}$.*
- *$dom_{=\mathbf{1}}(\Gamma)$ stands for the finite set of variables that are affine in typing context Γ.*
- *We write $\rho(\varsigma)$ for ς' such that: $dom(\varsigma') = dom(\varsigma)$; and, for any $x \in dom(\varsigma')$, $\varsigma'_{fst}(x) = \rho_{fst}(\varsigma_{fst}(x))$ and $\varsigma'_{snd}(x) = \rho_{snd}(\varsigma_{snd}(x))$.*

Definition 29 (Logical Relation)**.** *A logical relation for Λ^{open} is defined in Figure 8 with auxiliary definitions in Figure 7.*

$$\mathrm{Atom}\,[\Delta, A_1, A_2] \overset{\text{def}}{=} \{(M_1, M_2) \mid \Delta \vdash M_1 : A_1 \;\wedge\; \Delta \vdash M_2 : A_2\}$$

$$\mathrm{Atom}^{\mathrm{res}}\,[\Delta, A_1, A_2] \overset{\text{def}}{=} \{(R_1, R_2) \mid (R_1, R_2) \in \mathrm{Atom}\,[\Delta, A_1, A_2]\}$$

$$\mathrm{Atom}\,[W, A] \overset{\text{def}}{=} \mathrm{Atom}\,[W.\Delta,\, W.\rho_{\mathrm{fst}}(A),\, W.\rho_{\mathrm{snd}}(A)]$$

$$\mathrm{World}_n \overset{\text{def}}{=} \{(m, \Delta, \rho) \in \mathit{Nat} \times \mathit{TypCtx} \times (\mathit{TyVar} \rightharpoonup \mathit{Type} \times \mathit{Type} \times \mathrm{Rel}_m) \mid m < n \wedge \vdash (m, \Delta, \rho)\}$$

$$\begin{aligned}
\mathrm{Rel}_n[A_1, A_2] \overset{\text{def}}{=} \{ & r \in (W : \mathrm{World}_n) \rightharpoonup \mathcal{P}(\mathrm{Atom}^{\mathrm{res}}\,[W.\Delta, W.\rho_{\mathrm{fst}}(A_1), W.\rho_{\mathrm{snd}}(A_2)]) \mid \\
& \forall W_1.\ \forall W_2 \sqsupseteq W_1.\ \forall (R_1, R_2) \in r(W_1).\ (R_1, R_2)_{W_2} \in r(W_2) \\
& \wedge \forall W, \rho.\ \rho \uplus W \in \mathit{dom}(r) \wedge \mathit{dom}(\rho) \# \mathit{ftv}(A_1) \wedge \mathit{dom}(\rho) \# \mathit{ftv}(A_2) \Longrightarrow \\
& \qquad r(\rho \uplus W) \subseteq r(W) \\
& \wedge \forall W, \alpha.\ \{\alpha\} \# \mathit{ftv}(A_1) \wedge \{\alpha\} \# \mathit{ftv}(A_2) \wedge \vdash W \Longrightarrow \\
& \qquad \forall (R_1, R_2) \in r(W@\alpha).\ (\nu\alpha.\, R_1, \nu\alpha.\, R_2) \in r(W) \\
& \wedge \forall W, \alpha.\ \{\alpha\} \# W \Longrightarrow \\
& \qquad \forall (R_1, R_2) \in r(W).\ (\nu\alpha.\, R_1, R_2) \in r(W) \wedge (R_1, \nu\alpha.\, R_2) \in r(W) \\
\} &
\end{aligned}$$

$$\mathrm{Rel}_n \overset{\text{def}}{=} \textstyle\bigcup_{A_1, A_2} \mathrm{Rel}_n[A_1, A_2]$$

$$\dagger(\rho) \overset{\text{def}}{=} \omega\Delta \text{ such that } \mathit{dom}(\Delta) = \mathit{dom}(\rho)$$

$$\rho_2(\rho_1) \overset{\text{def}}{=} \{\alpha \Mapsto (\rho_{2\mathrm{fst}}(\rho_{1\mathrm{fst}}(\alpha)), \rho_{2\mathrm{snd}}(\rho_{1\mathrm{snd}}(\alpha)), \rho_1[\alpha]) \mid \alpha \in \mathit{dom}(\rho_1)\}$$

$$\rho_2 \circ \rho_1 \overset{\text{def}}{=} \rho_2 \uplus \rho_2(\rho_1)$$

$$\mathit{ftv}(\rho) \overset{\text{def}}{=} \textstyle\bigcup_{\alpha \in \mathit{dom}(\rho)} \mathit{ftv}(\rho_{\mathrm{fst}}(\alpha)) \cup \mathit{ftv}(\rho_{\mathrm{snd}}(\alpha))$$

$$\rho|_S \overset{\text{def}}{=} \{\alpha \Mapsto \rho(\alpha) \mid \alpha \in \mathit{dom}(\rho) \cap S\}$$

$$\Gamma \succ \rho \overset{\text{def}}{=} \forall \alpha \in \mathit{ftv}(\rho|_{\mathit{dom}(\Gamma)}) \cap \mathit{dom}(\Gamma).\ \alpha^{\mathbf{0}} \in \Gamma$$

$$\Delta_1 \ggcurly \Delta_2 \overset{\text{def}}{=} \exists \Delta, \Delta_0.\ \Delta_1 = (\Delta_2 + \Delta), \Delta_0$$

$$\begin{aligned}
W_1 \sqsupseteq W_2 \overset{\text{def}}{=} {} & \vdash W_1 \wedge \vdash W_2 \wedge W_1.n \leq W_2.n \wedge \\
& \exists \rho.\ (W_1.\Delta, \dagger(\rho)) \ggcurly W_2.\Delta \wedge W_1.\rho = \rho \circ W_2.\rho \wedge W_2.\Delta \succ \rho
\end{aligned}$$

$$(W_1, W_2) \Supset W_3 \overset{\text{def}}{=} W_1.n = W_2.n = W_3.n \wedge W_1.\Delta + W_2.\Delta = W_3.\Delta \wedge W_1.\rho = W_2.\rho = W_3.\rho$$

$$\rho \uplus W \overset{\text{def}}{=} (W.n,\ W.\Delta,\ \rho \uplus W.\rho) \quad (\text{if } \mathit{dom}(\rho) \# W)$$

$$\omega W \overset{\text{def}}{=} (W.n,\ \omega(W.\Delta),\ W.\rho)$$

$$W@\alpha \overset{\text{def}}{=} (W.n,\ (W.\Delta, \alpha^{\mathbf{1}}),\ W.\rho) \quad (\text{if } \{\alpha\} \# W)$$

$$S \# W \overset{\text{def}}{=} S \# \mathit{dom}(W.\Delta) \wedge S \# \mathit{dom}(W.\rho)$$

$$W \vdash (A_1, A_2, r) \overset{\text{def}}{=} W.\Delta \vdash A_1 \wedge W.\Delta \vdash A_2 \wedge r \in \mathrm{Rel}_{W.n}[A_1, A_2]$$

$$\vdash W \overset{\text{def}}{=} \mathit{dom}(W.\rho) \# \mathit{dom}(W.\Delta) \wedge \forall \alpha \in \mathit{dom}(W.\rho).\ W \vdash W.\rho(\alpha)$$

$$(n + m, \Delta, \rho) - m \overset{\text{def}}{=} (n, \Delta, \rho)$$

$$\blacktriangleright W \overset{\text{def}}{=} W - 1$$

$$(R_1, R_2)_W \overset{\text{def}}{=} (W.\rho_{\mathrm{fst}}(R_1),\ W.\rho_{\mathrm{snd}}(R_2))$$

Figure 7: Objects appearing in logical relation.

$$
\begin{array}{lll}
\mathcal{R}[\![\iota]\!]\, W & \stackrel{\text{def}}{=} & \{(\nu\overline{\alpha_1}.\, c, \nu\overline{\alpha_2}.\, c) \in \text{Atom}\,[W, \iota]\} \\
\mathcal{R}[\![\alpha]\!]\, W & \stackrel{\text{def}}{=} & W.\rho[\alpha](\blacktriangleright W) \\
\mathcal{R}[\![A \multimap B]\!]\, W & \stackrel{\text{def}}{=} & \{(R_1, R_2) \in \text{Atom}\,[W, A \multimap B] \mid \forall\, W' \sqsupseteq W.\ \forall\,(W_1, W_2) \Supset W'.\ W_1 \sqsupseteq W \Longrightarrow \\
 & & \quad \forall\,(R_1', R_2') \in \mathcal{R}[\![A]\!]\, W_2.\ (R_1\, R_1', R_2\, R_2')_{W'} \in \mathcal{E}[\![B]\!]\, W'\} \\
\mathcal{R}[\![\forall\alpha.A]\!]\, W & \stackrel{\text{def}}{=} & \{(R_1, R_2) \in \text{Atom}\,[W, \forall\alpha.A] \mid \forall\, W' \sqsupseteq W.\ \forall\, B_1, B_2, r. \\
 & & \quad \omega\, W' \vdash (B_1, B_2, r) \wedge \{\alpha\}\#\omega\, W' \Longrightarrow (R_1\, B_1, R_2\, B_2)_{\omega\, W'} \in \mathcal{E}[\![A]\!]\, \{\alpha \mapsto (B_1, B_2, r)\} \uplus \omega\, W'\} \\
\mathcal{R}[\![!A]\!]\, W & \stackrel{\text{def}}{=} & \{(R_1, R_2) \in \text{Atom}\,[W, !A] \mid (\mathsf{let}\,!x = R_1\,\mathsf{in}\,x, \mathsf{let}\,!x = R_2\,\mathsf{in}\,x) \in \mathcal{E}[\![A]\!]\,\omega\, W\} \\
\mathcal{E}[\![A]\!]\, W & \stackrel{\text{def}}{=} & \{(M_1, M_2) \in \text{Atom}\,[W, A] \mid \forall\, W' \sqsupseteq W.\ \forall\, n < W'.n.\ \forall\, R_1. \\
 & & \quad W'.\rho_{\text{fst}}(M_1) \longrightarrow^n R_1 \Longrightarrow \exists\, R_2.\ W'.\rho_{\text{snd}}(M_2) \longrightarrow^* R_2 \wedge (R_1, R_2) \in \mathcal{R}[\![A]\!]\,(W' - n)\} \\
\mathcal{G}[\![\Gamma]\!] & \stackrel{\text{def}}{=} & \{(W, \varsigma) \mid \exists\, \Delta.\, \exists \prod_{x \,\in\, dom_{=\mathbf{1}}(\Gamma)} \Delta_x. \\
 & & \qquad \vdash W \wedge \Gamma \succ W.\rho \wedge W.\Delta = \Delta + \sum_{x \in dom_{=\mathbf{1}}(\Gamma)} \Delta_x \\
 & & \quad \wedge\, \forall\, \alpha^{\pi} \in \Gamma.\ (\exists\, \pi' \geq \pi.\ \alpha^{\pi'} \in \Delta) \vee (\pi = \mathbf{0} \wedge \alpha \in dom(W.\rho)) \\
 & & \quad \wedge\, \forall\, x :^{\mathbf{1}} A \in \Gamma.\ (\varsigma_{\text{fst}}(x), \varsigma_{\text{snd}}(x)) \in \mathcal{R}[\![A]\!]\,(W.n,\, \Delta_x,\, W.\rho) \\
 & & \quad \wedge\, \forall\, x :^{\omega} A \in \Gamma.\ (\varsigma_{\text{fst}}(x), \varsigma_{\text{snd}}(x)) \in \mathcal{R}[\![A]\!]\,\omega\, W\} \\
\Gamma \vdash M_1 \preceq M_2 : A & \stackrel{\text{def}}{=} & \Gamma \vdash M_1 : A \wedge \Gamma \vdash M_2 : A \wedge \forall\,(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!].\ (\varsigma_{\text{fst}}(M_1), \varsigma_{\text{snd}}(M_2))_W \in \mathcal{E}[\![A]\!]\, W \\
\Gamma \vdash M_1 \approx M_2 : A & \stackrel{\text{def}}{=} & \Gamma \vdash M_1 \preceq M_2 : A \wedge \Gamma \vdash M_2 \preceq M_1 : A
\end{array}
$$

Figure 8: Logical relation.

3 Proofs

3.1 Type Soundness of $\Lambda^{\mathbf{open}}$

Lemma 1 (Uses as a commutative monoid)**.**

1. *$\pi_1 + \pi_2$ is well defined for any π_1 and π_2.*
2. *$\mathbf{0} + \pi = \pi + \mathbf{0} = \pi$ for any π.*
3. *$\pi_1 + (\pi_2 + \pi_3) = (\pi_1 + \pi_2) + \pi_3$ for any π_1, π_2, and π_3.*
4. *$\pi_1 + \pi_2 = \pi_2 + \pi_1$ for any π_1 and π_2.*

Proof. 1. By definition.

2. By definition.

3. By case analysis on π_1, π_2, and π_3.

   Case $\pi_1 = \omega$: $\pi_1 + (\pi_2 + \pi_3) = \omega = \omega + \pi_3 = (\omega + \pi_2) + \pi_3 = (\pi_1 + \pi_2) + \pi_3$.
   Case $\pi_1 = \mathbf{0}$: $\pi_1 + (\pi_2 + \pi_3) = \pi_2 + \pi_3 = (\mathbf{0} + \pi_2) + \pi_3 = (\pi_1 + \pi_2) + \pi_3$.
   Case $\pi_1 = \mathbf{1}$ and $\pi_2 = \omega$: $\pi_1 + (\pi_2 + \pi_3) = \omega = \omega + \pi_3 = (\pi_1 + \pi_2) + \pi_3$.
   Case $\pi_1 = \mathbf{1}$ and $\pi_2 = \mathbf{0}$: $\pi_1 + (\pi_2 + \pi_3) = \pi_1 + \pi_3 = (\pi_1 + \pi_2) + \pi_3$.
   Case $\pi_1 = \pi_2 = \mathbf{1}$ and $\pi_3 = \omega$: $\pi_1 + (\pi_2 + \pi_3) = \omega = (\pi_1 + \pi_2) + \pi_3$.
   Case $\pi_1 = \pi_2 = \mathbf{1}$ and $\pi_3 = \mathbf{0}$: $\pi_1 + (\pi_2 + \pi_3) = \pi_1 + \pi_2 = (\pi_1 + \pi_2) + \pi_3$.
   Case $\pi_1 = \pi_2 = \pi_3 = \mathbf{1}$: $\pi_1 + (\pi_2 + \pi_3) = \omega = (\pi_1 + \pi_2) + \pi_3$.

4. By definition.

□

Lemma 2 (Associativity of merging typing contexts)**.** $(\Gamma_1 + \Gamma_2) + \Gamma_3 = \Gamma_1 + (\Gamma_2 + \Gamma_3)$.

Proof. By induction on Γ_3. The cases for $\Gamma_3 = \Gamma_3', x :^\pi A$ and $\Gamma_3 = \Gamma_3', \alpha^\pi$ rest on Lemma 1 (3). □

Lemma 3 (Commutativity of merging typing contexts)**.** $\Gamma_1 + \Gamma_2 = \Gamma_2 + \Gamma_1$.

Proof. By induction on Γ_1 with Lemma 1 (4). □

Lemma 4. $\omega\Gamma + \omega\Gamma = \omega\Gamma$

Proof. Straightforward by induction on Γ. The proof depends on the fact that $\omega + \omega = \omega$ and $\mathbf{0} + \mathbf{0} = \mathbf{0}$. □

Lemma 5. *For any Γ, $\omega\omega\Gamma = \omega\Gamma$.*

Proof. Straightforward by induction on Γ. □

Lemma 6.

1. *If $\pi_1 = \omega$ or $\pi_2 = \omega$, then $\pi_1 + \pi_2 = \omega$.*
2. *if $\pi_1 + \pi_2 = \mathbf{0}$, then $\pi_1 = \pi_2 = \mathbf{0}$.*
3. *If $\pi_1 + \pi_2 + \pi_3 \neq \omega$, then $\pi_1 + \pi_3 \neq \omega$ and $\pi_2 + \pi_3 \neq \omega$.*
4. *If $\pi + \mathbf{0} = \omega$, then $\pi = \omega$.*
5. *If $\pi_1 \neq \mathbf{0}$ nor $\pi_2 \neq \mathbf{0}$, then $\pi_1 + \pi_2 = \omega$.*

Proof. 1. By definition.

2. Obvious.

3. By Lemma 1, it suffices to show $\pi_1 + \pi_3 \neq \omega$. Since $\pi_1 + \pi_2 + \pi_3 \neq \omega$, we can find $\pi_1 \neq \omega$ nor $\pi_3 \neq \omega$. If $\pi_1 = \mathbf{1}$ and $\pi_3 = \mathbf{1}$, then $\pi_1 + \pi_2 + \pi_3 = \pi_2 + (\mathbf{1} + \mathbf{1}) = \omega$, which is contradictory. Other remaining cases are: $\pi_1 = \mathbf{1}$ and $\pi_3 = \mathbf{0}$; $\pi_1 = \mathbf{0}$ and $\pi_3 = \mathbf{1}$; and $\pi_1 = \pi_3 = \mathbf{0}$. In all of the cases, $\pi_1 + \pi_3 \neq \omega$.

4. Obvious.

5. If $\pi_1 = \omega$ or $\pi_2 = \omega$, then $\pi_1 + \pi_2 = \omega$. Otherwise, $\pi_1 = \pi_2 = \mathbf{1}$, so we have the conclusion.

□

Lemma 7. $(\Gamma_1 + \Gamma_2), (\Gamma_3 + \Gamma_4) = (\Gamma_1, \Gamma_3) + (\Gamma_2, \Gamma_4)$.

Proof. Straightforward by induction on Γ_3. □

Lemma 8. $\omega(\Gamma, \Gamma') = \omega\Gamma, \omega\Gamma'$.

Proof. By induction on Γ'. □

Lemma 9. $dom(\Gamma) = dom(\omega\Gamma)$.

Proof. Both of the cases are shown by induction on Γ. □

Lemma 10. *If* $\vdash \Gamma$, *then* $\vdash \omega\Gamma$.

Proof. By induction on the derivation of $\vdash \Gamma$ with Lemma 9. □

Lemma 11. *If* $\Gamma_1 + \Gamma_2 = \Gamma'_1, \Gamma'_2$, *then there are exist some* Γ_{11}, Γ_{12}, Γ_{21}, *and* Γ_{22} *such that*

- $\Gamma_1 = \Gamma_{11}, \Gamma_{12}$,
- $\Gamma_2 = \Gamma_{21}, \Gamma_{22}$,
- $\Gamma'_1 = \Gamma_{11} + \Gamma_{21}$, *and*
- $\Gamma'_2 = \Gamma_{12} + \Gamma_{22}$.

Proof. By induction on Γ'_2. □

Lemma 12. $dom(\Gamma_1 + \Gamma_2) = dom(\Gamma_1) = dom(\Gamma_2)$.

Proof. By induction on Γ_1. □

Lemma 13. *If* $\vdash \Gamma_1 + \Gamma_2$, *then* $\vdash \Gamma_1$ *and* $\vdash \Gamma_2$.

Proof. By induction on Γ_1 with Lemma 12. The case for (WF_TyVar) relies on (the contraposition of) Lemma 6 (1). □

Lemma 14. *If* $\Gamma_1 \leq \Gamma_2$, *then* $dom(\Gamma_1) = dom(\Gamma_2)$.

Proof. Straightforward by induction on Γ_2. □

Lemma 15. *Suppose that* $\Gamma_1 \leq \Gamma_2$. $\vdash \Gamma_1$ *if and only if* $\vdash \Gamma_2$.

Proof. By induction on the derivation of $\Gamma_1 \leq \Gamma_2$ with Lemma 14. The right-to-left direction in the case for $\Gamma'_1, \alpha^{\pi_1} \leq \Gamma'_2, \alpha^{\pi_2}$ rests on the fact that, if $\pi_1 \leq \pi_2$ and $\pi_2 \neq \omega$, then $\pi_1 \neq \omega$. □

Lemma 16. *If* $\vdash \Gamma_1$ *and* $\vdash \Gamma_2$ *and* $\Gamma_1 + \Gamma_2$ *is well defined, then* $\vdash \Gamma_1 + \Gamma_2$.

Proof. By induction on the derivation of $\vdash \Gamma_1$ with Lemma 12. The case for (WF_TyVar) relies on the fact that, if $(\Gamma_1, \alpha^{\pi_1}) + (\Gamma_2, \alpha^{\pi_2})$ is well defined, then $\pi_1 + \pi_2 \neq \omega$. □

Lemma 17. *If* $\Gamma \vdash M : A$, *then* $\vdash \Gamma$.

Proof. By induction on the typing derivation of $\Gamma \vdash M : A$. The cases for (T_App) and (T_LetBang) rest on Lemma 16. The case for (T_Gen) rests on Lemma 15. □

Lemma 18. *If $\Gamma \vdash M : A$, then $\Gamma \vdash A$.*

Proof. By induction on the typing derivation of $\Gamma \vdash M : A$. The case for (T_ABS) rests on Lemma 17. The cases for (T_APP) and (T_LETBANG) rest on Lemma 12. The cases for (T_BANG) and (T_TABS) rest on Lemma 9. The case for (T_GEN) rests on Lemma 14. □

Lemma 19 (Idempotent typing contexts)**.** *Let Γ be a typing context such that, for any $\alpha^{\pi} \in \Gamma$, $\pi \neq \omega$. Then, there exists some Γ' such that $\Gamma + \Gamma' = \Gamma$.*

Proof. We can construct such a Γ' by $f(\Gamma)$ where f is a function defined inductively on Γ as follows.

$$\begin{array}{lcl} f(\emptyset) & \stackrel{\text{def}}{=} & \emptyset \\ f(\Gamma, x :^{\pi} A) & \stackrel{\text{def}}{=} & f(\Gamma), x :^{\mathbf{0}} A \\ f(\Gamma, \alpha^{\pi}) & \stackrel{\text{def}}{=} & f(\Gamma), \alpha^{\mathbf{0}} \ . \end{array}$$

It is easy to see $\Gamma + \Gamma' = \Gamma$. □

Lemma 20 (Weakening)**.** *Suppose that $\vdash \Gamma_1, \Gamma_2$ and $dom(\Gamma_2) \cap dom(\Gamma_3) = \emptyset$.*

1. *If $\vdash \Gamma_1, \Gamma_3$, then $\vdash \Gamma_1, \Gamma_2, \Gamma_3$.*
2. *If $\Gamma_1, \Gamma_3 \vdash M : A$, then $\Gamma_1, \Gamma_2, \Gamma_3 \vdash M : A$.*

Proof. 1. Straightforward by induction on the derivation of $\vdash \Gamma_1, \Gamma_3$ with case analysis on Γ_3.

2. By induction on the typing derivation of $\Gamma_1, \Gamma_3 \vdash M : A$.

Case (T_VAR) and (T_CONST): By the case (1).

Case (T_ABS), (T_NU), and (T_TAPP): By the IH(s).

Case (T_APP): We are given $\Gamma_{01}+\Gamma_{02} \vdash M_1\, M_2 : A$ for some Γ_{01}, Γ_{02}, M_1, and M_2 such that $\Gamma_1, \Gamma_3 = \Gamma_{01}+\Gamma_{02}$ and $M = M_1\, M_2$. By inversion,

- $\Gamma_{01} \vdash M_1 : B \multimap A$ and
- $\Gamma_{02} \vdash M_2 : B$

for some B. By Lemma 11, there are Γ_{11}, Γ_{12}, Γ_{31}, and Γ_{32} such that

- $\Gamma_{01} = \Gamma_{11}, \Gamma_{31}$,
- $\Gamma_{02} = \Gamma_{12}, \Gamma_{32}$,
- $\Gamma_1 = \Gamma_{11} + \Gamma_{12}$, and
- $\Gamma_3 = \Gamma_{31} + \Gamma_{32}$.

We can construct Γ_2' such that $\Gamma_2 + \Gamma_2' = \Gamma_2$ by Lemma 19, so $\vdash (\Gamma_{11}+\Gamma_{12}), (\Gamma_2+\Gamma_2')$ from $\vdash \Gamma_1, \Gamma_2$. Since $(\Gamma_{11}+\Gamma_{12}), (\Gamma_2+\Gamma_2') = (\Gamma_{11}, \Gamma_2) + (\Gamma_{12}, \Gamma_2')$ by Lemma 7, we have $\vdash \Gamma_{11}, \Gamma_2$ and $\vdash \Gamma_{12}, \Gamma_2'$ by Lemma 13. We also find

- $dom(\Gamma_2) \cap dom(\Gamma_{31}) = \emptyset$ and
- $dom(\Gamma_2') \cap dom(\Gamma_{32}) = \emptyset$

by Lemma 12 and $dom(\Gamma_2) \cap dom(\Gamma_3) = \emptyset$. Thus, by the IHs,

- $\Gamma_{11}, \Gamma_2, \Gamma_{31} \vdash M_1 : B \multimap A$ and
- $\Gamma_{12}, \Gamma_2', \Gamma_{32} \vdash M_2 : B$.

By (T_APP),

$$(\Gamma_{11}, \Gamma_2, \Gamma_{31}) + (\Gamma_{12}, \Gamma_2', \Gamma_{32}) \vdash M_1\, M_2 : A.$$

Since $(\Gamma_{11}, \Gamma_2, \Gamma_{31}) + (\Gamma_{12}, \Gamma_2', \Gamma_{32}) = \Gamma_1, \Gamma_2, \Gamma_3$ by Lemma 7, we have the conclusion.

Case (T_BANG): We are given $\Gamma_1, \Gamma_3 \vdash\, !M' : !A'$ for some M' and A' such that $M = !M'$ and $A = !A'$. By inversion,

$$\omega(\Gamma_1, \Gamma_3) \vdash M' : A'.$$

Since $\vdash \Gamma_1, \Gamma_2$, we have $\vdash \omega(\Gamma_1, \Gamma_2)$ by Lemma 10. By Lemma 8, $\omega\Gamma_1, \omega\Gamma_3 \vdash M' : A'$ and $\vdash \omega\Gamma_1, \omega\Gamma_2$. Since $dom(\Gamma_2) \cap dom(\Gamma_3) = \emptyset$, we have

$$dom(\omega\Gamma_2) \cap dom(\omega\Gamma_3) = \emptyset$$

by Lemma 9. Thus, by the IH,

$$\omega\Gamma_1, \omega\Gamma_2, \omega\Gamma_3 \vdash M' : A'.$$

By Lemma 8 and (T_BANG),

$$\Gamma_1, \Gamma_2, \Gamma_3 \vdash\, !M' : !A'.$$

Note that we have $\vdash \Gamma_1, \Gamma_2, \Gamma_3$ by the case (1).

Case (T_LETBANG): Similarly to the case for (T_APP).

Case (T_GEN): We are given $\Gamma_{01}, \alpha^{\mathbf{1}}, \Gamma_{02} \vdash \Lambda^\circ\langle\, \alpha, M'\,\rangle : !\forall\alpha.A'$ for some Γ_{01}, Γ_{02}, α, M', and A' such that

- $\Gamma_1, \Gamma_3 = \Gamma_{01}, \alpha^{\mathbf{1}}, \Gamma_{02}$,
- $M = \Lambda^\circ\langle\, \alpha, M'\,\rangle$, and
- $A = !\forall\alpha.A'$.

By inversion, $\Gamma_{01}, \alpha^{\mathbf{0}}, \Gamma_{02} \vdash M' : !A'$. We perform case analysis on $\Gamma_1, \Gamma_3 = \Gamma_{01}, \alpha^{\mathbf{1}}, \Gamma_{02}$.

Case $\Gamma_1 = \Gamma_{01}, \alpha^{\mathbf{1}}, \Gamma'_{02}$ for some Γ'_{02} such that $\Gamma_{02} = \Gamma'_{02}, \Gamma_3$: We have

$$\Gamma_{01}, \alpha^{\mathbf{0}}, \Gamma'_{02}, \Gamma_3 \vdash M' : !A'\ .$$

Since $\vdash \Gamma_1, \Gamma_2$, we have $\vdash \Gamma_{01}, \alpha^{\mathbf{1}}, \Gamma'_{02}, \Gamma_2$, so $\vdash \Gamma_{01}, \alpha^{\mathbf{0}}, \Gamma'_{02}, \Gamma_2$ by Lemma 15. Thus, by the IH,

$$\Gamma_{01}, \alpha^{\mathbf{0}}, \Gamma'_{02}, \Gamma_2, \Gamma_3 \vdash M' : !A'.$$

By (T_GEN), we have the conclusion

$$\Gamma_{01}, \alpha^{\mathbf{1}}, \Gamma'_{02}, \Gamma_2, \Gamma_3 \vdash \Lambda^\circ\langle\, \alpha, M'\,\rangle : !\forall\alpha.A'.$$

Case $\Gamma_3 = \Gamma'_{01}, \alpha^{\mathbf{1}}, \Gamma_{02}$ for some Γ'_{01} such that $\Gamma_{01} = \Gamma_1, \Gamma'_{01}$: We have

$$\Gamma_1, \Gamma'_{01}, \alpha^{\mathbf{0}}, \Gamma_{02} \vdash M' : !A'.$$

Since $dom(\Gamma_2) \cap dom(\Gamma_3) = \emptyset$, we have $dom(\Gamma_2) \cap dom(\Gamma'_{01}, \alpha^{\mathbf{0}}, \Gamma_{02}) = \emptyset$ by Lemma 14. Thus, by the IH,

$$\Gamma_1, \Gamma_2, \Gamma'_{01}, \alpha^{\mathbf{0}}, \Gamma_{02} \vdash M' : !A'.$$

By (T_GEN), we have the conclusion

$$\Gamma_1, \Gamma_2, \Gamma'_{01}, \alpha^{\mathbf{1}}, \Gamma_{02} \vdash \Lambda^\circ\langle\, \alpha, M'\,\rangle : !\forall\alpha.A'.$$

Case (T_TABS): We are given $\Gamma_1, \Gamma_3 \vdash \Lambda\alpha.M' : \forall\alpha.A'$ for some α, M', and A' such that $M = \Lambda\alpha.M'$ and $A = \forall\alpha.A'$. By inversion, $\vdash \Gamma_1, \Gamma_3$ and $\omega(\Gamma_1, \Gamma_3), \alpha^{\mathbf{0}} \vdash M' : A'$. Since $\vdash \Gamma_1, \Gamma_2$, we have $\vdash \omega(\Gamma_1, \Gamma_2)$ by Lemma 10. By Lemma 8, $\omega\Gamma_1, \omega\Gamma_3, \alpha^{\mathbf{0}} \vdash M' : A'$ and $\vdash \omega\Gamma_1, \omega\Gamma_2$. Since $dom(\Gamma_2) \cap dom(\Gamma_3) = \emptyset$, we have

$$dom(\omega\Gamma_2) \cap dom(\omega\Gamma_3) = \emptyset$$

by Lemma 9. Thus, by the IH,

$$\omega\Gamma_1, \omega\Gamma_2, \omega\Gamma_3, \alpha^{\mathbf{0}} \vdash M' : A'.$$

By Lemma 8 and (T_TABS), we have the conclusion

$$\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Lambda\alpha.M' : \forall\alpha.A'.$$

Note that we have $\vdash \Gamma_1, \Gamma_2, \Gamma_3$ by the case (1).

□

Lemma 21. *If* $(\Gamma_1 + \Gamma_2) + \Gamma_3$ *is well defined, then so are* $\Gamma_1 + \Gamma_3$ *and* $\Gamma_2 + \Gamma_3$*.*

Proof. By induction on Γ_3.

Case $\Gamma_3 = \emptyset$: Obvious because $\Gamma_1 = \Gamma_2 = \emptyset$.

Case $\Gamma_3 = \Gamma_3', x :^{\pi_3} A$: We can find $\Gamma_1 = \Gamma_1', x :^{\pi_1} A$ and $\Gamma_2 = \Gamma_2', x :^{\pi_2} A$ for some Γ_1', Γ_2', π_1, and π_2, and $(\Gamma_1' + \Gamma_2') + \Gamma_3'$ is well defined. Thus, by the IH, so are $\Gamma_1' + \Gamma_3'$ and $\Gamma_2' + \Gamma_3'$. By Lemma 1 (1), so are $(\Gamma_1', x :^{\pi_1} A) + (\Gamma_3', x :^{\pi_3} A)$ and $(\Gamma_2', x :^{\pi_2} A) + (\Gamma_3', x :^{\pi_3} A)$.

Case $\Gamma_3 = \Gamma_3', \alpha^{\pi_3}$: We can find $\Gamma_1 = \Gamma_1', \alpha^{\pi_1}$ and $\Gamma_2 = \Gamma_2', \alpha^{\pi_2}$ for some Γ_1', Γ_2', π_1, and π_2, and $(\Gamma_1' + \Gamma_2') + \Gamma_3'$ is well defined. Thus, by the IH, so are $\Gamma_1' + \Gamma_3'$ and $\Gamma_2' + \Gamma_3'$. By Lemma 6 (3), so are $(\Gamma_1', \alpha^{\pi_1}) + (\Gamma_3', \alpha^{\pi_3})$ and $(\Gamma_2', \alpha^{\pi_2}) + (\Gamma_3', \alpha^{\pi_3})$.

□

Lemma 22. *If* $\Gamma_1 + \Gamma_2 \leq \Gamma$*, then there exist some* Γ_1' *and* Γ_2' *such that* $\Gamma = \Gamma_1' + \Gamma_2'$ *and* $\Gamma_1 \leq \Gamma_1'$ *and* $\Gamma_2 \leq \Gamma_2'$*.*

Proof. By induction on Γ.

Case $\Gamma = \emptyset$: We finish by letting $\Gamma_1' = \emptyset$ and $\Gamma_2' = \emptyset$.

Case $\Gamma = \Gamma', x :^{\pi} A$: Since $\Gamma_1 + \Gamma_2 \leq \Gamma$, there exist some Γ_{01}, Γ_{02}, π_1, and π_2 such that

- $\Gamma_1 = \Gamma_{01}, x :^{\pi_1} A$,
- $\Gamma_2 = \Gamma_{02}, x :^{\pi_2} A$,
- $\pi_1 + \pi_2 \leq \pi$, and
- $\Gamma_{01} + \Gamma_{02} \leq \Gamma'$.

By the IH, there exist some Γ_{01}' and Γ_{02}' such that

- $\Gamma' = \Gamma_{01}' + \Gamma_{02}'$,
- $\Gamma_{01} \leq \Gamma_{01}'$, and
- $\Gamma_{02} \leq \Gamma_{02}'$.

If we have π_1' and π_2' such that

- $\pi_1 \leq \pi_1'$,
- $\pi_2 \leq \pi_2'$, and
- $\pi = \pi_1' + \pi_2'$,

then we finish by letting $\Gamma_1' = \Gamma_{01}', x :^{\pi_1'} A$ and $\Gamma_2' = \Gamma_{02}', x :^{\pi_2'} A$.
We find such π_1' and π_2' by case analysis on π_1 and π_2.

Case $\pi_1 = \omega$ or $\pi_2 = \omega$: We finish by letting $\pi_1' = \pi_1$ and $\pi_2' = \pi_2$ since $\pi = \omega$.
Case $\pi_1 = \mathbf{0}$: We finish by letting $\pi_1' = \mathbf{0}$ and $\pi_2' = \pi$ since $\pi_2 \leq \pi$.
Case $\pi_2 = \mathbf{0}$: We finish by letting $\pi_1' = \pi$ and $\pi_2' = \mathbf{0}$ since $\pi_1 \leq \pi$.
Case $\pi_1 = \pi_2 = \mathbf{1}$: We finish by letting $\pi_1' = \pi_2' = \mathbf{1}$ since $\pi = \omega$.

Case $\Gamma = \Gamma', \alpha^{\pi}$: Since $\Gamma_1 + \Gamma_2 \leq \Gamma$, $\pi \neq \omega$ and there exist some Γ_{01}, Γ_{02}, π_1, and π_2 such that

- $\Gamma_1 = \Gamma_{01}, \alpha^{\pi_1}$,
- $\Gamma_2 = \Gamma_{02}, \alpha^{\pi_2}$,
- $\pi_1 + \pi_2 \neq \omega$,
- $\pi_1 + \pi_2 \leq \pi$, and

- $\Gamma_{01} + \Gamma_{02} \leq \Gamma'$.

By the IH, there exist some Γ'_{01} and Γ'_{02} such that

- $\Gamma' = \Gamma'_{01} + \Gamma'_{02}$,
- $\Gamma_{01} \leq \Gamma'_{01}$, and
- $\Gamma_{02} \leq \Gamma'_{02}$.

If we have π'_1 and π'_2 such that

- $\pi_1 \leq \pi'_1 \neq \omega$,
- $\pi_2 \leq \pi'_2 \neq \omega$, and
- $\pi = \pi'_1 + \pi'_2$,

then we finish by letting $\Gamma'_1 = \Gamma'_{01}, \alpha^{\pi'_1}$ and $\Gamma'_2 = \Gamma'_{02}, \alpha^{\pi'_2}$.

We find such π'_1 and π'_2 by case analysis on π_1 and π_2.

Case $\pi_1 = \omega$, $\pi_2 = \omega$, or $\pi_1 = \mathbf{1}$ and $\pi_2 = \mathbf{1}$: Contradictory since $\pi_1 + \pi_2 = \omega$ but $\pi_1 + \pi_2 \neq \omega$ by the assumption.

Case $\pi_1 = \mathbf{0}$: We finish by letting $\pi'_1 = \mathbf{0}$ and $\pi'_2 = \pi$ since $\pi_2 \leq \pi$ and $\pi \neq \omega$.

Case $\pi_2 = \mathbf{0}$: We finish by letting $\pi'_1 = \pi$ and $\pi'_2 = \mathbf{0}$ since $\pi_1 \leq \pi$ and $\pi \neq \omega$.

□

Lemma 23. *If $\Gamma_1, \Gamma_2 \leq \Gamma$, then there exist some Γ'_1 and Γ'_2 such that $\Gamma = \Gamma'_1, \Gamma'_2$ and $\Gamma_1 \leq \Gamma'_1$ and $\Gamma_2 \leq \Gamma'_2$.*

Proof. Straightforward by induction on Γ_2. □

Lemma 24. *If $\Gamma_1 \leq \Gamma_2$, then $\omega\Gamma_1 \leq \omega\Gamma_2$.*

Proof. By induction on Γ_1.

Case $\Gamma_1 = \emptyset$: Obvious since $\Gamma_2 = \emptyset$.

Case $\Gamma_1 = \Gamma'_1, x :^{\pi_1} A$: By inversion of $\Gamma_1 \leq \Gamma_2$, there exist some Γ'_2 and π_2 such that

- $\Gamma_2 = \Gamma'_2, x :^{\pi_2} A$,
- $\Gamma'_1 \leq \Gamma'_2$, and
- $\pi_1 \leq \pi_2$.

By the IH, $\omega\Gamma'_1 \leq \omega\Gamma'_2$.

If $\pi_2 = \omega$, then we have $\omega\Gamma_1 \leq \omega\Gamma_2$ since $\omega\Gamma_2 = \omega\Gamma'_2, x :^{\omega} A$ and $\omega\Gamma'_1 \leq \omega\Gamma'_2$.

Otherwise, if $\pi_2 \neq \omega$, then $\pi_1 \neq \omega$ since $\pi_1 \leq \pi_2$. Thus, $\omega\Gamma_1 = \omega\Gamma'_1, x :^{\mathbf{0}} A$ and $\omega\Gamma_2 = \omega\Gamma'_1, x :^{\mathbf{0}} A$, and we have the conclusion.

Case $\Gamma_1 = \Gamma'_1, \alpha^{\pi_1}$: By inversion of $\Gamma_1 \leq \Gamma_2$, there exist some Γ'_2 and π_2 such that $\Gamma_2 = \Gamma'_2, \alpha^{\pi_2}$ and $\Gamma'_1 \leq \Gamma'_2$. By the IH, $\omega\Gamma'_1 \leq \omega\Gamma'_2$. We have $\omega\Gamma_1 = \omega\Gamma'_1, \alpha^{\mathbf{0}}$ and $\omega\Gamma_2 = \omega\Gamma'_2, \alpha^{\mathbf{0}}$, and also have $\omega\Gamma'_1, \alpha^{\mathbf{0}} \leq \omega\Gamma'_2, \alpha^{\mathbf{0}}$. Thus, we have the conclusion.

□

Lemma 25 (Increasing uses)**.** *If $\Gamma_1 \vdash M : B$ and $\Gamma_1 \leq \Gamma_2$, then $\Gamma_2 \vdash M : B$.*

Proof. By induction on the derivation of the typing judgment for M.

Case (T_Var) and (T_Const): By Lemma 15. Note that, for any π_1 and π_2 such that $\pi_1 \leq \pi_2$, if $\pi_1 \neq \mathbf{0}$, $\pi_2 \neq \mathbf{0}$ by Lemma 6 (2).

Case (T_ABS), (T_NU), and (T_TAPP): By the IH. (The cases for (T_NU) and (T_TAPP) use Lemma 14.)

Case (T_APP) and (T_LETBANG): We show the case for (T_APP): the case for (T_LETBANG) can be proven similarly.

We are given $\Gamma_{01} + \Gamma_{02} \vdash M_1\, M_2 : B$ for some Γ_{01}, Γ_{02}, M_1, and M_2 such that $\Gamma_1 = \Gamma_{01} + \Gamma_{02}$ and $M = M_1\, M_2$. By inversion, $\Gamma_{01} \vdash M_1 : C \multimap B$ and $\Gamma_{02} \vdash M_2 : C$ for some C. Since $\Gamma_1 \le \Gamma_2$, we have $\Gamma_{01} + \Gamma_{02} \le \Gamma_2$. By Lemma 22, there exist some Γ'_{01} and Γ'_{02} such that

- $\Gamma_2 = \Gamma'_{01} + \Gamma'_{02}$,
- $\Gamma_{01} \le \Gamma'_{01}$, and
- $\Gamma_{02} \le \Gamma'_{02}$.

By the IHs, $\Gamma'_{01} \vdash M_1 : C \multimap B$ and $\Gamma'_{02} \vdash M_2 : C$. Thus, by (T_APP), we have the conclusion.

Case (T_BANG): We are given $\Gamma_1 \vdash\, !M' : !B'$ for some M' and B' such that $M = !M'$ and $B = !B'$. By inversion, $\vdash \Gamma_1$ and $\omega\Gamma_1 \vdash M' : B'$.

By Lemma 15, $\vdash \Gamma_2$. By Lemma 24, $\omega\Gamma_1 \le \omega\Gamma_2$, so, by the IH, $\omega\Gamma_2 \vdash M' : B'$. Thus, we have the conclusion by (T_BANG).

Case (T_GEN): By Lemma 23, the IH, and the fact that $\mathbf{1} \le \pi$ and $\pi \ne \omega$ imply $\pi = \mathbf{1}$ for any π.

Case (T_TABS): By Lemmas 15 and 24, the IH, and (T_TABS), similarly to the case for (T_BANG).

□

Lemma 26. *If $\Gamma + \Gamma'$ is well defined, then $\Gamma \le \Gamma + \Gamma'$.*

Proof. Straightforward by induction on Γ with the fact that $\pi_1 \le \pi_1 + \pi_2$ for any π_1 and π_2. □

Definition 30. *We write $\pi \,|\, \Gamma$ if and only if $\Gamma = \omega\Gamma$ provided that $\pi = \omega$.*

Lemma 27.

1. *If $\vdash \Gamma_1, x :^{\pi} A, \Gamma_2$, then $\vdash \Gamma_1, \Gamma_2$.*
2. *If $\Gamma_1, x :^{\mathbf{0}} A, \Gamma_2 \vdash M : B$, then $\Gamma_1, \Gamma_2 \vdash M : B$.*

Proof. 1. Straightforward by induction on Γ_2.

2. Straightforward by induction on the typing derivation. The cases for (T_VAR), (T_CONST), (T_BANG), and (T_TABS) rest on case (1). Further, the cases for (T_BANG) and (T_TABS) rest on Lemma 8 as well. The cases for (T_APP) and (T_LETBANG) rest on Lemma 6 (2).

□

Lemma 28. *If $\Gamma_1 + \Gamma_2$ is well defined, so is $\Gamma_1 + \omega\Gamma_2$.*

Proof. Straightforward by induction on Γ_2. The case for $\Gamma_2 = \Gamma'_2, \alpha^{\pi}$ rests on (the contraposition of) Lemma 6 (1). □

Lemma 29. *If $\Gamma_1 = \omega\Gamma_1$, then $\omega(\Gamma_1 + \Gamma_2) = \omega\Gamma_1 + \omega\Gamma_2$.*

Proof. By induction on Γ_1.

Case $\Gamma_1 = \emptyset$: Obvious because $\Gamma_2 = \emptyset$.

Case $\Gamma_1 = \Gamma'_1, x :^{\pi_1} A$: We have $\Gamma'_1 = \omega\Gamma'_1$. There exist some Γ'_2 and π_2 such that $\Gamma_2 = \Gamma'_2, x :^{\pi_2} A$.

$$\begin{array}{rcll} \omega(\Gamma_1 + \Gamma_2) & = & \omega((\Gamma'_1 + \Gamma'_2), x :^{\pi_1+\pi_2} A) & \\ & = & \omega(\Gamma'_1 + \Gamma'_2), x :^{\pi} A & \text{(where } \pi = \omega \text{ if } \pi_1 + \pi_2 = \omega\text{; otherwise, } \pi = \mathbf{0}) \\ & = & (\omega\Gamma'_1 + \omega\Gamma'_2), x :^{\pi} A & \text{(by the IH).} \end{array}$$

If $\pi_1 = \omega$ or $\pi_2 = \omega$, then $\pi_1 + \pi_2 = \omega$, so we finish by:

$$(\omega\Gamma'_1 + \omega\Gamma'_2), x :^{\pi} A \;=\; (\omega\Gamma'_1 + \omega\Gamma'_2), x :^{\omega} A \;=\; \omega(\Gamma'_1, x :^{\pi_1} A) + \omega(\Gamma'_2, x :^{\pi_2} A).$$

Otherwise, if $\pi_1 \neq \omega$ nor $\pi_2 \neq \omega$, then $\pi_1 = \mathbf{0}$ since $\Gamma_1 = \omega\Gamma_1$. Thus, $\pi_1 + \pi_2 = \pi_2 \neq \omega$, and so $\pi = \mathbf{0}$. Thus, we finish by:

$$(\omega\Gamma'_1 + \omega\Gamma'_2), x :^{\pi} A \;=\; (\omega\Gamma'_1 + \omega\Gamma'_2), x :^{\mathbf{0}} A \;=\; \omega(\Gamma'_1, x :^{\pi_1} A) + \omega(\Gamma'_2, x :^{\pi_2} A).$$

Case $\Gamma_1 = \Gamma'_1, \alpha^{\pi_1}$: We have $\Gamma'_1 = \omega\Gamma'_1$. There exist some Γ'_2 and π_2 such that $\Gamma_2 = \Gamma'_2, \alpha^{\pi_2}$. We finish by:

$$\begin{array}{rcll} \omega(\Gamma_1 + \Gamma_2) & = & \omega((\Gamma'_1 + \Gamma'_2), \alpha^{\pi_1+\pi_2}) & \\ & = & \omega(\Gamma'_1 + \Gamma'_2), \alpha^{\mathbf{0}} & \\ & = & (\omega\Gamma'_1 + \omega\Gamma'_2), \alpha^{\mathbf{0}} & \text{(by the IH)} \\ & = & (\omega(\Gamma'_1, \alpha^{\pi_1}) + \omega(\Gamma'_2, \alpha^{\pi_2})). & \end{array}$$

Note that $\pi_1 + \pi_2 \neq \omega$, so $\pi_1 \neq \omega$ and $\pi_2 \neq \omega$ by (the contraposition of) Lemma 6 (1).

□

Lemma 30 (Term substitution). *Suppose that* $\Gamma_{11} + \Gamma_{12}$ *is well defined. If* $\Gamma_{11} \vdash M_1 : A$ *and* $\Gamma_{12}, x :^{\pi} A, \Gamma_2 \vdash M_2 : B$ *and* $\pi \mid \Gamma_{11}$*, then* $(\Gamma_{11} + \Gamma_{12}), \Gamma_2 \vdash M_2[M_1/x] : B$.

Proof. By induction on the typing derivation of $\Gamma_{12}, x :^{\pi} A, \Gamma_2 \vdash M_2 : B$.

We first show

$$\vdash (\Gamma_{11} + \Gamma_{12}), \Gamma_2. \tag{1}$$

By Lemma 17, we have $\vdash \Gamma_{11}$ and $\vdash \Gamma_{12}, x :^{\pi} A, \Gamma_2$. By Lemma 27 (1), $\vdash \Gamma_{12}, \Gamma_2$. By Lemma 19, there exists some Γ'_2 such that $\Gamma_2 + \Gamma'_2 = \Gamma_2$. Since $\Gamma_{11} + \Gamma_{12}$ is well defined, we have $(\Gamma_{11} + \Gamma_{12}), \Gamma_2 = (\Gamma_{11} + \Gamma_{12}), (\Gamma'_2 + \Gamma_2) = (\Gamma_{11}, \Gamma'_2) + (\Gamma_{12}, \Gamma_2)$ by Lemmas 3 and 7. We have $\Gamma_{12}, \Gamma_2 \leq (\Gamma_{11}, \Gamma'_2) + (\Gamma_{12}, \Gamma_2)$ by Lemmas 3 and 26. Thus, $\vdash (\Gamma_{11}, \Gamma'_2) + (\Gamma_{12}, \Gamma_2)$ by Lemma 15, so $\vdash (\Gamma_{11} + \Gamma_{12}), \Gamma_2$ by Lemma 7.

We perform case analysis on the typing rule last applied to derive $\Gamma_{12}, x :^{\pi} A, \Gamma_2 \vdash M_2 : B$.

Case (T_Var): We are given $\Gamma_{12}, x :^{\pi} A, \Gamma_2 \vdash y : B$ for some y such that $M_2 = y$ and $(\Gamma_{12}, x :^{\pi} A, \Gamma_2)(y) = B$.

Suppose that $x \neq y$, i.e., $M_2[M_1/x] = y$. It is easy to find $((\Gamma_{11} + \Gamma_{12}), \Gamma_2)(y) = B$. Thus, by (1) and (T_Var), we have the conclusion

$$(\Gamma_{11} + \Gamma_{12}), \Gamma_2 \vdash y : B.$$

Otherwise, if $x = y$, then we have $M_2[M_1/x] = M_1$ and $B = A$. We have $\Gamma_{11} \leq \Gamma_{11} + \Gamma_{12}$ by Lemma 26, so $\Gamma_{11} + \Gamma_{12} \vdash M_1 : A$ by Lemma 25. By (1) and Lemma 20 (2), we have the conclusion

$$(\Gamma_{11} + \Gamma_{12}), \Gamma_2 \vdash M_1 : A.$$

Case (T_Const): By (1) and (T_Const).

Case (T_Abs), (T_Nu), and (T_TApp): By the IH.

Case (T_App): We are given $\Gamma'_1 + \Gamma'_2 \vdash M'_1\, M'_2 : B$ for some Γ'_1, Γ'_2, M'_1, and M'_2 such that $\Gamma_{12}, x :^{\pi} A, \Gamma_2 = \Gamma'_1 + \Gamma'_2$ and $M_2 = M'_1\, M'_2$. By inversion, $\Gamma'_1 \vdash M'_1 : C \multimap B$ and $\Gamma'_2 \vdash M'_2 : C$ for some C. By Lemma 11,

$$\Gamma_{12}, x :^{\pi} A, \Gamma_2 \;=\; (\Gamma_{121} + \Gamma_{122}), (x :^{\pi_1} A + x :^{\pi_2} A), (\Gamma_{21} + \Gamma_{22})$$

for some Γ_{121}, Γ_{122}, Γ_{21}, Γ_{22}, π_1, and π_2 such that

- $\Gamma_{12} = \Gamma_{121} + \Gamma_{122}$,
- $\pi = \pi_1 + \pi_2$,
- $\Gamma_2 = \Gamma_{21} + \Gamma_{22}$,
- $\Gamma_1' = \Gamma_{121}, x :^{\pi_1} A, \Gamma_{21}$, and
- $\Gamma_2' = \Gamma_{122}, x :^{\pi_2} A, \Gamma_{22}$.

By Lemmas 3 and 21, $\Gamma_{11} + \Gamma_{121}$ and $\Gamma_{11} + \Gamma_{122}$ are well defined since so is $\Gamma_{11} + \Gamma_{12}$.

By case analysis on π, π_1, and π_2.

Case $\pi = \omega$: Since $\pi \mid \Gamma_{11}$, we have $\pi_1 \mid \Gamma_{11}$ and $\pi_2 \mid \Gamma_{11}$. Thus, by the IHs,

$$(\Gamma_{11} + \Gamma_{121}), \Gamma_{21} \vdash M_1'[M_1/x] : C \multimap B \quad \text{and}$$
$$(\Gamma_{11} + \Gamma_{122}), \Gamma_{22} \vdash M_2'[M_1/x] : C.$$

By (T_App) and Lemma 7,

$$((\Gamma_{11} + \Gamma_{121}) + (\Gamma_{11} + \Gamma_{122})), (\Gamma_{21} + \Gamma_{22}) \vdash (M_1' \, M_2')[M_1/x] : B.$$

Since $\pi \mid \Gamma_{11}$ and $\pi = \omega$, we have $\Gamma_{11} = \omega\Gamma_{11}$. Thus, $(\Gamma_{11} + \Gamma_{121}) + (\Gamma_{11} + \Gamma_{122}) = \Gamma_{11} + (\Gamma_{121} + \Gamma_{122})$ by Lemmas 2, 3, and 4. Since $\Gamma_{12} = \Gamma_{121} + \Gamma_{122}$ and $\Gamma_2 = \Gamma_{21} + \Gamma_{22}$, we have the conclusion

$$(\Gamma_{11} + \Gamma_{12}), \Gamma_2 \vdash (M_1' \, M_2')[M_1/x] : B.$$

Case $\pi = \pi_1 = \mathbf{1}$ and $\pi_2 = \mathbf{0}$: We have $\pi_1 \mid \Gamma_{11}$. Thus, by the IH,

$$(\Gamma_{11} + \Gamma_{121}), \Gamma_{21} \vdash M_1'[M_1/x] : C \multimap B.$$

From the inversion of the typing derivation for M, we have $\Gamma_{122}, x :^{\mathbf{0}} A, \Gamma_{22} \vdash M_2' : C$. Thus, $\Gamma_{122}, \Gamma_{22} \vdash M_2' : C$ by Lemma 27 (2), and so

$$\Gamma_{122}, \Gamma_{22} \vdash M_2'[M_1/x] : C$$

because x does not occur free in M_2'. By (T_App) and Lemmas 7 and 2, we have the conclusion

$$(\Gamma_{11} + \Gamma_{12}), \Gamma_2 \vdash (M_1' \, M_2')[M_1/x] : B.$$

Case $\pi = \pi_2 = \mathbf{1}$ and $\pi_1 = \mathbf{0}$: Similarly to the above case.

Case $\pi = \pi_1 = \pi_2 = \mathbf{0}$: Since $\Gamma_{12}, x :^{\mathbf{0}} A, \Gamma_2 \vdash M_1' \, M_2' : B$, we have $\Gamma_{12}, \Gamma_2 \vdash M_1' \, M_2' : B$ by Lemma 27 (2). Since $\Gamma_{12}, \Gamma_2 \le (\Gamma_{11} + \Gamma_{12}), \Gamma_2$ as discussed in the beginning of this proof, we have

$$(\Gamma_{11} + \Gamma_{12}), \Gamma_2 \vdash M_1' \, M_2' : B.$$

by Lemma 25. Since x does not occur free in M_1' nor M_2', we have the conclusion.

Case (T_Bang): We are given $\Gamma_{12}, x :^{\pi} A, \Gamma_2 \vdash !M' : !B'$ for some M' and B' such that $M_2 = !M'$ and $B = !B'$. By inversion, $\vdash \Gamma_{12}, x :^{\pi} A, \Gamma_2$ and $\omega(\Gamma_{12}, x :^{\pi} A, \Gamma_2) \vdash M' : B'$. By Lemma 8,

$$\omega\Gamma_{12}, x :^{\pi'} A, \omega\Gamma_2 \vdash M' : B'$$

for some π' such that: $\pi' = \omega$ if $\pi = \omega$; otherwise, $\pi' = \mathbf{0}$. We perform case analysis on π.

Case $\pi = \omega$: Since $\pi \mid \Gamma_{11}$, we have $\Gamma_{11} = \omega\Gamma_{11}$ and $\pi' \mid \Gamma_{11}$. Since $\Gamma_{11} + \Gamma_{12}$ is well defined, so is $\Gamma_{11} + \omega\Gamma_{12}$ by Lemma 28. Thus, by the IH,

$$(\Gamma_{11} + \omega\Gamma_{12}), \omega\Gamma_2 \vdash M'[M_1/x] : B'.$$

Since $\Gamma_{11} = \omega\Gamma_{11}$, we have

$$\omega((\Gamma_{11} + \Gamma_{12}), \Gamma_2) \vdash M'[M_1/x] : B'.$$

by Lemmas 29 and 8. By (1) and (T_Bang), we have the conclusion

$$(\Gamma_{11} + \Gamma_{12}), \Gamma_2 \vdash !M'[M_1/x] : !B'.$$

Case $\pi \neq \omega$: We have $\pi' = \mathbf{0}$, so

$$\omega\Gamma_{12}, x :^{\mathbf{0}} A, \omega\Gamma_2 \vdash M' : B'.$$

By Lemma 27 (2),

$$\omega\Gamma_{12}, \omega\Gamma_2 \vdash M' : B'.$$

Since $\vdash \Gamma_{12}, x :^{\pi} A, \Gamma_2$, we have $\vdash \Gamma_{12}, \Gamma_2$ by Lemma 27 (1). Thus, by Lemma 8 and (T_Bang),

$$\Gamma_{12}, \Gamma_2 \vdash !M' : !B'.$$

Since $\Gamma_{12}, \Gamma_2 \leq (\Gamma_{11} + \Gamma_{12}), \Gamma_2$, we have

$$(\Gamma_{11} + \Gamma_{12}), \Gamma_2 \vdash !M' : !B.$$

by Lemma 25. Since x does not occur free in M', we have the conclusion.

Case (T_LetBang): Similar to the case for (T_App).

Case (T_Gen): We are given $\Gamma_1', \alpha^{\mathbf{1}}, \Gamma_2' \vdash \Lambda^{\circ}\langle\, \alpha, M' \,\rangle : !\forall\alpha.C$ for some Γ_1', Γ_2', α, M', and C such that $\Gamma_{12}, x :^{\pi} A, \Gamma_2 = \Gamma_1', \alpha^{\mathbf{1}}, \Gamma_2'$ and $M_2 = \Lambda^{\circ}\langle\, \alpha, M' \,\rangle$ and $B = !\forall\alpha.C$. By inversion, $\Gamma_1', \alpha^{\mathbf{0}}, \Gamma_2' \vdash M' : !C$. We perform case analysis on $\Gamma_{12}, x :^{\pi} A, \Gamma_2 = \Gamma_1', \alpha^{\mathbf{1}}, \Gamma_2'$.

Case $\Gamma_{12} = \Gamma_1', \alpha^{\mathbf{1}}, \Gamma_2''$ for some Γ_2'': We can find that $\Gamma_2' = \Gamma_2'', x :^{\pi} A, \Gamma_2$. We have

$$\Gamma_1', \alpha^{\mathbf{0}}, \Gamma_2'', x :^{\pi} A, \Gamma_2 \vdash M' : !C \ .$$

Since $\Gamma_{11} + \Gamma_{12} = \Gamma_{11} + (\Gamma_1', \alpha^{\mathbf{1}}, \Gamma_2'')$ is well defined, we can find $\Gamma_{11} = \Gamma_{11}', \alpha^{\mathbf{0}}, \Gamma_{11}''$ for some Γ_{11}' and Γ_{11}'' such that

$$\Gamma_{11} + \Gamma_{12} \;=\; (\Gamma_{11}', \alpha^{\mathbf{0}}, \Gamma_{11}'') + (\Gamma_1', \alpha^{\mathbf{1}}, \Gamma_2'') \;=\; (\Gamma_{11}' + \Gamma_1'), \alpha^{\mathbf{1}}, (\Gamma_{11}'' + \Gamma_2'') \ . \tag{2}$$

It is found that

$$\Gamma_{11} + (\Gamma_1', \alpha^{\mathbf{0}}, \Gamma_2'') = (\Gamma_{11}', \alpha^{\mathbf{0}}, \Gamma_{11}'') + (\Gamma_1', \alpha^{\mathbf{0}}, \Gamma_2'') = (\Gamma_{11}' + \Gamma_1'), \alpha^{\mathbf{0}}, (\Gamma_{11}'' + \Gamma_2'')$$

is well defined. Thus, by the IH,

$$(\Gamma_{11} + (\Gamma_1', \alpha^{\mathbf{0}}, \Gamma_2'')), \Gamma_2 \vdash M'[M_1/x] : !C \ ,$$

i.e.,

$$(\Gamma_{11}' + \Gamma_1'), \alpha^{\mathbf{0}}, (\Gamma_{11}'' + \Gamma_2''), \Gamma_2 \vdash M'[M_1/x] : !C \ .$$

Thus, by (T_Gen),

$$(\Gamma_{11}' + \Gamma_1'), \alpha^{\mathbf{1}}, (\Gamma_{11}'' + \Gamma_2''), \Gamma_2 \vdash \Lambda^{\circ}\langle\, \alpha, M' \,\rangle[M_1/x] : !\forall\alpha.C \ .$$

By (2), we have the conclusion

$$(\Gamma_{11} + \Gamma_{12}), \Gamma_2 \vdash \Lambda^{\circ}\langle\, \alpha, M' \,\rangle[M_1/x] : !\forall\alpha.C \ .$$

Case $\Gamma_2 = \Gamma_1'', \alpha^{\mathbf{1}}, \Gamma_2'$ for some Γ_1'': We can find that $\Gamma_1' = \Gamma_{12}, x :^{\pi} A, \Gamma_1''$. We have

$$\Gamma_{12}, x :^{\pi} A, \Gamma_1'', \alpha^{\mathbf{0}}, \Gamma_2' \vdash M' : !C \ .$$

Thus, by the IH,

$$(\Gamma_{11} + \Gamma_{12}), \Gamma_1'', \alpha^{\mathbf{0}}, \Gamma_2' \vdash M'[M_1/x] : !C \ .$$

By (T_Gen),

$$(\Gamma_{11} + \Gamma_{12}), \Gamma_1'', \alpha^{\mathbf{1}}, \Gamma_2' \vdash \Lambda^{\circ}\langle\, \alpha, M' \,\rangle[M_1/x] : !\forall\alpha.C \ .$$

Since $\Gamma_2 = \Gamma_1'', \alpha^{\mathbf{1}}, \Gamma_2'$, we have the conclusion.

Case (T_TAbs): Similar to the case for (T_Bang).

□

Lemma 31. $(\Gamma_1[A/\alpha] + \Gamma_2[A/\alpha]) = (\Gamma_1 + \Gamma_2)[A/\alpha]$.

Proof. Straightforward by induction on Γ_1. □

Lemma 32. $(\omega\Gamma)[A/\alpha] = \omega(\Gamma[A/\alpha])$.

Proof. Straightforward by induction on Γ. □

Lemma 33. *If* $\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2 \vdash M : A$*, then* $M[B/\alpha]$ *is well defined for any* B*.*

Proof. Straightforward by induction on the typing derivation. The cases for (T_App) and (T_LetBang) rest on Lemma 6 (2). The cases for (T_Bang) and (T_TAbs) rest on Lemma 8. The case for (T_Gen) rests on the assumption that the use given to α is $\mathbf{0}$. □

Lemma 34 (Type substitution)**.**

1. *If* $\Gamma_1 \vdash A$ *and* $\vdash \Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2$*, then* $\vdash \Gamma_1, \Gamma_2[A/\alpha]$*.*
2. *Suppose that, for any* $\alpha^\pi \in \Gamma_1$*,* $\pi = \mathbf{0}$*. If* $\Gamma_1 \vdash A$ *and* $\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2 \vdash M : B$*, then* $\Gamma_1, \Gamma_2[A/\alpha] \vdash M[A/\alpha] : B[A/\alpha]$*.*

Proof. 1. Straightforward by induction on Γ_2.

2. By induction on the typing derivation of $\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2 \vdash M : B$. Note that $M[A/\alpha]$ is well defined by Lemma 33.

Case (T_Var) and (T_Const): By the case (1).

Case (T_Abs), (T_Nu), and (T_TApp): By the IH.

Case (T_App): We are given $\Gamma_{01} + \Gamma_{02} \vdash M'_1\, M'_2 : B$ for some Γ_{01}, Γ_{02}, M'_1, and M'_2 such that $\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2 = \Gamma_{01} + \Gamma_{02}$ and $M = M'_1\, M'_2$. By inversion, $\Gamma_{01} \vdash M'_1 : C \multimap B$ and $\Gamma_{02} \vdash M'_2 : C$ for some C. By Lemmas 11 and 6 (2), there exist some Γ_{11}, Γ_{12}, Γ_{21}, and Γ_{22} such that

- $\Gamma_1 = \Gamma_{11} + \Gamma_{12}$,
- $\Gamma_2 = \Gamma_{21} + \Gamma_{22}$,
- $\Gamma_{01} = \Gamma_{11}, \alpha^{\mathbf{0}}, \Gamma_{21}$, and
- $\Gamma_{02} = \Gamma_{12}, \alpha^{\mathbf{0}}, \Gamma_{22}$.

Since $\Gamma_1 \vdash A$, we have $\Gamma_{11} \vdash A$ and $\Gamma_{12} \vdash A$ by Lemma 12. We can find that, for any $\alpha^\pi \in \Gamma_{11}$ or $\alpha^\pi \in \Gamma_{12}$, $\pi = \mathbf{0}$ by Lemma 6 (2). Thus, by the IHs, $\Gamma_{11}, \Gamma_{21}[A/\alpha] \vdash M'_1[A/\alpha] : C[A/\alpha] \multimap B[A/\alpha]$ and $\Gamma_{12}, \Gamma_{22}[A/\alpha] \vdash M'_2[A/\alpha] : C[A/\alpha]$. By (T_App) and Lemma 7,

$$(\Gamma_{11} + \Gamma_{12}), (\Gamma_{21}[A/\alpha] + \Gamma_{22}[A/\alpha]) \vdash (M'_1\, M'_2)[A/\alpha] : B[A/\alpha].$$

Since $\Gamma_{11} + \Gamma_{12} = \Gamma_1$ and $(\Gamma_{21}[A/\alpha] + \Gamma_{22}[A/\alpha]) = (\Gamma_{21} + \Gamma_{22})[A/\alpha] = \Gamma_2[A/\alpha]$ by Lemma 31, we have the conclusion.

Case (T_Bang): We are given $\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2 \vdash !M' : !B'$ for some M' and B' such that $M = !M'$ and $B = !B'$. By inversion, $\vdash \Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2$ and $\omega(\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2) \vdash M' : B'$. By the case (1), $\vdash \Gamma_1, \Gamma_2[A/\alpha]$. By Lemma 8, $\omega\Gamma_1, \alpha^{\mathbf{0}}, \omega\Gamma_2 \vdash M' : B'$. Since $\omega\Gamma_1 \vdash A$ by Lemma 9, we have $\omega\Gamma_1, \omega\Gamma_2[A/\alpha] \vdash M'[A/\alpha] : B'[A/\alpha]$ by the IH. By Lemmas 32 and 8, $\omega(\Gamma_1, \Gamma_2[A/\alpha]) \vdash M'[A/\alpha] : B'[A/\alpha]$. By (T_Bang), we have the conclusion

$$\Gamma_1, \Gamma_2[A/\alpha] \vdash !M'[A/\alpha] : !B'[A/\alpha].$$

Case (T_LetBang): Similar to the case for (T_App).

Case (T_Gen): We are given $\Gamma_{01}, \beta^{\mathbf{1}}, \Gamma_{02} \vdash \Lambda^\circ\langle\, \beta, M'\,\rangle : !\forall\beta.C$ for some Γ_{01}, Γ_{02}, β, M', and C such that $\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2 = \Gamma_{01}, \beta^{\mathbf{1}}, \Gamma_{02}$ and $M = \Lambda^\circ\langle\, \beta, M'\,\rangle$ and $B = !\forall\beta.C$. By inversion, $\Gamma_{01}, \beta^{\mathbf{0}}, \Gamma_{02} \vdash M' : !C$.

We perform case analysis on $\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2 = \Gamma_{01}, \beta^{\mathbf{1}}, \Gamma_{02}$.

Case $\Gamma_1 = \Gamma_{01}, \beta^{\mathbf{1}}, \Gamma'_{02}$ for some Γ'_{02}: This is contradictory with the assumption that $\alpha^\pi \in \Gamma_1$ implies $\pi = \mathbf{0}$.

Case $\Gamma_2 = \Gamma'_{01}, \beta^{\mathbf{1}}, \Gamma_{02}$ for some Γ'_{01}: We have $\Gamma_{01} = \Gamma_1, \alpha^{\mathbf{0}}, \Gamma'_{01}$, so $\Gamma_1, \alpha^{\mathbf{0}}, \Gamma'_{01}, \beta^{\mathbf{0}}, \Gamma_{02} \vdash M' : !C$. By the IH,

$$\Gamma_1, \Gamma'_{01}[A/\alpha], \beta^{\mathbf{0}}, \Gamma_{02}[A/\alpha] \vdash M'[A/\alpha] : !C[A/\alpha].$$

Thus, by (T_Gen),

$$\Gamma_1, \Gamma'_{01}[A/\alpha], \beta^{\mathbf{1}}, \Gamma_{02}[A/\alpha] \vdash \Lambda^{\circ}\langle \beta, M' \rangle[A/\alpha] : !\forall\beta.(C[A/\alpha]).$$

Since $\Gamma_1 \vdash A$, β does not occur free in A. Since $\alpha \neq \beta$, we have $!\forall\beta.(C[A/\alpha]) = (!\forall\beta.C)[A/\alpha]$. Thus, we have the conclusion.

Case (T_TAbs): Similar to the case for (T_Bang). We are given $\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2 \vdash \Lambda\beta.M' : \forall\beta.B'$ for some M' and B' such that $M = \Lambda\beta.M'$ and $B = \forall\beta.B'$. By inversion, $\vdash \Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2$ and $\omega(\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2), \beta^{\mathbf{0}} \vdash M' : B'$. By the case (1), $\vdash \Gamma_1, \Gamma_2[A/\alpha]$. By Lemma 8, $\omega\Gamma_1, \alpha^{\mathbf{0}}, \omega\Gamma_2, \beta^{\mathbf{0}} \vdash M' : B'$. Since $\omega\Gamma_1 \vdash A$ by Lemma 9, we have $\omega\Gamma_1, \omega\Gamma_2[A/\alpha], \beta^{\mathbf{0}} \vdash M'[A/\alpha] : B'[A/\alpha]$ by the IH. By Lemmas 32 and 8, $\omega(\Gamma_1, \Gamma_2[A/\alpha]), \beta^{\mathbf{0}} \vdash M'[A/\alpha] : B'[A/\alpha]$. By (T_TAbs), we have

$$\Gamma_1, \Gamma_2[A/\alpha] \vdash \Lambda\beta.M'[A/\alpha] : \forall\beta.B'[A/\alpha] .$$

Since we can assume that $\beta \neq \alpha$ and $\beta \notin ftv(A)$ without loss of generality, we have the conclusion.

□

Lemma 35 (Canonical forms)**.** *Suppose that $\Gamma \vdash V : A$.*

1. *If $A = \iota$, then $V = c$ for some c such that $ty(c) = \iota$.*
2. *If $A = B \multimap C$, then:*
   - *$V = c$ for some c such that $ty(c) = B \multimap C$; or*
   - *$V = \lambda x.M$ for some x and M.*
3. *If $A = \forall\alpha.B$, then $V = \Lambda\alpha.M$ for some M.*
4. *If $A = !B$, then $V = !R$ for some R.*

Proof. Straightforward by case analysis on the typing rule applied last to derive $\Gamma \vdash V : A$. □

Lemma 36 (Progress)**.** *If $\Delta \vdash M : A$, then:*

- *$M = R$ for some R; or*
- *$M \longrightarrow M'$ for some M'.*

Proof. By induction on the typing derivation of $\Delta \vdash M : A$.

Case (T_Var): Contradictory.

Case (T_Const), (T_Abs), and (T_TAbs): M is a value.

Case (T_App): We are given $\Delta_1 + \Delta_2 \vdash M_1\, M_2 : A$ for some Δ_1, Δ_2, M_1, and M_2 such that $\Delta = \Delta_1 + \Delta_2$ and $M = M_1\, M_2$. By inversion, $\Delta_1 \vdash M_1 : B \multimap A$ and $\Delta_2 \vdash M_2 : B$ for some B.

By case analysis on the IHs for M_1 and M_2.

Case $M_1 \longrightarrow M'_1$ for some M'_1: By (E_Eval).

Case $M_1 = R_1$ and $M_2 \longrightarrow M'_2$ for some R_1 and M'_2: By (E_Eval).

Case $M_1 = \nu\alpha.\, R_1$ and $M_2 = R_2$ for some α, R_1, and R_2: By (E_Extr).

Case $M_1 = V_1$ and $M_2 = R_2$ for some V_1 and R_2: Since $\Delta_1 \vdash V_1 : B \multimap A$, $V_1 = \lambda x.M'_1$ for some x and M'_1, or $V_1 = c_1$ for some c_1 such that $ty(c_1) = B \multimap A$ by Lemma 35.

If $V_1 = \lambda x.M$, then we have the conclusion by (R_Beta)/(E_Red).

If $V_1 = c_1$, then, by Assumption 1, $B = \iota$ for some ι. Since $\Delta_2 \vdash R_2 : B$, we have $R_2 = \nu\overline{\alpha}.\, c_2$ for some $\overline{\alpha}$ and c_2 such that $ty(c_2) = \iota$ by Lemma 35. By Assumption 1, $\zeta(c_1, c_2)$ is well defined. Thus, we have the conclusion by (R_Const)/(E_Red).

Case (T_Bang): We are given $\Delta \vdash !M_0 : !B$ for some M_0 and B such that $M = !M_0$ and $A = !B$. By inversion, $\omega\Delta \vdash M_0 : B$. By case analysis on the IH.

Case $M_0 \longrightarrow M_0'$ for some M_0': By (E_Bang).

Case $M_0 = R_0$ for some R_0: We have the conclusion because $M = !R_0$ is a value.

Case (T_LetBang): Similar to the case for (T_App). This case uses (R_Bang) for reducing M.

Case (T_Nu): By the IH and (E_Eval).

Case (T_Gen): We are given $\Delta_1, \alpha^{\mathbf{1}}, \Delta_2 \vdash \Lambda^{\circ}\langle \alpha, M_0 \rangle : !\forall\alpha.B$ for some Δ_1, Δ_2, α, M_0, and B such that $\Delta = \Delta_1, \alpha^{\mathbf{1}}, \Delta_2$ and $M = \Lambda^{\circ}\langle \alpha, M_0 \rangle$ and $A = !\forall\alpha.B$. By inversion, $\Delta_1, \alpha^{\mathbf{0}}, \Delta_2 \vdash M_0 : !B$. By case analysis on the IH.

Case $M_0 \longrightarrow M_0'$ for some M_0': By (E_Eval).

Case $M_0 = \nu\beta.\, R$ for some β and R: By (E_Extr).

Case $M_0 = V$ for some V: By Lemma 35 and (R_Closing)/(E_Red).

Case (T_TApp): We are given $\Delta \vdash M_0\, B : C[B/\alpha]$ for some M_0, B, C, and α such that $M = M_0\, B$ and $A = C[B/\alpha]$. By inversion, $\Delta \vdash M_0 : \forall\alpha.C$ and $\Delta \vdash B$. By case analysis on the IH.

Case $M_0 \longrightarrow M_0'$ for some M_0': By (E_Eval).

Case $M_0 = \nu\beta.\, R_0$ for some β and R_0: By (E_Extr).

Case $M_0 = V$ for some V: By Lemma 35 and (R_TBeta)/(E_Red).

□

Lemma 37. *If* $\vdash \Gamma, \alpha^{\pi}, \Delta$*, then* $\vdash \Gamma, \Delta$.

Proof. Straightforward by induction on Δ. □

Lemma 38. *If* $\Gamma_1 + \Gamma_2$ *is well defined, then* $\omega\Gamma_1 + \Gamma_2 \leq \Gamma_1 + \Gamma_2$.

Proof. Straightforward by induction on Γ_1. □

Lemma 39.

1. *If* $\vdash \Gamma_1, \alpha^{\pi_1}, \beta^{\pi_2}, \Gamma_2$*, then* $\vdash \Gamma_1, \beta^{\pi_2}, \alpha^{\pi_1}, \Gamma_2$.
2. *If* $\Gamma_1, \alpha^{\pi_1}, \beta^{\pi_2}, \Gamma_2 \vdash M : A$*, then* $\Gamma_1, \beta^{\pi_2}, \alpha^{\pi_1}, \Gamma_2 \vdash M : A$.

Proof. 1. Straightforward by induction on Γ_2.

2. Straightforward by induction on the typing derivation. The cases for (T_Var), (T_Const), (T_Bang), and (T_TAbs) rest on case (1). Further, the cases for (T_Bang) and (T_TAbs) rest on Lemma 8 as well. The cases for (T_App) and (T_LetBang) rest on Lemma 6 (2).

□

Lemma 40 (Subject reduction)**.**

1. *If* $\Delta \vdash M_1 : A$ *and* $M_1 \rightsquigarrow M_2$*, then* $\Delta \vdash M_2 : A$.
2. *If* $\Delta \vdash M_1 : A$ *and* $M_1 \longrightarrow M_2$*, then* $\Delta \vdash M_2 : A$.

Proof. 1. By case analysis on the typing rule applied last to derive $\Delta \vdash M_1 : A$.

Case (T_Var): Contradictory.

Case (T_Const), (T_Abs), (T_Bang), (T_Nu), and (T_TAbs): No reduction rule to be applied.

Case (T_APP): We are given $\Delta_1 + \Delta_2 \vdash M'_1\, M'_2 : A$ for some Δ_1, Δ_2, M'_1, and M'_2 such that $\Delta = \Delta_1 + \Delta_2$ and $M_1 = M'_1\, M'_2$. By inversion, $\Delta_1 \vdash M'_1 : B \multimap A$ and $\Delta_2 \vdash M'_2 : B$ for some B. We perform case analysis on the reduction rules applicable to $M_1 = M'_1\, M'_2$.

Case (R_CONST): We have

- $M'_1 = c_1$,
- $M'_2 = \nu\overline{\alpha}.\, c_2$,
- $M_2 = \nu\overline{\alpha}.\, \zeta(c_1, c_2)$ (i.e., the reduction takes the form $c_1\, \nu\overline{\alpha}.\, c_2 \rightsquigarrow \nu\overline{\alpha}.\, \zeta(c_1, c_2)$)

for some c_1, c_2, and $\overline{\alpha}$. By inversion of the judgment $\Delta_1 \vdash c_1 : B \multimap A$, we have $ty(c_1) = B \multimap A$, so $ty(\zeta(c_1, c_2)) = A$ by Assumption 1. Since $\vdash \Delta$ by Lemma 17, we have the conclusion

$$\Delta \vdash \nu\overline{\alpha}.\, \zeta(c_1, c_2) : A$$

by (T_CONST) and (T_NU).

Case (R_BETA): We have

- $M'_1 = \lambda x.M$,
- $M'_2 = R$, and
- $M_2 = M[R/x]$ (i.e., the reduction takes the form $(\lambda x.M)\, R \rightsquigarrow M[R/x]$)

for some x, M, and R. By inversion of the judgment $\Delta_1 \vdash \lambda x.M : B \multimap A$, we can find $\Delta_1, x :^{\mathbf{1}} B \vdash M : A$. By Lemmas 30 and 3, we have the conclusion

$$\Delta_1 + \Delta_2 \vdash M[R/x] : A \ .$$

Case (T_LETBANG): We are given $\Delta_1 + \Delta_2 \vdash \mathsf{let}\,!x = M'_1\,\mathsf{in}\, M'_2 : A$ for some Δ_1, Δ_2, x, M'_1, and M'_2 such that $\Delta = \Delta_1 + \Delta_2$ and $M_1 = \mathsf{let}\,!x = M'_1\,\mathsf{in}\, M'_2$. By inversion, $\Delta_1 \vdash M'_1 : !B$ and $\Delta_2, x :^{\omega} B \vdash M'_2 : A$ for some B. Reduction rules applicable to $M_1 = \mathsf{let}\,!x = M'_1\,\mathsf{in}\, M'_2$ are only (R_BANG). Thus,

- $M'_1 = \nu\overline{\alpha}.\,!R$ and
- $M_2 = M'_2[\nu\overline{\alpha}.\, R/x]$ (i.e., the reduction takes the form $\mathsf{let}\,!x = \nu\overline{\alpha}.\,!R\,\mathsf{in}\, M'_2 \rightsquigarrow M'_2[\nu\overline{\alpha}.\, R/x]$)

for some $\overline{\alpha}$ and R. By inversion of the judgment $\Delta_1 \vdash \nu\overline{\alpha}.\,!R : !B$, we can find that $\overline{\alpha}$ do not occur in B and $\omega(\Delta_1, \Delta') \vdash R : B$ where $\Delta' = {\alpha_1}^{\mathbf{1}}, \cdots, {\alpha_n}^{\mathbf{1}}$ when $\overline{\alpha} = \alpha_1, \cdots \alpha_n$. By Lemma 25 and (T_NU), $\omega\Delta_1 \vdash \nu\overline{\alpha}.\, R : B$. By Lemmas 3 and 28, $\omega\Delta_1 + \Delta_2$ is well defined. By Lemma 5, $\omega \,|\, \omega\Delta_1$. Thus, by Lemma 30,

$$\omega\Delta_1 + \Delta_2 \vdash M'_2[\nu\overline{\alpha}.\, R/x] : A \ .$$

By Lemmas 38 and 25, we have the conclusion

$$\Delta_1 + \Delta_2 \vdash M'_2[\nu\overline{\alpha}.\, R/x] : A \ .$$

Case (T_GEN): We are given $\Delta_1, \alpha^{\mathbf{1}}, \Delta_2 \vdash \Lambda^{\circ}\langle\, \alpha, M' \,\rangle : !\forall\alpha.B$ for some Δ_1, Δ_2, α, M', and B such that $\Delta = \Delta_1, \alpha^{\mathbf{1}}, \Delta_2$ and $M_1 = \Lambda^{\circ}\langle\, \alpha, M' \,\rangle$ and $A = !\forall\alpha.B$. By inversion, $\Delta_1, \alpha^{\mathbf{0}}, \Delta_2 \vdash M' : !B$. Reduction rules applicable to $M_1 = \Lambda^{\circ}\langle\, \alpha, M' \,\rangle$ are only (R_CLOSING). Thus,

- $M' = !R$ and
- $M_2 = !\Lambda\alpha.R$ (i.e., the reduction takes the form $\Lambda^{\circ}\langle\, \alpha, !R \,\rangle \rightsquigarrow !\Lambda\alpha.R$)

for some R. By inversion of $\Delta_1, \alpha^{\mathbf{0}}, \Delta_2 \vdash !R : !B$, we have $\vdash \Delta_1, \alpha^{\mathbf{0}}, \Delta_2$ and $\omega(\Delta_1, \alpha^{\mathbf{0}}, \Delta_2) \vdash R : B$. By Lemmas 8, 39, and 5, $\omega\omega(\Delta_1, \Delta_2), \alpha^{\mathbf{0}} \vdash R : B$. By Lemmas 37 and 10, $\vdash \Delta_1, \Delta_2$ and $\vdash \omega(\Delta_1, \Delta_2)$. By (T_TABS) and (T_BANG), we have $\Delta_1, \Delta_2 \vdash !\Lambda\alpha.R : !\forall\alpha.B$. By Lemma 20 (2), we have the conclusion

$$\Delta_1, \alpha^{\mathbf{1}}, \Delta_2 \vdash !\Lambda\alpha.R : !\forall\alpha.B \ .$$

Case (T_TAPP): We are given $\Delta \vdash M'_1\, B : C[B/\alpha]$ for some M'_1, B, C, and α such that $M_1 = M'_1\, B$ and $A = C[B/\alpha]$. By inversion, $\Delta \vdash M'_1 : \forall\alpha.C$ and $\Delta \vdash B$. Reduction rules applicable to $M_1 = M'_1\, B$ are only (R_TBETA). Thus, without loss of generality, we can suppose

- $M'_1 = \Lambda\alpha.M'$,
- $M_2 = M'[B/\alpha]$ (i.e., the reduction takes the form $(\Lambda\alpha.M')\, B \rightsquigarrow M'[B/\alpha]$)

for some M'.

By inversion of the judgment $\Delta \vdash \Lambda\alpha.M' : \forall\alpha.C$, we have $\omega\Delta, \alpha^{\mathbf{0}} \vdash M' : C$. Since $\Delta \vdash B$, we have $\omega\Delta \vdash B$ by Lemma 9. Thus, by Lemma 34 (2), $\omega\Delta \vdash M'[B/\alpha] : C[B/\alpha]$. Since $\omega\Delta \leq \Delta$, we have the conclusion

$$\Delta \vdash M'[B/\alpha] : C[B/\alpha]$$

by Lemma 25.

2. By induction on the derivation of $M_1 \longrightarrow M_2$ with case analysis on the evaluation rule applied last.

Case (E_RED): By the case (1).

Case (E_EVAL): We are given $E[M'_1] \longrightarrow E[M'_2]$ for some E, M'_1, and M'_2 such that $M_1 = E[M'_1]$ and $M_2 = E[M'_2]$. By inversion, $M'_1 \longrightarrow M'_2$. We perform case analysis on the typing rule applied last to derive $\Delta \vdash M_1 : A$.

Case (T_VAR), (T_CONST), (T_ABS), and (T_TABS): Contradictory because there is no E such that $M_1 = E[M'_1]$.

Case (T_APP): We are given $\Delta_1 + \Delta_2 \vdash M_{11}\, M_{12} : A$ for some Δ_1, Δ_2, M_{11}, and M_{12} such that $\Delta = \Delta_1 + \Delta_2$ and $M_1 = M_{11}\, M_{12}$. By inversion, $\Delta_1 \vdash M_{11} : B \multimap A$ and $\Delta_2 \vdash M_{12} : B$ for some B. We perform case analysis on E.

Case $E = [\,]\, M_{12}$: We are given $M'_1 = M_{11}$. Since $\Delta_1 \vdash M'_1 : B \multimap A$ and $M'_1 \longrightarrow M'_2$, we have $\Delta_1 \vdash M'_2 : B \multimap A$ by the IH. By (T_APP),

$$\Delta_1 + \Delta_2 \vdash M'_2\, M_{12} : A\ .$$

Since $M'_2\, M_{12} = E[M'_2] = M_2$, we have the conclusion.

Case $E = R_{11}\, [\,]$ for some R_{11} such that $R_{11} = M_{11}$: We are given $M'_1 = M_{12}$. Since $\Delta_2 \vdash M'_1 : B$ and $M'_1 \longrightarrow M'_2$, we have $\Delta_2 \vdash M'_2 : B$ by the IH. By (T_APP),

$$\Delta_1 + \Delta_2 \vdash R_{11}\, M'_2 : A\ .$$

Since $R_{11}\, M'_2 = E[M'_2] = M_2$, we have the conclusion.

Case (T_LETBANG), (T_GEN), (T_TAPP), (T_BANG), and (T_NU): Similar to the case for (T_APP).

Case (E_EXTR): We are given $\mathbb{E}[\nu\alpha.\, R] \longrightarrow \nu\alpha.\, \mathbb{E}[R]$ for some $\mathbb{E}$, α, and R such that $M_1 = \mathbb{E}[\nu\alpha.\, R]$ and $M_2 = \nu\alpha.\, \mathbb{E}[R]$ and $\alpha \notin \mathit{ftv}(\mathbb{E})$. We perform case analysis on the typing rule applied last to derive $\Delta \vdash M_1 : A$.

Case (T_VAR), (T_CONST), (T_ABS), (T_BANG), (T_LETBANG), (T_NU), and (T_TABS): Contradictory because there is no $\mathbb{E}$ such that $M_1 = \mathbb{E}[\nu\alpha.\, R]$.

Case (T_APP): We are given $\Delta_1 + \Delta_2 \vdash M_{11}\, M_{12} : A$ for some Δ_1, Δ_2, M_{11}, and M_{12} such that $\Delta = \Delta_1 + \Delta_2$ and $M_1 = M_{11}\, M_{12}$. By inversion, $\Delta_1 \vdash M_{11} : B \multimap A$ and $\Delta_2 \vdash M_{12} : B$ for some B. By case analysis on $\mathbb{E}$, we can find $\mathbb{E} = [\,]\, R_{12}$ for some R_{12} such that $R_{12} = M_{12}$. We are also given $M_{11} = \nu\alpha.\, R$. By inversion of the judgment $\Delta_1 \vdash \nu\alpha.\, R : B \multimap A$, we have $\Delta_1, \alpha^{\mathbf{1}} \vdash R : B \multimap A$ and $\Delta_1 \vdash A$. By Lemmas 17 and 12, $\vdash \Delta_2, \alpha^{\mathbf{0}}$. Thus, by Lemma 20 (2), $\Delta_2, \alpha^{\mathbf{0}} \vdash R_{12} : B$. Thus, by (T_APP) and Lemma 7,

$$(\Delta_1 + \Delta_2), \alpha^{\mathbf{1}} \vdash R\, R_{12} : A.$$

Since $\Delta_1 + \Delta_2 \vdash A$ by Lemma 12, we have the conclusion

$$\Delta_1 + \Delta_2 \vdash \nu\alpha.\, (R\, R_{12}) : A$$

by (T_NU).

Case (T_GEN), and (T_TAPP): Similar to the case for (T_APP).

□

Theorem 1 (Type soundness)**.** *If $\Delta \vdash M : A$ and $M \longrightarrow^* M'$ and $M' \not\longrightarrow$, then $M' = R$ for some R such that $\Delta \vdash R : A$.*

Proof. By induction on the number of the steps of $M \longrightarrow^* M'$.

If the number of the steps is zero, then $M = M'$. We have $\Delta \vdash M : A$ and $M \not\longrightarrow$, so M is a result by Lemma 36.

If the number of the steps is more than zero, we have $M \longrightarrow M''$ and $M'' \longrightarrow^* M'$ for some M''. By Lemma 40, $\Delta \vdash M'' : A$. By the IH, we have the conclusion. □

3.2 Properties of Reductions in $\lambda_v^\forall$

Lemma 41. *If $w_1 \rightrightarrows_{\overline{\aleph}} w_2$, then $e[w_1/x] \rightrightarrows_{\overline{\aleph}} e[w_2/x]$.*

Proof. Straightforward by induction on e. □

Lemma 42. *If $e_1 \rightrightarrows_{\overline{\aleph}} e_2$ and $w_1 \rightrightarrows_{\overline{\aleph}} w_2$, then $e_1[w_1/x] \rightrightarrows_{\overline{\aleph}} e_2[w_2/x]$.*

Proof. By induction on the derivation of $e_1 \rightrightarrows_{\overline{\aleph}} e_2$.

Case (P_Refl): By Lemma 41.

Case (P_Beta): We are given $(\lambda y.e_1'')\, w_1'' \rightrightarrows_{\overline{\aleph}} e_2''[w_2''/y]$ for some y, e_1'', e_2'', w_1'', and w_2'' such that $e_1 = (\lambda y.e_1'')\, w_1''$ and $e_2 = e_2''[w_2''/y]$. By inversion, $e_1'' \rightrightarrows_{\overline{\aleph}} e_2''$ and $w_1'' \rightrightarrows_{\overline{\aleph}} w_2''$ and $\beta_v \in \{\overline{\aleph}\}$. Without loss of generality, we can suppose that $y \neq x$ and $y \notin fv\,(w_1) \cup fv\,(w_2)$.

By the IHs, $e_1''[w_1/x] \rightrightarrows_{\overline{\aleph}} e_2''[w_2/x]$ and $w_1''[w_1/x] \rightrightarrows_{\overline{\aleph}} w_2''[w_2/x]$. Thus, we have the conclusion

$$e_1[w_1/x] = (\lambda y.e_1''[w_1/x])\, w_1''[w_1/x] \rightrightarrows_{\overline{\aleph}} e_2''[w_2/x][w_2''[w_2/x]/y] = e_2[w_2/x]\ .$$

Case (P_Eta): By the IH.

Case (P_Delta): By the IH.

Case (P_Abs): By the IH.

Case (P_App): By the IHs.

□

Lemma 43. *If $e_1 \longrightarrow_F e_2$, then $e_1[w/x] \longrightarrow_F e_2[w/x]$.*

Proof. By induction on the derivation of $e_1 \longrightarrow_F e_2$.

Case $c_1\, c_2 \rightsquigarrow_\delta \zeta(c_1, c_2)$: Obvious.

Case $(\lambda y.e')\, w' \rightsquigarrow_{\beta_v} e'[w'/x]$: We have $e_1 = (\lambda y.e')\, w'$ and $e_2 = e'[w'/y]$. Without loss of generality, we can suppose that $y \neq x$ and $y \notin fv\,(w)$. Then:

$$e_1[w/x] = (\lambda y.e'[w/x])\, w'[w/x] \longrightarrow_F e'[w/x][w'[w/x]/y] = e'[w'/y][w/x] = e_2[w/x]\ .$$

Case $e_1'\, e_2' \longrightarrow_F e_1''\, e_2'$ and $e_1' \longrightarrow_F e_1''$: We have $e_1 = e_1'\, e_2'$ and $e_2 = e_1''\, e_2'$. By the IH, $e_1'[w/x] \longrightarrow_F e_1''[w/x]$. Thus:

$$e_1[w/x] = e_1'[w/x]\, e_2'[w/x] \longrightarrow_F e_1''[w/x]\, e_2'[w/x] = e_2[w/x]\ .$$

Case $w_1'\, e_2' \longrightarrow_F w_1'\, e_2''$ and $e_2' \longrightarrow_F e_2''$: We have $e_1 = w_1'\, e_2'$ and $e_2 = w_1'\, e_2''$. By the IH, $e_2'[w/x] \longrightarrow_F e_2''[w/x]$. Thus:

$$e_1[w/x] = w_1'[w/x]\, e_2'[w/x] \longrightarrow_F w_1'[w/x]\, e_2''[w/x] = e_2[w/x]\ .$$

□

Lemma 44. *If $e_1 \rightrightarrows_{\overline{\aleph_1}} e_2$ and $\{\overline{\aleph_1}\} \subseteq \{\overline{\aleph_2}\}$, then $e_1 \rightrightarrows_{\overline{\aleph_2}} e_2$.*

Proof. Straightforward by induction on the derivation of $e_1 \rightrightarrows_{\aleph_0} e_2$. □

Lemma 45. *If* $e_1 \Longmapsto_{\overline{\aleph}} e_2$*, then* $e_1 \rightrightarrows_{\overline{\aleph}} e_2$*.*

Proof. By Lemma 44, it suffices to show that: for any e_1, e_2, $\mathcal{C}$, and $\aleph_0 \in \{\overline{\aleph}\}$, if $e_1 \rightsquigarrow_{\aleph_0} e_2$, then $\mathcal{C}[e_1] \rightrightarrows_{\aleph_0} \mathcal{C}[e_2]$. We proceed by induction on $\mathcal{C}$.

Case $\mathcal{C} = [\,]$: By case analysis on $\aleph_0$.

Case $\aleph_0 = \beta_v$: We can find $\mathcal{C}[e_1] = (\lambda x.e)\, w$ and $\mathcal{C}[e_2] = e[w/x]$ for some x, e, and w. We have $(\lambda x.e)\, w \rightrightarrows_{\beta_v} e[w/x]$ by (P_Refl) and (P_Beta).

Case $\aleph_0 = \eta_v$: We can find $\mathcal{C}[e_1] = \lambda x.w\, x$ and $\mathcal{C}[e_2] = w$ for some x and w such that $x \notin fv\,(w)$. We have $\lambda x.w\, x \rightrightarrows_{\eta_v} w$ by (P_Refl) and (P_Eta).

Case $\aleph_0 = \delta$: We can find $\mathcal{C}[e_1] = c_1\, c_2$ and $\mathcal{C}[e_2] = \zeta(c_1, c_2)$ for some c_1 and c_2. We have $c_1\, c_2 \rightrightarrows_{\delta} \zeta(c_1, c_2)$ by (P_Refl) and (P_Delta).

Case $\mathcal{C} = \lambda x.\mathcal{C}'$: By the IH and (P_Abs).

Case $\mathcal{C} = e\, \mathcal{C}', \mathcal{C}'\, e$: By the IH, (P_Refl), and (P_App).

□

Lemma 46. *If* $e_1 \Longmapsto^*_{\overline{\aleph}} e_2$*, then* $\mathcal{C}[e_1] \Longmapsto^*_{\overline{\aleph}} \mathcal{C}[e_2]$ *for any* $\mathcal{C}$*.*

Proof. Straightforward by induction on the number of the steps of the reduction $e_1 \Longmapsto^*_{\overline{\aleph}} e_2$. □

Lemma 47. *If* $e_1 \rightrightarrows_{\overline{\aleph}} e_2$*, then* $e_1 \Longmapsto^*_{\overline{\aleph}} e_2$*.*

Proof. By induction on the derivation of $e_1 \rightrightarrows_{\overline{\aleph}} e_2$.

Case (P_Refl): Obvious.

Case (P_Beta): We are given $(\lambda x.e'_1)\, w'_1 \rightrightarrows_{\overline{\aleph}} e'_2[w'_2/x]$ for some x, e'_1, e'_2, w'_1, and w'_2 such that $e_1 = (\lambda x.e'_1)\, w'_1$ and $e_2 = e'_2[w'_2/x]$. By inversion, $e'_1 \rightrightarrows_{\overline{\aleph}} e'_2$ and $w'_1 \rightrightarrows_{\overline{\aleph}} w'_2$ and $\beta_v \in \{\overline{\aleph}\}$. By the IHs, $e'_1 \Longmapsto^*_{\overline{\aleph}} e'_2$ and $w'_1 \Longmapsto^*_{\overline{\aleph}} w'_2$. Thus:

$$e_1 = (\lambda x.e'_1)\, w'_1 \Longmapsto^*_{\overline{\aleph}} (\lambda x.e'_2)\, w'_1 \Longmapsto^*_{\overline{\aleph}} (\lambda x.e'_2)\, w'_2 \Longmapsto_{\overline{\aleph}} e'_2[w'_2/x] = e_2$$

by Lemma 46 and $\beta_v \in \{\overline{\aleph}\}$.

Case (P_Eta): We are given $\lambda x.w_1\, x \rightrightarrows_{\overline{\aleph}} w_2$ for some x, w_1, and w_2 such that $e_1 = \lambda x.w_1\, x$ and $e_2 = w_2$. By inversion, $w_1 \rightrightarrows_{\overline{\aleph}} w_2$ and $x \notin fv\,(w_1)$ and $\eta_v \in \{\overline{\aleph}\}$. By the IH, $w_1 \Longmapsto^*_{\overline{\aleph}} w_2$. Thus:

$$e_1 = \lambda x.w_1\, x \Longmapsto_{\overline{\aleph}} w_1 \Longmapsto^*_{\overline{\aleph}} w_2 = e_2$$

by $\eta_v \in \{\overline{\aleph}\}$.

Case (P_Delta): We are given $c_1\, c_2 \rightrightarrows_{\overline{\aleph}} \zeta(c_1, c_2)$ for some c_1 and c_2 such that $e_1 = c_1\, c_2$ and $e_2 = \zeta(c_1, c_2)$. By inversion, $\delta \in \{\overline{\aleph}\}$. Thus, we have the conclusion by δ-reduction.

Case (P_Abs): By the IH and Lemma 46.

Case (P_App): By the IHs and Lemma 46.

□

Lemma 48. *If* $w \rightrightarrows_{\overline{\aleph}} e$*, then* e *is a value.*

Proof. Straightforward by case analysis on the derivation of $w \rightrightarrows_{\overline{\aleph}} e$. □

Lemma 49. *Suppose that* e_1 *or* e_2 *is not a value. If* $e_1\, e_2 \rightrightarrows_{\overline{\aleph}} e$*, then there exist some* e'_1 *and* e'_2 *such that* $e = e'_1\, e'_2$ *and* $e_1 \rightrightarrows_{\overline{\aleph}} e'_1$ *and* $e_2 \rightrightarrows_{\overline{\aleph}} e'_2$*.*

Proof. Straightforward by case analysis on the derivation of $e_1\, e_2 \rightrightarrows_{\overline{\aleph}} e$. □

Lemma 50. *If $c \rightrightarrows_{\overline{\aleph}} e$, then $e = c$.*

Proof. Straightforward by case analysis on the derivation of $c \rightrightarrows_{\overline{\aleph}} e$. □

Lemma 51. *If $e_1 \rightrightarrows_{\overline{\aleph}} e_2$ and $e_1 \longrightarrow_F e_1'$, then there exists some e_2' such that $e_2 \longrightarrow_F^* e_2'$ and $e_1' \rightrightarrows_{\overline{\aleph}} e_2'$.*

Proof. By induction on the derivation of $e_1 \longrightarrow_F e_1'$ with case analysis on that derivation.

Case $(\lambda x.e)\, w \rightsquigarrow_{\beta_v} e[w/x]$: We have $e_1 = (\lambda x.e)\, w$ and $e_1' = e[w/x]$. We perform case analysis on $(\lambda x.e)\, w = e_1 \rightrightarrows_{\overline{\aleph}} e_2$.

Case (P_Refl) and (P_App): With Lemma 48, we have $e_2 = w_{21}\, w_{22}$ for some w_{21} and w_{22} such that $\lambda x.e \rightrightarrows_{\overline{\aleph}} w_{21}$ and $w \rightrightarrows_{\overline{\aleph}} w_{22}$. By case analysis on $\lambda x.e \rightrightarrows_{\overline{\aleph}} w_{21}$.

Case (P_Refl) and (P_Abs): We have $w_{21} = \lambda x.e_{21}$ for some e_{21} such that $e \rightrightarrows_{\overline{\aleph}} e_{21}$. We have the conclusion by letting $e_2' = e_{21}[w_{22}/x]$ because: $e_2 = w_{21}\, w_{22} = (\lambda x.e_{21})\, w_{22} \longrightarrow_F e_{21}[w_{22}/x] = e_2'$; and $e_1' = e[w/x] \rightrightarrows_{\overline{\aleph}} e_{21}[w_{22}/x] = e_2'$ by Lemma 42.

Case (P_Eta): We have $e = w_{11}\, x$ for some w_{11} such that $w_{11} \rightrightarrows_{\overline{\aleph}} w_{21}$ and $x \notin fv(w_{11})$. (We also have $\eta_v \in \{\overline{\aleph}\}$.) We have the conclusion by letting $e_2' = w_{21}\, w_{22}$ because: $e_2 = w_{21}\, w_{22} = e_2'$; and $e_1' = e[w/x] = w_{11}\, w \rightrightarrows_{\overline{\aleph}} w_{21}\, w_{22} = e_2'$ by (P_App).

Case (P_Beta), (P_Delta), and (P_App): Contradictory.

Case (P_Beta): We have $e_2 = e'[w'/x]$ for some e' and w' such that $e \rightrightarrows_{\overline{\aleph}} e'$ and $w \rightrightarrows_{\overline{\aleph}} w'$. (We also have $\beta_v \in \{\overline{\aleph}\}$). We have the conclusion by letting $e_2' = e'[w'/x]$ because: $e_2 = e'[w'/x] = e_2'$; and $e_1' = e[w/x] \rightrightarrows_{\overline{\aleph}} e'[w'/x] = e_2'$ by Lemma 42.

Case (P_Eta), (P_Delta), and (P_Abs): Contradictory.

Case $c_1\, c_2 \rightsquigarrow_\delta \zeta(c_1, c_2)$: We have $e_1 = c_1\, c_2$ and $e_1' = \zeta(c_1, c_2)$. By case analysis on $c_1\, c_2 = e_1 \rightrightarrows_{\overline{\aleph}} e_2$.

Case (P_Refl) and (P_App): We can find $e_2 = e_{21}\, e_{22}$ for some e_{21} and e_{22} such that $c_1 \rightrightarrows_{\overline{\aleph}} e_{21}$ and $c_2 \rightrightarrows_{\overline{\aleph}} e_{22}$. By Lemma 50, $e_{21} = c_1$ and $e_{22} = c_2$. We have the conclusion by letting $e_2' = \zeta(c_1, c_2)$ because: $e_2 = e_{21}\, e_{22} = c_1\, c_2 \longrightarrow_F \zeta(c_1, c_2) = e_2'$; and $e_1' = \zeta(c_1, c_2) \rightrightarrows_{\overline{\aleph}} \zeta(c_1, c_2) = e_2'$ by (P_Refl).

Case (P_Delta): We are given $e_2 = \zeta(c_1, c_2)$. (We also have $\delta \in \{\overline{\aleph}\}$). We have the conclusion by letting $e_2' = \zeta(c_1, c_2)$ because: $e_2 = \zeta(c_1, c_2) = e_2'$; and $e_1' = \zeta(c_1, c_2) \rightrightarrows_{\overline{\aleph}} \zeta(c_1, c_2) = e_2'$ by (P_Refl).

Case (P_Beta), (P_Eta), and (P_Abs): Contradictory.

Case $e_{11}\, e_{12} \longrightarrow_F e_{11}'\, e_{12}$ and $e_{11} \longrightarrow_F e_{11}'$: We have $e_1 = e_{11}\, e_{12}$ and $e_1' = e_{11}'\, e_{12}$. Since $e_1 = e_{11}\, e_{12} \rightrightarrows_{\overline{\aleph}} e_2$, there exist some e_{21} and e_{22} such that $e_2 = e_{21}\, e_{22}$ and $e_{11} \rightrightarrows_{\overline{\aleph}} e_{21}$ and $e_{12} \rightrightarrows_{\overline{\aleph}} e_{22}$ by Lemma 49. By the IH, there exists some e_{21}' such that $e_{21} \longrightarrow_F^* e_{21}'$ and $e_{11}' \rightrightarrows_{\overline{\aleph}} e_{21}'$. We have the conclusion by letting $e_2' = e_{21}'\, e_{22}$ because: $e_2 = e_{21}\, e_{22} \longrightarrow_F^* e_{21}'\, e_{22} = e_2'$; and $e_1' = e_{11}'\, e_{12} \rightrightarrows_{\overline{\aleph}} e_{21}'\, e_{22} = e_2'$ by (P_App).

Case $w_{11}\, e_{12} \longrightarrow_F w_{11}\, e_{12}'$ and $e_{12} \longrightarrow_F e_{12}'$: We have $e_1 = w_{11}\, e_{12}$ and $e_1' = w_{11}\, e_{12}'$. Since $e_1 = w_{11}\, e_{12} \rightrightarrows_{\overline{\aleph}} e_2$, there exist some w_{21} and e_{22} such that $e_2 = w_{21}\, e_{22}$ and $w_{11} \rightrightarrows_{\overline{\aleph}} w_{21}$ and $e_{12} \rightrightarrows_{\overline{\aleph}} e_{22}$ by Lemmas 49 and 48. By the IH, there exists some e_{22}' such that $e_{22} \longrightarrow_F^* e_{22}'$ and $e_{12}' \rightrightarrows_{\overline{\aleph}} e_{22}'$. We have the conclusion by letting $e_2' = w_{21}\, e_{22}'$ because: $e_2 = w_{21}\, e_{22} \longrightarrow_F^* w_{21}\, e_{22}' = e_2'$; and $e_1' = w_{11}\, e_{12}' \rightrightarrows_{\overline{\aleph}} w_{21}\, e_{22}' = e_2'$ by (P_App).

□

Lemma 52. *If $e_1 \rightrightarrows_{\overline{\aleph}} e_2$ and $e_1 \longrightarrow_F^* e_1'$, then there exists some e_2' such that $e_2 \longrightarrow_F^* e_2'$ and $e_1' \rightrightarrows_{\overline{\aleph}} e_2'$.*

Proof. By induction on the number of the steps of $e_1 \longrightarrow_F^* e_1'$.

If the number of the steps is zero, then $e_1 = e_1'$, so we have the conclusion by letting $e_2' = e_2$.

If the number of the steps is more than zero, there exists some e_1'' such that $e_1 \longrightarrow_F e_1'' \longrightarrow_F^* e_1'$. By Lemma 51, there exists some e_2'' such that $e_2 \longrightarrow_F^* e_2''$ and $e_1'' \rightrightarrows_{\overline{\aleph}} e_2''$. By the IH, there exists some e_2' such that $e_2'' \longrightarrow_F^* e_2'$ and $e_1' \rightrightarrows_{\overline{\aleph}} e_2'$. Since $e_2 \longrightarrow_F^* e_2'' \longrightarrow_F^* e_2'$, we have the conclusion. □

Lemma 53. *If $e_1 \Longmapsto_{\overline{\aleph}}^* e_2$ and $e_1 \longrightarrow_F^* e_1'$, then there exists some e_2' such that $e_2 \longrightarrow_F^* e_2'$ and $e_1' \Longmapsto_{\overline{\aleph}}^* e_2'$.*

Proof. By induction on the number of the steps of $e_1 \Longmapsto^*_{\overline{\aleph}} e_2$.

If the number of the steps is zero, then $e_1 = e_2$, so we have the conclusion by letting $e'_2 = e'_1$.

If the number of the steps is more than zero, then there exists some e such that $e_1 \Longmapsto_{\overline{\aleph}} e \Longmapsto^*_{\overline{\aleph}} e_2$. By Lemmas 45 and 52, there exists some e' such that $e \longrightarrow^*_F e'$ and $e'_1 \rightrightarrows_{\overline{\aleph}} e'$. By the IH, there exists some e'_2 such that $e_2 \longrightarrow^*_F e'_2$ and $e' \Longmapsto^*_{\overline{\aleph}} e'_2$. Since $e'_1 \rightrightarrows_{\overline{\aleph}} e'$, we have $e'_1 \Longmapsto^*_{\overline{\aleph}} e'$ by Lemma 47. Thus, we have the conclusion because $e'_1 \Longmapsto^*_{\overline{\aleph}} e' \Longmapsto^*_{\overline{\aleph}} e'_2$. □

Lemma 54. *If $w \Longmapsto^*_{\overline{\aleph}} e$, then e is a value.*

Proof. By induction on the number of the steps of $w \Longmapsto^*_{\overline{\aleph}} e$.

If the number of the steps is zero, then $w = e$, so we have the conclusion.

Otherwise, if the number of the steps is more than zero, then there exists some e' such that $w \Longmapsto_{\overline{\aleph}} e' \Longmapsto^*_{\overline{\aleph}} e$. By Lemmas 45 and 48, e' is a value. Thus, by the IH, e is a value. □

Lemma 55. *If $e \rightrightarrows_{\overline{\aleph}} w$, then there exists some w' such that $e \longrightarrow^*_F w'$ and $w' \rightrightarrows_{\overline{\aleph}} w$.*

Proof. By induction on the derivation of $e \rightrightarrows_{\overline{\aleph}} w$.

Case (P_Refl): We are given $e = w$. We have the conclusion by letting $w' = w$ because $e = w \longrightarrow^*_F w = w'$; and $w' = w \rightrightarrows_{\overline{\aleph}} w$ by (P_Refl).

Case (P_Beta): We are given $(\lambda x.e_1)\, w_1 \rightrightarrows_{\overline{\aleph}} e_2[w_2/x]$ for some x, e_1, e_2, w_1, and w_2 such that $e = (\lambda x.e_1)\, w_1$ and $w = e_2[w_2/x]$. By inversion, $e_1 \rightrightarrows_{\overline{\aleph}} e_2$ and $w_1 \rightrightarrows_{\overline{\aleph}} w_2$ and $\beta_v \in \{\overline{\aleph}\}$. Since $e_2[w_2/x] = w$ is a value, e_2 is also a value. By the IH, there exists some w'_1 such that $e_1 \longrightarrow^*_F w'_1$ and $w'_1 \rightrightarrows_{\overline{\aleph}} e_2$.

We have the conclusion by letting $w' = w'_1[w_1/x]$ because: $e = (\lambda x.e_1)\, w_1 \longrightarrow_F e_1[w_1/x] \longrightarrow^*_F w'_1[w_1/x] = w'$ by Lemma 43; and $w' = w'_1[w_1/x] \rightrightarrows_{\overline{\aleph}} e_2[w_2/x] = w$ by Lemma 42.

Case (P_Eta): We are given $\lambda x.w_1\, x \rightrightarrows_{\overline{\aleph}} w$ for some x and w_1 such that $e = \lambda x.w_1\, x$. We have the conclusion by letting $w' = \lambda x.w_1\, x$.

Case (P_Delta): We are given $c_1\, c_2 \rightrightarrows_{\overline{\aleph}} \zeta(c_1, c_2)$ for some c_1 and c_2 such that $e = c_1\, c_2$ and $w = \zeta(c_1, c_2)$. We have the conclusion by letting $w' = \zeta(c_1, c_2)$.

Case (P_Abs): We are given $\lambda x.e_1 \rightrightarrows_{\overline{\aleph}} \lambda x.e_2$ for some x, e_1, and e_2 such that $e = \lambda x.e_1$ and $w = \lambda x.e_2$. We have the conclusion by letting $w' = \lambda x.e_1$.

Case (P_App): Contradictory.

□

Lemma 56. *If $e \rightrightarrows_{\overline{\aleph}} e_1\, e_2$, then there exist some e'_1 and e'_2 such that $e \longrightarrow^*_F e'_1\, e'_2$ and $e'_1 \rightrightarrows_{\overline{\aleph}} e_1$ and $e'_2 \rightrightarrows_{\overline{\aleph}} e_2$.*

Proof. By induction on $e \rightrightarrows_{\overline{\aleph}} e_1\, e_2$.

Case (P_Refl): We are given $e = e_1\, e_2$. Obvious by letting $e'_1 = e_1$ and $e'_2 = e_2$.

Case (P_Beta): We are given $(\lambda x.e''_1)\, w''_1 \rightrightarrows_{\overline{\aleph}} e''_2[w''_2/x]$ for some x, e''_1, e''_2, w''_1, and w''_2 such that $e = (\lambda x.e''_1)\, w''_1$ and $e_1\, e_2 = e''_2[w''_2/x]$. By inversion, $e''_1 \rightrightarrows_{\overline{\aleph}} e''_2$ and $w''_1 \rightrightarrows_{\overline{\aleph}} w''_2$ and $\beta_v \in \{\overline{\aleph}\}$. We can see $e''_2 = e''_{21}\, e''_{22}$ for some e''_{21} and e''_{22} such that $e_1 = e''_{21}[w''_2/x]$ and $e_2 = e''_{22}[w''_2/x]$. Since $e''_1 \rightrightarrows_{\overline{\aleph}} e''_2 = e''_{21}\, e''_{22}$, there exist some e''_{11} and e''_{12} such that $e''_1 \longrightarrow^*_F e''_{11}\, e''_{12}$ and $e''_{11} \rightrightarrows_{\overline{\aleph}} e''_{21}$ and $e''_{12} \rightrightarrows_{\overline{\aleph}} e''_{22}$ by the IH. By Lemma 43, $e''_1[w''_1/x] \longrightarrow^*_F (e''_{11}\, e''_{12})[w''_1/x]$. We have the conclusion by letting $e'_1 = e''_{11}[w''_1/x]$ and $e'_2 = e''_{12}[w''_1/x]$ because: $e = (\lambda x.e''_1)\, w''_1 \longrightarrow_F e''_1[w''_1/x] \longrightarrow^*_F (e''_{11}\, e''_{12})[w''_1/x] = e'_1\, e'_2$; $e'_1 = e''_{11}[w''_1/x] \rightrightarrows_{\overline{\aleph}} e''_{21}[w''_2/x] = e_1$ by Lemma 42; and $e'_2 = e''_{12}[w''_1/x] \rightrightarrows_{\overline{\aleph}} e''_{22}[w''_2/x] = e_2$ by Lemma 42.

Case (P_Eta), (P_Delta), and (P_Abs): Contradictory.

Case (P_App): Obvious.

□

Lemma 57. *If $w \rightrightarrows_{\overline{\aleph}} \lambda x.e$, then, for any w', $w\,w' \longrightarrow_F^* (\lambda x.e')\,w'$ for some e' such that $e' \rightrightarrows_{\overline{\aleph}} e$.*

Proof. By induction on the derivation of $w \rightrightarrows_{\overline{\aleph}} \lambda x.e$.

Case (P_Refl): Obvious by letting $e' = e$.

Case (P_Beta), (P_Delta), and (P_App): Contradictory.

Case (P_Eta): We are given $\lambda y.w''\,y \rightrightarrows_{\overline{\aleph}} \lambda x.e$ for some y and w'' such that $w = \lambda y.w''\,y$. By inversion, $w'' \rightrightarrows_{\overline{\aleph}} \lambda x.e$ and $y \notin fv\,(w'')$. By the IH, there exists some e' such that $w''\,w' \longrightarrow_F^* (\lambda x.e')\,w'$ and $e' \rightrightarrows_{\overline{\aleph}} e$. We have the conclusion because: $w\,w' = (\lambda y.w''\,y)\,w' \longrightarrow_F w''\,w' \longrightarrow_F^* (\lambda x.e')\,w'$.

Case (P_Abs): Obvious.

□

Lemma 58. *Let $w_2 = c$ or x. If $w_1 \rightrightarrows_{\overline{\aleph}} w_2$, then $w_1\,w \longrightarrow_F^* w_2\,w$ for any w.*

Proof. By induction on the derivation of $w_1 \rightrightarrows_{\overline{\aleph}} w_2$.

Case (P_Refl): Obvious.

Case (P_Beta), (P_Delta), (P_Abs), and (P_App): Contradictory.

Case (P_Eta): We are given $\lambda y.w_1'\,y \rightrightarrows_{\overline{\aleph}} w_2$ for some y and w_1' such that $w_1 = \lambda y.w_1'\,y$. By inversion, $w_1' \rightrightarrows_{\overline{\aleph}} w_2$ and $y \notin fv\,(w_1')$. By the IH, $w_1'\,w \longrightarrow_F^* w_2\,w$. We have the conclusion because: $w_1\,w = (\lambda y.w_1'\,y)\,w \longrightarrow_F w_1'\,w \longrightarrow_F^* w_2\,w$.

□

Lemma 59. *Suppose that e_1 does not get stuck. If $e_1 \rightrightarrows_{\overline{\aleph}} e_2$ and $e_2 \longrightarrow_F e_2'$, then there exists some e_1' such that $e_1 \longrightarrow_F^* e_1'$ and $e_1' \rightrightarrows_{\overline{\aleph}} e_2'$.*

Proof. By induction on the derivation of $e_2 \longrightarrow_F e_2'$ with case analysis on that derivation.

Case $(\lambda x.e)\,w \rightsquigarrow_{\beta_v} e[w/x]$: We have $e_2 = (\lambda x.e)\,w$ and $e_2' = e[w/x]$. Since $e_1 \rightrightarrows_{\overline{\aleph}} e_2 = (\lambda x.e)\,w$, there exist some w_{11} and w_{12} such that $e_1 \longrightarrow_F^* w_{11}\,w_{12}$ and $w_{11} \rightrightarrows_{\overline{\aleph}} \lambda x.e$ and $w_{12} \rightrightarrows_{\overline{\aleph}} w$ by Lemmas 56 and 55. By Lemma 57, there exists some e_{11} such that $w_{11}\,w_{12} \longrightarrow_F^* (\lambda x.e_{11})\,w_{12}$ and $e_{11} \rightrightarrows_{\overline{\aleph}} e$. Thus, we have the conclusion by letting $e_1' = e_{11}[w_{12}/x]$ because: $e_1 \longrightarrow_F^* w_{11}\,w_{12} \longrightarrow_F^* (\lambda x.e_{11})\,w_{12} \longrightarrow_F e_{11}[w_{12}/x] = e_1'$; and $e_1' = e_{11}[w_{12}/x] \rightrightarrows_{\overline{\aleph}} e[w/x] = e_2'$ by Lemma 42.

Case $c_1\,c_2 \rightsquigarrow_\delta \zeta(c_1, c_2)$: We have $e_2 = c_1\,c_2$ and $e_2' = \zeta(c_1, c_2)$. Since $e_1 \rightrightarrows_{\overline{\aleph}} e_2 = c_1\,c_2$, there exist some w_{11} and w_{12} such that $e_1 \longrightarrow_F^* w_{11}\,w_{12}$ and $w_{11} \rightrightarrows_{\overline{\aleph}} c_1$ and $w_{12} \rightrightarrows_{\overline{\aleph}} c_2$ by Lemmas 56 and 55. By Lemma 58, $w_{11}\,w_{12} \longrightarrow_F^* c_1\,w_{12}$. Since e_1 does not get stuck and $e_1 \longrightarrow_F^* c_1\,w_{12}$, we have $w_{12} = c_2'$ for some c_2' such that $\zeta(c_1, c_2')$ is well defined. Since $c_2' = w_{12} \rightrightarrows_{\overline{\aleph}} c_2$, we can see $c_2' = c_2$. Thus, we have the conclusion by letting $e_1' = \zeta(c_1, c_2)$ because: $e_1 \longrightarrow_F^* c_1\,w_{12} = c_1\,c_2 \longrightarrow_F \zeta(c_1, c_2) = e_1'$; and $e_1' = \zeta(c_1, c_2) \rightrightarrows_{\overline{\aleph}} \zeta(c_1, c_2) = e_2'$ by (P_Refl).

Case $e_{21}\,e_{22} \longrightarrow_F e_{21}'\,e_{22}$ and $e_{21} \longrightarrow_F e_{21}'$: We have $e_2 = e_{21}\,e_{22}$ and $e_2' = e_{21}'\,e_{22}$. Since $e_1 \rightrightarrows_{\overline{\aleph}} e_2 = e_{21}\,e_{22}$, there exist some e_{11} and e_{12} such that $e_1 \longrightarrow_F^* e_{11}\,e_{12}$ and $e_{11} \rightrightarrows_{\overline{\aleph}} e_{21}$ and $e_{12} \rightrightarrows_{\overline{\aleph}} e_{22}$ by Lemma 56. By the IH, there exists some e_{11}' such that $e_{11} \longrightarrow_F^* e_{11}'$ and $e_{11}' \rightrightarrows_{\overline{\aleph}} e_{21}'$. We have the conclusion by letting $e_1' = e_{11}'\,e_{12}$ because: $e_1 \longrightarrow_F^* e_{11}\,e_{12} \longrightarrow_F^* e_{11}'\,e_{12} = e_1'$; and $e_1' = e_{11}'\,e_{12} \rightrightarrows_{\overline{\aleph}} e_{21}'\,e_{22} = e_2'$ by (P_App).

Case $w_{21}\,e_{22} \longrightarrow_F w_{21}\,e_{22}'$ and $e_{22} \longrightarrow_F e_{22}'$: We have $e_2 = w_{21}\,e_{22}$ and $e_2' = w_{21}\,e_{22}'$. Since $e_1 \rightrightarrows_{\overline{\aleph}} e_2 = w_{21}\,e_{22}$, there exist some w_{11} and e_{12} such that $e_1 \longrightarrow_F^* w_{11}\,e_{12}$ and $w_{11} \rightrightarrows_{\overline{\aleph}} w_{21}$ and $e_{12} \rightrightarrows_{\overline{\aleph}} e_{22}$ by Lemmas 56 and 55. By the IH, there exists some e_{12}' such that $e_{12} \longrightarrow_F^* e_{12}'$ and $e_{12}' \rightrightarrows_{\overline{\aleph}} e_{22}'$. We have the conclusion by letting $e_1' = w_{11}\,e_{12}'$ because: $e_1 \longrightarrow_F^* w_{11}\,e_{12} \longrightarrow_F^* w_{11}\,e_{12}' = e_1'$; and $e_1' = w_{11}\,e_{12}' \rightrightarrows_{\overline{\aleph}} w_{21}\,e_{22}' = e_2'$ by (P_App).

□

Lemma 60. *Suppose that e_1 does not get stuck. If $e_1 \rightrightarrows_{\overline{\aleph}} e_2$ and $e_2 \longrightarrow_F^* e_2'$, then there exists some e_1' such that $e_1 \longrightarrow_F^* e_1'$ and $e_1' \rightrightarrows_{\overline{\aleph}} e_2'$.*

Proof. By induction on the number of the steps of $e_2 \longrightarrow_F^* e_2'$.

If the number of the steps is zero, then $e_2 = e_2'$, so we have the conclusion by letting $e_1' = e_1$.

If the number of the steps is more than zero, there exists some e_2'' such that $e_2 \longrightarrow_F e_2'' \longrightarrow_F^* e_2'$. By Lemma 59, there exists some e_1'' such that $e_1 \longrightarrow_F^* e_1''$ and $e_1'' \rightrightarrows_{\overline{\aleph}} e_2''$. Since e_1 does not get suck, e_1'' does not either. Thus, by the IH, there exists some e_1' such that $e_1'' \longrightarrow_F^* e_1'$ and $e_1' \rightrightarrows_{\overline{\aleph}} e_2'$. Since $e_1 \longrightarrow_F^* e_1'' \longrightarrow_F^* e_1'$, we have the conclusion. □

Lemma 61. *If e_1 does not get stuck and $e_1 \rightrightarrows_{\overline{\aleph}} e_2$ and $e_2 \not\longrightarrow_F$, then e_2 is a value.*

Proof. By induction on e_2.

If e_2 is a value, then we have the conclusion.

Otherwise, we show a contradiction. Suppose that e_2 is not a value, i.e., $e_2 = e_{21}\, e_{22}$ for some e_{21} and e_{22}. Since $e_1 \rightrightarrows_{\overline{\aleph}} e_2 = e_{21}\, e_{22}$, there exist some e_{11} and e_{12} such that $e_1 \longrightarrow_F^* e_{11}\, e_{12}$ and $e_{11} \rightrightarrows_{\overline{\aleph}} e_{21}$ and $e_{12} \rightrightarrows_{\overline{\aleph}} e_{22}$ by Lemma 56. Since e_1 does not get stuck, e_{11} does not either. Since $e_{21}\, e_{22} = e_2 \not\longrightarrow_F$, we have $e_{21} \not\longrightarrow_F$. Thus, by the IH, $e_{21} = w_{21}$ for some w_{21}. Since $e_{11} \rightrightarrows_{\overline{\aleph}} e_{21} = w_{21}$, there exists some w_{11} such that $e_{11} \longrightarrow_F^* w_{11}$ and $w_{11} \rightrightarrows_{\overline{\aleph}} w_{21}$ by Lemma 55. Thus, $e_1 \longrightarrow_F^* e_{11}\, e_{12} \longrightarrow_F^* w_{11}\, e_{12}$. Since e_1 does not get stuck, e_{12} does not either. Since $w_{21}\, e_{22} = e_2 \not\longrightarrow_F$, we have $e_{22} \not\longrightarrow_F$. Thus, by the IH, $e_{22} = w_{22}$ for some w_{22}. Since $e_{12} \rightrightarrows_{\overline{\aleph}} e_{22} = w_{22}$, there exists some w_{12} such that $e_{12} \longrightarrow_F^* w_{12}$ and $w_{12} \rightrightarrows_{\overline{\aleph}} w_{22}$ by Lemma 55. Thus, $e_1 \longrightarrow_F^* w_{11}\, e_{12} \longrightarrow_F^* w_{11}\, w_{12}$. By case analysis on w_{21}.

Case $w_{21} = \lambda x.e_{21}'$: Contradictory because $e_2 \not\longrightarrow_F$ by the assumption but $e_2 = (\lambda x.e_{21}')\, w_{22}$ can be evaluated.

Case $w_{21} = c_1$ or x: Since $w_{11} \rightrightarrows_{\overline{\aleph}} w_{21}$, we have $e_1 \longrightarrow_F^* w_{11}\, w_{12} \longrightarrow_F^* w_{21}\, w_{12}$ by Lemma 58.

If $w_{21} = x$, then contradictory to the assumption that e_1 does not get stuck.

Otherwise, if $w_{21} = c_1$, then, since e_1 does not get stuck, we can see $w_{12} = c_2$ for some c_2 such that $\zeta(c_1, c_2)$ is well defined, and $w_{21}\, w_{12} = c_1\, c_2 \rightsquigarrow_\delta \zeta(c_1, c_2)$. Since $c_2 = w_{12} \rightrightarrows_{\overline{\aleph}} w_{22}$, we have $w_{22} = c_2$ by Lemma 50. Thus, $e_2 = w_{21}\, w_{22} = c_1\, c_2 \longrightarrow_F \zeta(c_1, c_2)$, which is contradictory to the assumption that $e_2 \not\longrightarrow_F$.

□

Lemma 62. *If e_1 does not get stuck and $e_1 \Longmapsto_{\overline{\aleph}} e_2$, then e_2 does not either.*

Proof. Suppose that e_2 gets stuck, i.e., there exists some e_2' such that $e_2 \longrightarrow_F^* e_2'$ and $e_2' \not\longrightarrow_F$ and e_2' is not a value. By Lemmas 45 and 60, there exists some e_1' such that $e_1 \longrightarrow_F^* e_1'$ and $e_1' \rightrightarrows_{\overline{\aleph}} e_2'$. Since e_1 does not get stuck, e_1' does not either. By Lemma 61, e_2' is a value, which is contradictory to the assumption that e_2' is not a value. □

Lemma 63. *Suppose that e_1 does not get stuck. If $e_1 \Longmapsto_{\overline{\aleph}}^* e_2$ and $e_2 \longrightarrow_F^* e_2'$, then there exists some e_1' such that $e_1 \longrightarrow_F^* e_1'$ and $e_1' \Longmapsto_{\overline{\aleph}}^* e_2'$.*

Proof. By induction on the number of the steps of $e_1 \Longmapsto_{\overline{\aleph}}^* e_2$.

If the number of the steps is zero, then $e_1 = e_2$, so we have the conclusion by letting $e_1' = e_2'$.

If the number of the steps is more than zero, then there exists some e such that $e_1 \Longmapsto_{\overline{\aleph}} e \Longmapsto_{\overline{\aleph}}^* e_2$. By Lemma 62, e does not get stuck. Thus, by the IH, there exists some e' such that $e \longrightarrow_F^* e'$ and $e' \Longmapsto_{\overline{\aleph}}^* e_2'$. Since $e_1 \Longmapsto_{\overline{\aleph}} e$, we have $e_1 \rightrightarrows_{\overline{\aleph}} e$ by Lemma 45. By Lemma 60, there exists some e_1' such that $e_1 \longrightarrow_F^* e_1'$ and $e_1' \rightrightarrows_{\overline{\aleph}} e'$. By Lemma 47, $e_1' \Longmapsto_{\overline{\aleph}}^* e'$. Thus, we have the conclusion because $e_1' \Longmapsto_{\overline{\aleph}}^* e' \Longmapsto_{\overline{\aleph}}^* e_2'$. □

Lemma 64. *If e does not get stuck and $e \Longmapsto_{\overline{\aleph}}^* w$, then $e \longrightarrow_F^* w'$ for some w' such that $w' \Longmapsto_{\overline{\aleph}}^* w$.*

Proof. By induction on the number of the steps of $e \Longmapsto_{\overline{\aleph}}^* w$.

If the number of the steps is zero, then $e = w$, so we have the conclusion by letting $w' = w$.

If the number of the steps is more than zero, then there exists some e'' such that $e \Longmapsto_{\overline{\aleph}} e'' \Longmapsto_{\overline{\aleph}}^* w$. By Lemma 62, e'' does not get stuck. Thus, by the IH, there exists some w'' such that $e'' \longrightarrow_F^* w''$ and $w'' \Longmapsto_{\overline{\aleph}}^* w$. By Lemmas 45 and 60, there exists some e' such that $e \longrightarrow_F^* e'$ and $e' \rightrightarrows_{\overline{\aleph}} w''$. By Lemma 55, there exists some w' such that $e' \longrightarrow_F^* w'$ and $w' \rightrightarrows_{\overline{\aleph}} w''$. By Lemma 47, $w' \Longmapsto_{\overline{\aleph}}^* w''$. Now, we have the conclusion because: $e \longrightarrow_F^* e' \longrightarrow_F^* w'$; and $w' \Longmapsto_{\overline{\aleph}}^* w'' \Longmapsto_{\overline{\aleph}}^* w$. □

3.3 Type Erasure

Lemma 65. *For any erasable result R, $\mathsf{erase}(R)$ is a value in $\lambda_v^{\forall}$.*

Proof. By induction on R. If $R = \Lambda\alpha.M$ for some α and M, then $M = R'$ for some R' because R is erasable. Thus, $\mathsf{erase}(R) = \mathsf{erase}(R')$ is a value by the IH. □

Lemma 66. *For any M_1, M_2, and x, $\mathsf{erase}(M_1)[\mathsf{erase}(M_2)/x] = \mathsf{erase}(M_1[M_2/x])$.*

Proof. By induction on M_1.

Case $M_1 = y$: Obvious.

Case $M_1 = c$: Obvious.

Case $M_1 = \lambda y.M_1'$: Without loss of generality, we can suppose that $y \neq x$ and y does not occur free in M_2 and $\mathsf{erase}(M_2)$. Then:

$$\begin{array}{rcl} \mathsf{erase}(M_1)[\mathsf{erase}(M_2)/x] & = & (\lambda y.\mathsf{erase}(M_1'))[\mathsf{erase}(M_2)/x] \\ & = & \lambda y.(\mathsf{erase}(M_1')[\mathsf{erase}(M_2)/x]) \\ & = & (\lambda y.\mathsf{erase}(M_1'[M_2/x])) \quad \text{(by the IH)} \\ & = & \mathsf{erase}(\lambda y.M_1'[M_2/x]) \\ & = & \mathsf{erase}(M_1[M_2/x]) \ . \end{array}$$

Case $M_1 = M_{11}\, M_{12}$: By the IHs.

Case $M_1 = !M_1'$: By the IH,

$$\mathsf{erase}(M_1)[\mathsf{erase}(M_2)/x] = \mathsf{erase}(M_1')[\mathsf{erase}(M_2)/x] = \mathsf{erase}(M_1'[M_2/x]) = \mathsf{erase}(!M_1'[M_2/x]) = \mathsf{erase}(M_1[M_2/x]) \ .$$

Case $M_1 = \mathsf{let}\,!y = M_{11}\,\mathsf{in}\,M_{12}$: Without loss of generality, we can suppose that $y \neq x$ and y does not occur free in M_2 and $\mathsf{erase}(M_2)$. Then:

$$\begin{array}{rcl} \mathsf{erase}(M_1)[\mathsf{erase}(M_2)/x] & = & ((\lambda y.\mathsf{erase}(M_{12}))\,\mathsf{erase}(M_{11}))[\mathsf{erase}(M_2)/x] \\ & = & (\lambda y.(\mathsf{erase}(M_{12})[\mathsf{erase}(M_2)/x]))\,\mathsf{erase}(M_{11})[\mathsf{erase}(M_2)/x] \\ & = & (\lambda y.\mathsf{erase}(M_{12}[M_2/x]))\,\mathsf{erase}(M_{11}[M_2/x]) \quad \text{(by the IHs)} \\ & = & \mathsf{erase}(\mathsf{let}\,!y = M_{11}[M_2/x]\,\mathsf{in}\,M_{12}[M_2/x]) \\ & = & \mathsf{erase}(M_1[M_2/x]) \ . \end{array}$$

Case $M_1 = \nu\alpha.\,M_1'$: By the IH.

Case $M_1 = \Lambda^\circ\langle\,\alpha, M_1'\,\rangle$: By the IH.

Case $M_1 = \Lambda\alpha.M_1'$: By the IH.

Case $M_1 = M_1'\,A$: By the IH.

□

Lemma 67. *For any M, A, and α, $\mathsf{erase}(M[A/\alpha]) = \mathsf{erase}(M)$.*

Proof. Straightforward by induction on M. □

Lemma 68. *If M_1 is erasable and $M_1 \rightsquigarrow M_2$, then $\mathsf{erase}(M_1) = \mathsf{erase}(M_2)$ or $\mathsf{erase}(M_1) \rightsquigarrow_{\aleph} \mathsf{erase}(M_2)$ for some $\aleph \in \{\beta_v, \delta\}$.*

Proof. By case analysis on the reduction rule applied to derive $M_1 \rightsquigarrow M_2$.

Case (R_Const): By δ-reduction.

Case (R_BETA): We are given $(\lambda x.M)\,R \rightsquigarrow M[R/x]$ for some x, M, and R such that $M_1 = (\lambda x.M)\,R$ and $M_2 = M[R/x]$. Since M_1 is erasable, so is R. Thus, by Lemma 65, $\mathsf{erase}(R)$ is a value in $\lambda_v^\forall$. Thus:

$$\mathsf{erase}(M_1) = (\lambda x.\mathsf{erase}(M))\,\mathsf{erase}(R) \rightsquigarrow_{\beta_v} \mathsf{erase}(M)[\mathsf{erase}(R)/x]\ .$$

By Lemma 66, $\mathsf{erase}(M)[\mathsf{erase}(R)/x] = \mathsf{erase}(M[R/x]) = \mathsf{erase}(M_2)$. Thus, we have the conclusion.

Case (R_BANG): We are given $\mathsf{let}\,!x = \nu\overline{\alpha}.\,!R\,\mathsf{in}\,M \rightsquigarrow M[\nu\overline{\alpha}.\,R/x]$ for some x, $\overline{\alpha}$, R, and M such that $M_1 = \mathsf{let}\,!x = \nu\overline{\alpha}.\,!R\,\mathsf{in}\,M$ and $M_2 = M[\nu\overline{\alpha}.\,R/x]$. By Lemma 65, $\mathsf{erase}(\nu\overline{\alpha}.\,R)$ is a value in $\lambda_v^\forall$. Thus:

$$\mathsf{erase}(M_1) = (\lambda x.\mathsf{erase}(M))\,\mathsf{erase}(\nu\overline{\alpha}.\,R) \rightsquigarrow_{\beta_v} \mathsf{erase}(M)[\mathsf{erase}(\nu\overline{\alpha}.\,R)/x]\ .$$

By Lemma 66, $\mathsf{erase}(M)[\mathsf{erase}(\nu\overline{\alpha}.\,R)/x] = \mathsf{erase}(M[\nu\overline{\alpha}.\,R/x]) = \mathsf{erase}(M_2)$. Thus, we have the conclusion.

Case (R_CLOSING): We are given $\Lambda^\circ\langle\,\alpha, !R\,\rangle \rightsquigarrow\ !\Lambda\alpha.R$ for some α and R such that $M_1 = \Lambda^\circ\langle\,\alpha, !R\,\rangle$ and $M_2 =\ !\Lambda\alpha.R$. By definition, $\mathsf{erase}(M_1) = \mathsf{erase}(M_2)$.

Case (R_TBETA): We are given $(\Lambda\alpha.M)\,A \rightsquigarrow M[A/\alpha]$ for some α, M, and A such that $M_1 = (\Lambda\alpha.M)\,A$ and $M_2 = M[A/\alpha]$. We have the conclusion by:

$$\mathsf{erase}(M_1) = \mathsf{erase}(M) = \mathsf{erase}(M[A/\alpha]) = \mathsf{erase}(M_2)$$

with Lemma 67.

□

Lemma 69. *If M_1 is erasable and $M_1 \longrightarrow M_2$, then* $\mathsf{erase}(M_1) \longrightarrow_F{}^{0,1} \mathsf{erase}(M_2)$*.*

Proof. By induction on the derivation of $M_1 \longrightarrow M_2$.

Case (E_RED): By Lemma 68.

Case (E_EVAL): We are given $E[M_1'] \longrightarrow E[M_2']$ for some E, M_1', and M_2' such that $M_1 = E[M_1']$ and $M_2 = E[M_2']$. By inversion, we have $M_1' \longrightarrow M_2'$.

By the IH, $\mathsf{erase}(M_1') \longrightarrow_F{}^{0,1} \mathsf{erase}(M_2')$. We perform case analysis on E.

Case $E = [\,]\,M$: We have $M_1 = M_1'\,M$ and $M_2 = M_2'\,M$. Since $\mathsf{erase}(M_1') \longrightarrow_F{}^{0,1} \mathsf{erase}(M_2')$, we have the conclusion by:

$$\mathsf{erase}(M_1) = \mathsf{erase}(M_1')\,\mathsf{erase}(M) \longrightarrow_F{}^{0,1} \mathsf{erase}(M_2')\,\mathsf{erase}(M) = \mathsf{erase}(M_2)\ .$$

Case $E = R\,[\,]$: We have $M_1 = R\,M_1'$ and $M_2 = R\,M_2'$. Since M_1 is erasable, so is R. Thus, by Lemma 65, $\mathsf{erase}(R)$ is a value in $\lambda_v^\forall$. Since $\mathsf{erase}(M_1') \longrightarrow_F{}^{0,1} \mathsf{erase}(M_2')$, we have the conclusion by:

$$\mathsf{erase}(M_1) = \mathsf{erase}(R)\,\mathsf{erase}(M_1') \longrightarrow_F{}^{0,1} \mathsf{erase}(R)\,\mathsf{erase}(M_2') = \mathsf{erase}(M_2)\ .$$

Case $E = \mathsf{let}\,!x = [\,]\,\mathsf{in}\,M$: We have $M_1 = \mathsf{let}\,!x = M_1'\,\mathsf{in}\,M$ and $M_2 = \mathsf{let}\,!x = M_2'\,\mathsf{in}\,M$. Since $\mathsf{erase}(M_1') \longrightarrow_F{}^{0,1} \mathsf{erase}(M_2')$, we have the conclusion by:

$$\mathsf{erase}(M_1) = (\lambda x.\mathsf{erase}(M))\,\mathsf{erase}(M_1') \longrightarrow_F{}^{0,1} (\lambda x.\mathsf{erase}(M))\,\mathsf{erase}(M_2') = \mathsf{erase}(M_2)\ .$$

Case $E = \Lambda^\circ\langle\,\beta, [\,]\,\rangle$, $[\,]\,A$, $\nu\alpha.\,[\,]$, and $!\,[\,]$: We have the conclusion by:

$$\mathsf{erase}(M_1) = \mathsf{erase}(M_1') \longrightarrow_F{}^{0,1} \mathsf{erase}(M_2') = \mathsf{erase}(M_2)$$

with the IH $\mathsf{erase}(M_1') \longrightarrow_F{}^{0,1} \mathsf{erase}(M_2')$.

Case (E_EXTR): We are given $\mathbb{E}[\nu\beta.\,R] \longrightarrow \nu\beta.\,\mathbb{E}[R]$ for some $\mathbb{E}$, β, and R such that $M_1 = \mathbb{E}[\nu\beta.\,R]$ and $M_2 = \nu\beta.\,\mathbb{E}[R]$ and $\alpha \notin \mathit{ftv}(\mathbb{E})$.

We show that $\mathsf{erase}(M_1) = \mathsf{erase}(M_2)$ by case analysis on $\mathbb{E}$.

Case $\mathbb{E} = [\,]\ R_2$: We find $M_1 = (\nu\beta.\, R)\, R_2$ and $M_2 = \nu\beta.\, (R\, R_2)$. We have the conclusion by:

$$\mathsf{erase}(M_1) = \mathsf{erase}((\nu\beta.\, R)\, R_2) = \mathsf{erase}(R\, R_2) = \mathsf{erase}(\nu\beta.\, (R\, R_2)) = \mathsf{erase}(M_2)\ .$$

Case $\mathbb{E} = \Lambda^\circ\langle\, \gamma,\ [\,]\,\rangle$ and $[\,]\ A$: We have the conclusion by:

$$\mathsf{erase}(M_1) = \mathsf{erase}(\mathbb{E}[\nu\beta.\, R]) = \mathsf{erase}(\nu\beta.\, R) = \mathsf{erase}(R) = \mathsf{erase}(\mathbb{E}[R]) = \mathsf{erase}(\nu\beta.\, \mathbb{E}[R]) = \mathsf{erase}(M_2)\ .$$

□

Lemma 70. *If R is erasable and $\Delta \vdash R : \forall\alpha.A$, then, for any B, $R\, B \longrightarrow^* R'$ for some R' such that* $\mathsf{erase}(R) = \mathsf{erase}(R')$.

Proof. By induction on the derivation of $\Delta \vdash R : \forall\alpha.A$.

Case (T_Var): Contradictory.

Case (T_Const), (T_Abs), and (T_Bang): Contradictory because the type of R is a polymorphic type.

Case (T_App), (T_TApp), (T_LetBang), and (T_Gen): Contradictory because terms accepted by those typing rules are not results.

Case (T_Nu): We are given $\Delta \vdash \nu\beta.\, R_0 : \forall\alpha.A$ for some β and R_0 such that $R = \nu\beta.\, R_0$. By inversion, $\Delta, \beta^{\mathbf{1}} \vdash R_0 : \forall\alpha.A$. Without loss of generality, we can suppose that $\beta \notin \mathit{ftv}(B)$.

By the IH, $R_0\, B \longrightarrow^* R''$ for some R'' such that $\mathsf{erase}(R_0) = \mathsf{erase}(R'')$. By (E_Eval), $\nu\beta.\, (R_0\, B) \longrightarrow^* \nu\beta.\, R''$. We have the conclusion by letting $R' = \nu\beta.\, R''$ because: $R\, B = (\nu\beta.\, R_0)\, B \longrightarrow \nu\beta.\, (R_0\, B) \longrightarrow^* \nu\beta.\, R'' = R'$; and $\mathsf{erase}(R) = \mathsf{erase}(R_0) = \mathsf{erase}(R'') = \mathsf{erase}(\nu\beta.\, R'') = \mathsf{erase}(R')$.

Case (T_TAbs): We are given $\Delta \vdash \Lambda\alpha.R_0 : \forall\alpha.A$ for some Δ, Δ, and R_0 such that $R = \Lambda\alpha.R_0$. Note that the body of the type abstraction is a result because R is erasable. We have the conclusion by letting $R' = R_0[B/\alpha]$ because: $R\, B = (\Lambda\alpha.R_0)\, B \longrightarrow R_0[B/\alpha] = R'$ by (R_TBeta)/(E_Red); and $\mathsf{erase}(R) = \mathsf{erase}(R_0) = \mathsf{erase}(R')$ by Lemma 67.

□

Lemma 71. *If $\Delta \vdash R : !A$, then $\Lambda^\circ\langle\, \alpha, R\,\rangle \longrightarrow^* R'$ for some R' such that* $\mathsf{erase}(R) = \mathsf{erase}(R')$.

Proof. By induction on the derivation of $\Delta \vdash R : !A$.

Case (T_Var): Contradictory.

Case (T_Const), (T_Abs), and (T_TAbs): Contradictory because the type of R is $!A$.

Case (T_App), (T_LetBang), (T_TApp), and (T_Gen): Contradictory because terms accepted by those typing rules are not results.

Case (T_Bang): We are given $\Delta \vdash !R_0 : !A$ for some R_0 such that $R = !R_0$. We have the conclusion by letting $R' = !\Lambda\alpha.R_0$ because: $\Lambda^\circ\langle\, \alpha, R\,\rangle = \Lambda^\circ\langle\, \alpha, !R_0\,\rangle \longrightarrow !\Lambda\alpha.R_0 = R'$; and $\mathsf{erase}(R) = \mathsf{erase}(R_0) = \mathsf{erase}(!\Lambda\alpha.R_0) = \mathsf{erase}(R')$.

Case (T_Nu): We are given $\Delta \vdash \nu\beta.\, R_0 : !A$ for some β and R_0 such that $R = \nu\beta.\, R_0$. By inversion, $\Delta, \beta^{\mathbf{1}} \vdash R_0 : !A$. Without loss of generality, we can suppose that $\beta \neq \alpha$.

By the IH, $\Lambda^\circ\langle\, \alpha, R_0\,\rangle \longrightarrow^* R''$ for some R'' such that $\mathsf{erase}(R_0) = \mathsf{erase}(R'')$. By (E_Eval), $\nu\beta.\, \Lambda^\circ\langle\, \alpha, R_0\,\rangle \longrightarrow^* \nu\beta.\, R''$. We have the conclusion by letting $R' = \nu\beta.\, R''$ because: $\Lambda^\circ\langle\, \alpha, R\,\rangle = \Lambda^\circ\langle\, \alpha, \nu\beta.\, R_0\,\rangle \longrightarrow \nu\beta.\, \Lambda^\circ\langle\, \alpha, R_0\,\rangle \longrightarrow^* \nu\beta.\, R'' = R'$; and $\mathsf{erase}(R) = \mathsf{erase}(R_0) = \mathsf{erase}(R'') = \mathsf{erase}(R')$.

□

Lemma 72. *If M is erasable and $M \longrightarrow M'$, then M' is also erasable.*

Proof. Straightforward by induction on the evaluation derivation of $M \longrightarrow M'$. The case for (E_RED) depends on the fact that substitution preserves erasability (which can be proven easily using the fact that substitution for a variable in a result produces a result). ☐

Lemma 73. *Suppose that M is erasable. If $\Delta \vdash M : A$ and $\mathsf{erase}(M)$ is a value, then $M \longrightarrow^* R$ for some R such that $\mathsf{erase}(R) = \mathsf{erase}(M)$.*

Proof. By induction on the typing derivation for M.

Case (T_VAR): Contradictory.

Case (T_CONST), (T_ABS), and (T_TABS): Obvious by letting $R = M$.

Case (T_APP): Contradictory because $\mathsf{erase}(M)$ is not a value.

Case (T_BANG): We are given $\Delta \vdash\, !M' : !B$ for some M' and B such that $M = !M'$ and $A = !B$. By inversion, $\omega\Delta \vdash M' : B$.

Since $\mathsf{erase}(M) = \mathsf{erase}(M')$, we find $\mathsf{erase}(M')$ is a value. Since M is erasable, so is M'. Thus, by the IH, $M' \longrightarrow^* R'$ for some R' such that $\mathsf{erase}(R') = \mathsf{erase}(M')$. We have the conclusion by letting $R = !R'$ because: $M = !M' \longrightarrow^* !R' = R$ by (E_EVAL); and $\mathsf{erase}(R) = \mathsf{erase}(R') = \mathsf{erase}(M') = \mathsf{erase}(M)$.

Case (T_LETBANG): Contradictory because $\mathsf{erase}(M)$ is not a value.

Case (T_NU): We are given $\Delta \vdash \nu\alpha.\, M' : A$ for some α and M' such that $M = \nu\alpha.\, M'$. By inversion, $\Delta, \alpha^{\mathbf{1}} \vdash M' : A$.

Since $\mathsf{erase}(M) = \mathsf{erase}(M')$, we find $\mathsf{erase}(M')$ is a value. Since M is erasable, so is M'. Thus, by the IH, $M' \longrightarrow^* R'$ for some R' such that $\mathsf{erase}(R') = \mathsf{erase}(M')$. We have the conclusion by letting $R = \nu\alpha.\, R'$ because: $M = \nu\alpha.\, M' \longrightarrow^* \nu\alpha.\, R' = R$ by (E_EVAL); and $\mathsf{erase}(R) = \mathsf{erase}(R') = \mathsf{erase}(M') = \mathsf{erase}(M)$.

Case (T_GEN): We are given $\Delta_1, \alpha^{\mathbf{1}}, \Delta_2 \vdash \Lambda^{\circ}\langle\, \alpha, M'\,\rangle : !\forall\alpha.B$ for some Δ_1, Δ_2, α, M', and B such that $\Delta = \Delta_1, \alpha^{\mathbf{1}}, \Delta_2$ and $M = \Lambda^{\circ}\langle\, \alpha, M'\,\rangle$ and $A = !\forall\alpha.B$. By inversion, $\Delta_1, \alpha^{\mathbf{0}}, \Delta_2 \vdash M' : !B$.

Since $\mathsf{erase}(M) = \mathsf{erase}(M')$, we find $\mathsf{erase}(M')$ is a value. Since M is erasable, so is M'. Thus, by the IH, $M' \longrightarrow^* R'$ for some R' such that $\mathsf{erase}(R') = \mathsf{erase}(M')$. We also have $\Delta_1, \alpha^{\mathbf{0}}, \Delta_2 \vdash R' : !B$ by Lemma 40. By Lemma 71, $\Lambda^{\circ}\langle\, \alpha, R'\,\rangle \longrightarrow^* R$ for some R such that $\mathsf{erase}(R') = \mathsf{erase}(R)$. We have the conclusion by: $M = \Lambda^{\circ}\langle\, \alpha, M'\,\rangle \longrightarrow^* \Lambda^{\circ}\langle\, \alpha, R'\,\rangle \longrightarrow^* R$; and $\mathsf{erase}(R) = \mathsf{erase}(R') = \mathsf{erase}(M') = \mathsf{erase}(M)$.

Case (T_TAPP): We are given $\Delta \vdash M'\, B : C[B/\alpha]$ for some M', B, C, and α such that $M = M'\, B$ and $A = C[B/\alpha]$. By inversion, $\Delta \vdash M' : \forall\alpha.C$.

Since $\mathsf{erase}(M) = \mathsf{erase}(M')$, we find $\mathsf{erase}(M')$ is a value. Since M is erasable, so is M'. Thus, by the IH, $M' \longrightarrow^* R'$ for some R' such that $\mathsf{erase}(R') = \mathsf{erase}(M')$. By Lemma 40, $\Delta \vdash R' : \forall\alpha.C$. By Lemma 72, R' is erasable. Thus, by Lemma 70, $R'\, B \longrightarrow^* R$ for some R such that $\mathsf{erase}(R') = \mathsf{erase}(R)$. We have the conclusion because: $M = M'\, B \longrightarrow^* R'\, B \longrightarrow^* R$; and $\mathsf{erase}(R) = \mathsf{erase}(R') = \mathsf{erase}(M') = \mathsf{erase}(M)$.

☐

Lemma 74. *If $\Delta_1 \vdash R_1 : A \multimap B$ and $\Delta_2 \vdash R_2 : A$ and R_2 is erasable and $\mathsf{erase}(R_1\, R_2) \rightsquigarrow_{\aleph} e$ for some $\aleph \in \{\beta_v, \delta\}$, then $R_1\, R_2 \longrightarrow^* M$ for some M such that $\mathsf{erase}(M) = e$.*

Proof. By induction on the derivation of $\Delta_1 \vdash R_1 : A \multimap B$ with case analysis on the typing rule applied last to derive $\Delta_1 \vdash R_1 : A \multimap B$.

Case (T_VAR), (T_APP), (T_BANG), (T_LETBANG), (T_GEN), (T_TABS), and (T_TAPP): Contradictory.

Case (T_CONST): We are given $\Delta_1 \vdash c_1 : ty(c_1)$ for some c_1 such that $R_1 = c_1$ and $A \multimap B = ty(c_1)$. By Assumption 1, $A = \iota$ for some ι. Since $\Delta_2 \vdash R_2 : \iota$, we have $R_2 = \nu\overline{\alpha}.\, c_2$ for some $\overline{\alpha}$ and c_2 such that $ty(c_2) = \iota$ by Lemma 35. Again by Assumption 1, $\zeta(c_1, c_2)$ is well defined, and $R_1\, R_2 = c_1\, \nu\overline{\alpha}.\, c_2 \longrightarrow \nu\overline{\alpha}.\, \zeta(c_1, c_2)$ by (R_CONST)/(E_RED). We also have $\mathsf{erase}(R_1\, R_2) = c_1\, c_2 \rightsquigarrow_{\aleph} e$, so $e = \zeta(c_1, c_2)$. Since $\zeta(c_1, c_2)$ is a constant, we have $\mathsf{erase}(\nu\overline{\alpha}.\, \zeta(c_1, c_2)) = \zeta(c_1, c_2)$. Thus, we have the conclusion by letting $M = \nu\overline{\alpha}.\, \zeta(c_1, c_2)$.

Case (T_ABS): We have $R_1 = \lambda x.M_1$ for some x and M_1. By (R_BETA)/(E_RED), $R_1\,R_2 \longrightarrow M_1[R_2/x]$. Let $M = M_1[R_2/x]$. Since R_2 is erasable, $\mathsf{erase}(R_2)$ is a value by Lemma 65. Thus, $\mathsf{erase}(R_1\,R_2) = (\lambda x.\mathsf{erase}(M_1))\,\mathsf{erase}(R_2) \rightsquigarrow_{\aleph} \mathsf{erase}(M_1)[\mathsf{erase}(R_2)/x] = e$. By Lemma 66, $e = \mathsf{erase}(M_1[R_2/x]) = \mathsf{erase}(M)$. Thus, we have the conclusion.

Case (T_NU): We are given $\Delta_1 \vdash \nu\alpha.\,R_1' : A \multimap B$ for some α and R_1' such that $R_1 = \nu\alpha.\,R_1'$. By inversion, $\Delta_1, \alpha^{\mathbf{1}} \vdash R_1' : A \multimap B$. By (R_EXTR)/(E_RED), $R_1\,R_2 \longrightarrow \nu\alpha.\,(R_1'\,R_2)$. Since $\mathsf{erase}(R_1'\,R_2) = \mathsf{erase}(R_1\,R_2) \rightsquigarrow_{\aleph} e$, there exists some M' such that $R_1'\,R_2 \longrightarrow^* M'$ and $\mathsf{erase}(M') = e$ by the IH. We have the conclusion by letting $M = \nu\alpha.\,M'$ because: $R_1\,R_2 = (\nu\alpha.\,R_1')\,R_2 \longrightarrow \nu\alpha.\,(R_1'\,R_2) \longrightarrow^* \nu\alpha.\,M' = M$; and $\mathsf{erase}(M) = \mathsf{erase}(M') = e$.

□

Lemma 75. *Suppose that M_1 and M_2 are erasable. If $\Delta_1 \vdash M_1 : A \multimap B$ and $\Delta_2 \vdash M_2 : A$ and $\mathsf{erase}(M_1\,M_2) \rightsquigarrow_{\aleph} e$ for some $\aleph \in \{\beta_v, \delta\}$, then $M_1\,M_2 \longrightarrow^* M$ for some M such that $\mathsf{erase}(M) = e$.*

Proof. Since $\mathsf{erase}(M_1\,M_2) = \mathsf{erase}(M_1)\,\mathsf{erase}(M_2)$ and $\mathsf{erase}(M_1\,M_2) \rightsquigarrow_{\aleph} e$, we find $\mathsf{erase}(M_1)$ and $\mathsf{erase}(M_2)$ are values. Thus, by Lemma 73, there exist some R_1 and R_2 such that

- $M_1 \longrightarrow^* R_1$ and $\mathsf{erase}(R_1) = \mathsf{erase}(M_1)$, and
- $M_2 \longrightarrow^* R_2$ and $\mathsf{erase}(R_2) = \mathsf{erase}(M_2)$.

We also have $\Delta_1 \vdash R_1 : A \multimap B$ and $\Delta_2 \vdash R_2 : A$ by Lemma 40. Since $\mathsf{erase}(R_1\,R_2) = \mathsf{erase}(M_1\,M_2)$, we have $\mathsf{erase}(R_1\,R_2) \rightsquigarrow_{\aleph} e$. By Lemma 72, R_2 is erasable. Thus, by Lemma 74, there exists some M such that $R_1\,R_2 \longrightarrow^* M$ and $\mathsf{erase}(M) = e$. Since $M_1\,M_2 \longrightarrow^* R_1\,M_2 \longrightarrow^* R_1\,R_2 \longrightarrow^* M$, we have the conclusion. □

Lemma 76. *Suppose that M_1 is erasable. If $\Delta_1 \vdash M_1 : !B$ and $\Delta_2, x :^{\omega} B \vdash M_2 : A$ and $\mathsf{erase}(\mathsf{let}\,!x = M_1\,\mathsf{in}\,M_2) \rightsquigarrow_{\aleph} e$ for some $\aleph \in \{\beta_v, \delta\}$, then $\mathsf{let}\,!x = M_1\,\mathsf{in}\,M_2 \longrightarrow^* M$ for some M such that $\mathsf{erase}(M) = e$.*

Proof. Since $(\lambda x.\mathsf{erase}(M_2))\,\mathsf{erase}(M_1) = \mathsf{erase}(\mathsf{let}\,!x = M_1\,\mathsf{in}\,M_2) \rightsquigarrow_{\aleph} e$, we can find $\mathsf{erase}(M_1)$ is a value. Thus, by Lemma 73, $M_1 \longrightarrow^* R_1$ for some R_1 such that $\mathsf{erase}(R_1) = \mathsf{erase}(M_1)$. We also have $\Delta_1 \vdash R_1 : !B$ by Lemma 40. By Lemma 35, $R_1 = \nu\overline{\alpha}.\,!R_1'$ for some $\overline{\alpha}$ and R_1'. Now, we have the conclusion by letting $M = M_2[\nu\overline{\alpha}.\,R_1'/x]$ because:

- $\mathsf{let}\,!x = M_1\,\mathsf{in}\,M_2 \longrightarrow^* \mathsf{let}\,!x = R_1\,\mathsf{in}\,M_2 = \mathsf{let}\,!x = \nu\overline{\alpha}.\,!R_1'\,\mathsf{in}\,M_2 \longrightarrow M_2[\nu\overline{\alpha}.\,R_1'/x]$; and
- $\mathsf{erase}(M) = \mathsf{erase}(M_2[\nu\overline{\alpha}.\,R_1'/x]) = \mathsf{erase}(M_2)[\mathsf{erase}(\nu\overline{\alpha}.\,R_1')/x] = \mathsf{erase}(M_2)[\mathsf{erase}(R_1)/x] = \mathsf{erase}(M_2)[\mathsf{erase}(M_1)/x] = e$ with Lemma 66 and the fact that $(\lambda x.\mathsf{erase}(M_2))\,\mathsf{erase}(M_1) \rightsquigarrow_{\aleph} e$, so $e = \mathsf{erase}(M_2)[\mathsf{erase}(M_1)/x]$.

□

Lemma 77. *Suppose that M_1 is erasable. If $\Delta \vdash M_1 : A$ and $\mathsf{erase}(M_1) \rightsquigarrow_{\aleph} e$ for $\aleph \in \{\beta_v, \delta\}$, then there exists some M_2 such that $M_1 \longrightarrow^* M_2$ and $\mathsf{erase}(M_2) = e$.*

Proof. By induction on the typing derivation of $\Delta \vdash M_1 : A$.

Case (T_VAR): Contradictory.

Case (T_CONST) and (T_ABS): Contradictory because there is no reduction allowing $\mathsf{erase}(M_1) \rightsquigarrow_{\aleph} e$.

Case (T_TABS): Since M_1 is erasable, we have $M_1 = \Lambda\alpha.R$ for some α and erasable R. By Lemma 65, $\mathsf{erase}(R)$ is a value in $\lambda_v^{\forall}$. Thus, there is no reduction allowing $\mathsf{erase}(R) = \mathsf{erase}(M_1) \rightsquigarrow_{\aleph} e$.

Case (T_APP): By Lemma 75.

Case (T_BANG): We are given $\Delta \vdash !M_1' : !B$ for some M_1' and B such that $M_1 = !M_1'$ and $A = !B$. By inversion, $\omega\Delta \vdash M_1' : B$. We have $\mathsf{erase}(M_1') = \mathsf{erase}(M_1) \rightsquigarrow_{\aleph} e$. Since M_1 is erasable, so is M_1'. Thus, by the IH, $M_1' \longrightarrow^* M_2'$ for some M_2' such that $\mathsf{erase}(M_2') = e$. We have the conclusion by letting $M_2 = !M_2'$ because: $M_1 = !M_1' \longrightarrow^* !M_2' = M_2$ by (E_EVAL); and $\mathsf{erase}(M_2) = \mathsf{erase}(M_2') = e$.

Case (T_LETBANG): By Lemma 76.

Case (T_Nu): We are given $\Delta \vdash \nu\alpha.\, M_1' : A$ for some α and M_1' such that $M_1 = \nu\alpha.\, M_1'$. By inversion, $\Delta, \alpha^{\mathbf{1}} \vdash M_1' : A$. We have $\mathsf{erase}(M_1') = \mathsf{erase}(M_1) \rightsquigarrow_{\aleph} e$. Since M_1 is erasable, so is M_1'. Thus, by the IH, $M_1' \longrightarrow^* M_2'$ for some M_2' such that $\mathsf{erase}(M_2') = e$. We have the conclusion by letting $M_2 = \nu\alpha.\, M_2'$ because: $M_1 = \nu\alpha.\, M_1' \longrightarrow^* \nu\alpha.\, M_2' = M_2$ by (E_Eval); and $\mathsf{erase}(M_2) = \mathsf{erase}(M_2') = e$.

Case (T_Gen) and (T_TApp): By the IH, similarly to the cases of (T_Bang) and (T_Nu).

□

Lemma 78. *Suppose that M_1 is erasable. If $\Delta \vdash M_1 : A$ and $\mathsf{erase}(M_1) \longrightarrow_F e$, then there exists some M_2 such that $M_1 \longrightarrow^* M_2$ and $\mathsf{erase}(M_2) = e$.*

Proof. By induction on the derivation of $\Delta \vdash M_1 : A$ with case analysis on the typing rule last to derive $\Delta \vdash M_1 : A$.

Case (T_Var): Contradictory.

Case (T_Const) and (T_Abs): Contradictory because there is no reduction allowing $\mathsf{erase}(M_1) \longrightarrow_F e$.

Case (T_TAbs): Since M_1 is erasable, we have $M_1 = \Lambda\alpha.R$ for some α and erasable R. By Lemma 65, $\mathsf{erase}(R) = \mathsf{erase}(M_1)$ is a value in $\lambda_v^{\forall}$. Thus, there is no reduction allowing $\mathsf{erase}(M_1) \longrightarrow_F e$, so there is a contradiction.

Case (T_App): We are given $\Delta_1 + \Delta_2 \vdash M_{11}\, M_{12} : A$ for some Δ_1, Δ_2, M_{11}, and M_{12} such that $\Delta = \Delta_1 + \Delta_2$ and $M_1 = M_{11}\, M_{12}$. By inversion, $\Delta_1 \vdash M_{11} : B \multimap A$ and $\Delta_2 \vdash M_{12} : B$ for some B. We perform case analysis on how the evaluation $\mathsf{erase}(M_{11})\, \mathsf{erase}(M_{12}) = \mathsf{erase}(M_1) \longrightarrow_F e$ proceeds.

Case $\mathsf{erase}(M_{11})\, \mathsf{erase}(M_{12}) \rightsquigarrow_{\aleph} e$ for some $\aleph \in \{\beta_v, \delta\}$: By Lemma 77.

Case $\mathsf{erase}(M_{11}) \longrightarrow_F e_1$ and $e = e_1\, \mathsf{erase}(M_{12})$: Since $\Delta_1 \vdash M_{11} : B \multimap A$ and M_{11} is erasable, there exists some M_{21} such that $M_{11} \longrightarrow^* M_{21}$ and $\mathsf{erase}(M_{21}) = e_1$ by the IH. We have the conclusion by letting $M_2 = M_{21}\, M_{12}$ because: $M_1 = M_{11}\, M_{12} \longrightarrow^* M_{21}\, M_{12} = M_2$; and $\mathsf{erase}(M_2) = \mathsf{erase}(M_{21})\, \mathsf{erase}(M_{12}) = e_1\, \mathsf{erase}(M_{12}) = e$.

Case $\mathsf{erase}(M_{11}) = w_1$ and $\mathsf{erase}(M_{12}) \longrightarrow_F e_2$ and $e = w_1\, e_2$: Since $\mathsf{erase}(M_{11})$ is a value and M_{11} is erasable, there exists some R_{11} such that $M_{11} \longrightarrow^* R_{11}$ and $\mathsf{erase}(R_{11}) = \mathsf{erase}(M_{11})$ by Lemma 73.
By the IH on M_{12}, there exists some M_{22} such that $M_{12} \longrightarrow^* M_{22}$ and $\mathsf{erase}(M_{22}) = e_2$. We have the conclusion by letting $M_2 = R_{11}\, M_{22}$ because: $M_1 = M_{11}\, M_{12} \longrightarrow^* R_{11}\, M_{12} \longrightarrow^* R_{11}\, M_{22} = M_2$; and $\mathsf{erase}(M_2) = \mathsf{erase}(R_{11}\, M_{22}) = \mathsf{erase}(M_{11})\, e_2 = e$.

Case (T_Bang), (T_Nu), (T_TApp), and (T_Gen): By the IH and (E_Eval).

Case (T_LetBang): We are given $\Delta_1 + \Delta_2 \vdash \mathsf{let}\, !x = M_{11}\, \mathsf{in}\, M_{12} : A$ for some Δ_1, Δ_2, x, M_{11}, and M_{12} such that $\Delta = \Delta_1 + \Delta_2$ and $M_1 = \mathsf{let}\, !x = M_{11}\, \mathsf{in}\, M_{12}$. By inversion, $\Delta_1 \vdash M_{11} : !B$ and $\Delta_2, x :^{\omega} B \vdash M_{12} : A$ for some B. We perform case analysis on how the evaluation $(\lambda x.\mathsf{erase}(M_{12}))\, \mathsf{erase}(M_{11}) = \mathsf{erase}(M_1) \longrightarrow_F e$ proceeds.

Case $(\lambda x.\mathsf{erase}(M_{12}))\, \mathsf{erase}(M_{11}) \rightsquigarrow_{\aleph} e$ for some $\aleph \in \{\beta_v, \delta\}$: By Lemma 77.

Case $\mathsf{erase}(M_{11}) \longrightarrow_F e_1$ and $e = (\lambda x.\mathsf{erase}(M_{12}))\, e_1$: By the IH, there exists some M_{21} such that $M_{11} \longrightarrow^* M_{21}$ and $\mathsf{erase}(M_{21}) = e_1$. We have the conclusion by letting $M_2 = \mathsf{let}\, !x = M_{21}\, \mathsf{in}\, M_{12}$ because: $M_1 = \mathsf{let}\, !x = M_{11}\, \mathsf{in}\, M_{12} \longrightarrow^* \mathsf{let}\, !x = M_{21}\, \mathsf{in}\, M_{12} = M_2$; and $\mathsf{erase}(M_2) = (\lambda x.\mathsf{erase}(M_{12}))\, \mathsf{erase}(M_{21}) = (\lambda x.\mathsf{erase}(M_{12}))\, e_1 = e$.

□

Theorem 2 (Meaning preservation of type erasure)**.** *Suppose that M is erasable.*

1. *If $M \longrightarrow^* M'$, then $\mathsf{erase}(M) \longrightarrow_F^* \mathsf{erase}(M')$. Furthermore, if M' is a result, then $\mathsf{erase}(M')$ is a value.*
2. *If $\Delta \vdash M : A$ and $\mathsf{erase}(M) \longrightarrow_F^* e$, then $M \longrightarrow^* M'$ for some M' such that $\mathsf{erase}(M') = e$. Furthermore, if $e = w$, then $M' \longrightarrow^* R$ for some R such that $\mathsf{erase}(R) = w$.*

Proof. 1. We first show that $M \longrightarrow^* M'$ implies $\mathsf{erase}(M) \longrightarrow_F^* \mathsf{erase}(M')$ by induction on the number of the steps of $M \longrightarrow^* M'$.

If the number of the steps is zero, i.e., $M = M'$, then we have the conclusion because $\mathsf{erase}(M) \longrightarrow_F^* \mathsf{erase}(M) = \mathsf{erase}(M')$.

If the number of the steps is more than zero, there exists some M'' such that $M \longrightarrow M'' \longrightarrow^* M'$. We have the conclusion because $\mathsf{erase}(M) \longrightarrow_F{}^{0,1} \mathsf{erase}(M'') \longrightarrow_F^* \mathsf{erase}(M')$ by Lemmas 69 and 72 and the IH.

Finally, by Lemma 72, M' is erasable. Thus, if $M' = R$ for some R, then $\mathsf{erase}(R)$ is a value by Lemma 65. Thus, we have the conclusion.

2. We first show that there exists a desired M' by induction on the number of the steps of $\mathsf{erase}(M) \longrightarrow_F^* e$.

If the number of the steps is zero, i.e., $\mathsf{erase}(M) = e$, then we have the conclusion by letting $M' = M$.

If the number of the steps is more than zero, there exists some e'' such that $\mathsf{erase}(M) \longrightarrow_F e'' \longrightarrow_F^* e$. By Lemma 78, there exists some M'' such that $M \longrightarrow^* M''$ and $\mathsf{erase}(M'') = e''$. By Lemma 40, $\Delta \vdash M'' : A$. By Lemma 72, M'' is erasable. Thus, by the IH, $M'' \longrightarrow^* M'$ for some M' such that $\mathsf{erase}(M') = e$. M' is a desired term since $M \longrightarrow^* M'' \longrightarrow^* M'$.

Next, we show that, if $\mathsf{erase}(M') = w$, then $M' \longrightarrow^* R$ for some R such that $\mathsf{erase}(R) = w$. Since $\Delta \vdash M' : A$ by Lemma 40 and M' is erasable by Lemma 72, this is proven by Lemma 73.

□

Lemma 79. *Suppose that V_2 is erasable. If $\Delta_1 \vdash V_1 : A \multimap B$ and $\Delta_2 \vdash V_2 : A$, then $\mathsf{erase}(V_1\ V_2) \longrightarrow_F e$ for some e.*

Proof. By inversion of $\Delta_1 \vdash V_1 : A \multimap B$.

Case (T_Const): We are given $\Delta_1 \vdash c_1 : ty(c_1)$ for some c_1 such that $V_1 = c_1$ and $A \multimap B = ty(c_1)$. By Assumption 1, $A = \iota$ for some ι. By inversion of $\Delta_2 \vdash V_2 : \iota$, we can find $V_2 = c_2$ for some c_2 such that $ty(c_2) = \iota$. Thus, by Assumption 1 and Definition 22, $\mathsf{erase}(V_1\ V_2) = c_1\ c_2 \longrightarrow_F \zeta(c_1, c_2)$.

Case (T_Abs): We are given $V_1 = \lambda x.M_1$ for some x and M_1. Since $\mathsf{erase}(V_1) = \lambda x.\mathsf{erase}(M_1)$ and $\mathsf{erase}(V_2)$ is a value by Lemma 65, we have the conclusion by letting $e = \mathsf{erase}(M_1)[\mathsf{erase}(V_2)/x]$ because: $\mathsf{erase}(V_1\ V_2) = (\lambda x.\mathsf{erase}(M_1))\ \mathsf{erase}(V_2) \longrightarrow_F \mathsf{erase}(M_1)[\mathsf{erase}(V_2)/x] = e$.

Otherwise: Contradictory.

□

Lemma 80. *Suppose that R_2 is erasable. If $\Delta_1 \vdash V_1 : A \multimap B$ and $\Delta_2 \vdash R_2 : A$, then $\mathsf{erase}(V_1\ R_2) \longrightarrow_F e$ for some e.*

Proof. By induction on the derivation of $\Delta_2 \vdash R_2 : A$.

Case (T_Var), (T_App), (T_LetBang), (T_TApp), and (T_Gen): Contradictory.

Case (T_Const), (T_Abs), (T_Bang), and (T_TAbs): By Lemma 79.

Case (T_Nu): We are given $\Delta_2 \vdash \nu\alpha.\ R_2' : A$ for some α and R_2' such that $R_2 = \nu\alpha.\ R_2'$. By inversion, $\Delta, \alpha^{\mathbf{1}} \vdash R_2' : A$. By the IH, $\mathsf{erase}(V_1\ R_2') \longrightarrow_F e$ for some e. Since $\mathsf{erase}(V_1\ R_2') = \mathsf{erase}(V_1\ R_2)$, we have the conclusion.

□

Lemma 81. *Suppose that R_2 is erasable. If $\Delta_1 \vdash R_1 : A \multimap B$ and $\Delta_2 \vdash R_2 : A$, then $\mathsf{erase}(R_1\ R_2) \longrightarrow_F e$ for some e.*

Proof. By induction on the derivation of $\Delta_1 \vdash R_1 : A \multimap B$.

Case (T_Var), (T_App), (T_Bang), (T_LetBang), (T_Gen), (T_TAbs), and (T_TApp): Contradictory.

Case (T_Const) and (T_Abs): By Lemma 80.

Case (T_NU): We are given $\Delta_1 \vdash \nu\alpha.\, R_1' : A \multimap B$ for some α and R_1' such that $R_1 = \nu\alpha.\, R_1'$. By inversion, $\Delta, \alpha^{\mathbf{1}} \vdash R_1' : A \multimap B$. By the IH, $\mathsf{erase}(R_1'\, R_2) \longrightarrow_F e$ for some e. Since $\mathsf{erase}(R_1'\, R_2) = \mathsf{erase}(R_1\, R_2)$, we have the conclusion.

□

Lemma 82. *Suppose that M is erasable. If $\Delta \vdash M : A$ and $\mathsf{erase}(M) \not\longrightarrow_F$, then $\mathsf{erase}(M)$ is a value in $\lambda_v^\forall$.*

Proof. By induction on the derivation of $\Delta \vdash M : A$.

Case (T_VAR): Contradictory.

Case (T_CONST) and (T_ABS): Obvious.

Case (T_APP): We are given $\Delta_1 + \Delta_2 \vdash M_1\, M_2 : A$ for some Δ_1, Δ_2, M_1, and M_2 such that $\Delta = \Delta_1 + \Delta_2$ and $M = M_1\, M_2$. By inversion, $\Delta_1 \vdash M_1 : B \multimap A$ and $\Delta_2 \vdash M_2 : B$ for some B.

Since $\mathsf{erase}(M_1)\, \mathsf{erase}(M_2) = \mathsf{erase}(M) \not\longrightarrow_F$, we can find $\mathsf{erase}(M_1) \not\longrightarrow_F$. Thus, by the IH, $\mathsf{erase}(M_1)$ is a value. By Lemma 73, $M_1 \longrightarrow^* R_1$ for some R_1 such that $\mathsf{erase}(R_1) = \mathsf{erase}(M_1)$. Since $\mathsf{erase}(M_1)$ is a value and $\mathsf{erase}(M_1)\, \mathsf{erase}(M_2) \not\longrightarrow_F$, we can find $\mathsf{erase}(M_2) \not\longrightarrow_F$. Thus, by the IH, $\mathsf{erase}(M_2)$ is a value. By Lemma 73, $M_2 \longrightarrow^* R_2$ for some R_2 such that $\mathsf{erase}(R_2) = \mathsf{erase}(M_2)$.

By Lemma 40, $\Delta_1 \vdash R_1 : B \multimap A$ and $\Delta_2 \vdash R_2 : B$. By Lemma 72, R_2 is erasable. By Lemma 81, $\mathsf{erase}(M) = \mathsf{erase}(M_1\, M_2) = \mathsf{erase}(R_1\, R_2) \longrightarrow_F e$ for some e. However, it is contradictory to the assumption that $\mathsf{erase}(M) \not\longrightarrow_F$.

Case (T_BANG), (T_NU), (T_GEN), and (T_TAPP): By the IH.

Case (T_LETBANG): We are given $\Delta_1 + \Delta_2 \vdash \mathsf{let}\, !x = M_1\, \mathsf{in}\, M_2 : A$ for some Δ_1, Δ_2, x, M_1, and M_2 such that $\Delta = \Delta_1 + \Delta_2$ and $M = \mathsf{let}\, !x = M_1\, \mathsf{in}\, M_2$. By inversion, $\Delta_1 \vdash M_1 : !B$ and $\Delta_2, x :^\omega B \vdash M_2 : A$ for some B.

Since $(\lambda x.\mathsf{erase}(M_2))\, \mathsf{erase}(M_1) = \mathsf{erase}(M) \not\longrightarrow_F$, we can find $\mathsf{erase}(M_1) \not\longrightarrow_F$. Thus, by the IH, $\mathsf{erase}(M_1)$ is a value. Thus, we have $\mathsf{erase}(M) = (\lambda x.\mathsf{erase}(M_2))\, \mathsf{erase}(M_1) \longrightarrow_F \mathsf{erase}(M_2)[\mathsf{erase}(M_1)/x]$, which is contradictory to the assumption that $\mathsf{erase}(M) \not\longrightarrow_F$.

Case (T_TABS): Since M is erasable, $M = \Lambda\alpha.R$ for some α and erasable R. By Lemma 65, $\mathsf{erase}(M) = \mathsf{erase}(R)$ is a value in $\lambda_v^\forall$.

□

Lemma 83. *If M is erasable and $\Delta \vdash M : A$, then $\mathsf{erase}(M)$ does not get stuck.*

Proof. Suppose that $\mathsf{erase}(M)$ gets stuck, i.e., there exists some e such that $\mathsf{erase}(M) \longrightarrow_F^* e$ and $e \not\longrightarrow_F$ and e is not a value. By Theorem 2, there exists some M' such that $M \longrightarrow^* M'$ and $\mathsf{erase}(M') = e$. By Lemma 40, $\Delta \vdash M' : A$. By Lemma 72, M' is erasable. Since $\mathsf{erase}(M') = e \not\longrightarrow_F$, we can find e is a value by Lemma 82. However, it is contradictory to the assumption that e is not a value. □

3.4 CPS Transformation for $\lambda_v^\forall$

3.4.1 Type Preservation

Lemma 84. *For any τ, $ftv(\tau) = ftv(\llbracket\tau\rrbracket) = ftv(\llbracket\tau\rrbracket_{\mathsf{v}})$.*

Proof. Straightforward by induction on τ. □

Lemma 85. *For any Θ, $dom(\Theta) = dom(\llbracket\Theta\rrbracket)$.*

Proof. Straightforward by induction on Θ. □

Lemma 86. *If $\vdash \Theta$, then $\vdash \llbracket\Theta\rrbracket$.*

Proof. Straightforward by induction on the derivation of $\vdash \Theta$ with Lemmas 84 and 85. □

Lemma 87. *For any* Θ, $[\![\Theta]\!] = \omega([\![\Theta]\!])$.

Proof. Straightforward by induction on Θ. ☐

Lemma 88. *For any* Θ *and* x, *if* $\Theta(x)$ *is well defined, then* $[\![\Theta(x)]\!]_{\mathsf{v}} = [\![\Theta]\!](x)$.

Proof. Obvious. ☐

Lemma 89. *If* $[\![\Theta \vdash e : \tau]\!] \Rightarrow R$, *then* $\vdash \Theta$ *and* $\Theta \vdash \tau$.

Proof. Straightforward by induction on the derivation of $[\![\Theta \vdash e : \tau]\!] \Rightarrow R$. ☐

Lemma 90. *For any* τ_1, τ_2, *and* α, $[\![\tau_1]\!]_{\mathsf{v}}[[\![\tau_2]\!]_{\mathsf{v}}/\alpha] = [\![\tau_1[\tau_2/\alpha]]\!]_{\mathsf{v}}$.

Proof. By induction on τ_1.

Case $\tau_1 = \beta, \iota$: Obvious.

Case $\tau_1 = \tau_{11} \to \tau_{12}$:

$$\begin{array}{rcl} [\![\tau_1]\!]_{\mathsf{v}}[[\![\tau_2]\!]_{\mathsf{v}}/\alpha] & = & (![\![\tau_{11}]\!]_{\mathsf{v}} \multimap \forall\beta.(![\![\tau_{12}]\!]_{\mathsf{v}} \multimap \beta) \multimap \beta)[[\![\tau_2]\!]_{\mathsf{v}}/\alpha] \quad (\beta \notin \mathit{ftv}(\tau_{12})) \\ & = & !([\![\tau_{11}]\!]_{\mathsf{v}}[[\![\tau_2]\!]_{\mathsf{v}}/\alpha]) \multimap \forall\beta.(!([\![\tau_{12}]\!]_{\mathsf{v}}[[\![\tau_2]\!]_{\mathsf{v}}/\alpha]) \multimap \beta) \multimap \beta \\ & & \quad \text{(since we can suppose } \beta \neq \alpha \text{ and } \beta \notin \mathit{ftv}(\tau_2) = \mathit{ftv}([\![\tau_2]\!]_{\mathsf{v}}) \text{ (Lemma 84) w.l.o.g.)} \\ & = & ![\![\tau_{11}[\tau_2/\alpha]]\!]_{\mathsf{v}} \multimap \forall\beta.(![\![\tau_{12}[\tau_2/\alpha]]\!]_{\mathsf{v}} \multimap \beta) \multimap \beta \quad \text{(by the IHs)} \\ & = & [\![\tau_{11}[\tau_2/\alpha] \to \tau_{12}[\tau_2/\alpha]]\!]_{\mathsf{v}} \\ & = & [\![\tau_1[\tau_2/\alpha]]\!]_{\mathsf{v}} . \end{array}$$

Case $\tau_1 = \forall\beta.\tau_0$: Without loss of generality, we can suppose that $\beta \neq \alpha$ and $\beta \notin \mathit{ftv}(\tau_2) = \mathit{ftv}([\![\tau_2]\!]_{\mathsf{v}})$ (Lemma 84). Then:

$$\begin{array}{rcl} [\![\tau_1]\!]_{\mathsf{v}}[[\![\tau_2]\!]_{\mathsf{v}}/\alpha] & = & (\forall\beta.[\![\tau_0]\!]_{\mathsf{v}})[[\![\tau_2]\!]_{\mathsf{v}}/\alpha] \\ & = & \forall\beta.([\![\tau_0]\!]_{\mathsf{v}}[[\![\tau_2]\!]_{\mathsf{v}}/\alpha]) \\ & = & \forall\beta.[\![\tau_0[\tau_2/\alpha]]\!]_{\mathsf{v}} \quad \text{(by the IH)} \\ & = & [\![\forall\beta.\tau_0[\tau_2/\alpha]]\!]_{\mathsf{v}} \\ & = & [\![\tau_1[\tau_2/\alpha]]\!]_{\mathsf{v}} . \end{array}$$

☐

Lemma 91. *If* $\omega\Gamma, \alpha^{\mathbf{0}} \vdash M : A$, *then* $\omega\Gamma \vdash \Lambda\alpha.M : \forall\alpha.A$.

Proof. By Lemma 5, $\omega\omega\Gamma, \alpha^{\mathbf{0}} \vdash M : A$. By Lemma 17, $\vdash \omega\Gamma$. Thus, by (T_TAbs), we have the conclusion. ☐

Lemma 92. *If* $[\![\Theta]\!], \alpha^{\mathbf{0}} \vdash M : A$, *then* $[\![\Theta]\!] \vdash \Lambda\alpha.M : \forall\alpha.A$.

Proof. By Lemmas 87 and 91. ☐

Lemma 93. *Suppose that* $[\![x : \tau]\!]$ *is well defined. Let* A *be a type obtained by replacing* $\to$ *in* τ *by* $\multimap$. *Then,* $x :^{\omega} A \vdash [\![x : \tau]\!] : ![\![\tau]\!]_{\mathsf{v}}$.

Proof. By induction on the derivation of $[\![x : \tau]\!]$.

Case $[\![x : \iota]\!] = !x$: We have $\tau = A = \iota$. We have the conclusion $x :^{\omega} \iota \vdash !x : !\iota$.

Case $[\![x : \iota \to \tau']\!] = !(\lambda x'.\mathsf{let}\,!y = x' \,\mathsf{in}\, \mathsf{let}\,!z = !(x\,y) \,\mathsf{in}\, \Lambda\alpha.\lambda k.k\,[\![z : \tau']\!])$: We have $\tau = \iota \to \tau'$ and $A = \iota \multimap B$ for some B obtained by replacing $\to$ in τ' by $\multimap$. By the IH,

$$z :^{\omega} B \vdash [\![z : \tau']\!] : ![\![\tau']\!]_{\mathsf{v}} .$$

Thus,

$$x :^{\omega} \iota \multimap B \vdash !(\lambda x'.\mathsf{let}\,!y = x' \,\mathsf{in}\, \mathsf{let}\,!z = !(x\,y) \,\mathsf{in}\, \Lambda\alpha.\lambda k.k\,[\![z : \tau']\!]) : !(!\iota \multimap \forall\alpha.(![\![\tau']\!]_{\mathsf{v}} \multimap \alpha) \multimap \alpha)$$

with Lemmas20 (2) and 91. Since $[\![\tau]\!]_{\mathsf{v}} = [\![\iota \to \tau']\!]_{\mathsf{v}} = !\iota \multimap \forall\alpha.(![\![\tau']\!]_{\mathsf{v}} \multimap \alpha) \multimap \alpha$, we have the conclusion.

□

Lemma 94. *For any c, $\emptyset \vdash [\![c : ty^{\rightarrow}(c)]\!] : ![\![ty^{\rightarrow}(c)]\!]_{\mathsf{v}}$.*

Proof. It is easy to show that $[\![c : ty^{\rightarrow}(c)]\!]$ is well defined by induction on $ty^{\rightarrow}(c)$. By case analysis on $[\![c : ty^{\rightarrow}(c)]\!]$.

Case $[\![c : \iota]\!] = \,!c$: We have $ty^{\rightarrow}(c) = \iota$. We have the conclusion $\emptyset \vdash !c : !\iota$ by (T_Const) and (T_Bang).

Case $[\![c : \iota \rightarrow \tau]\!] = \,!(\lambda x.\mathsf{let}\,!y = x \,\mathsf{in}\, \mathsf{let}\,!z = !(c\, y) \,\mathsf{in}\, \Lambda\alpha.\lambda k.k\, [\![z : \tau]\!])$: We have $ty^{\rightarrow}(c) = \iota \rightarrow \tau$, so $ty(c) = \iota \multimap A$ for some A obtained by replacing $\rightarrow$ in τ by $\multimap$ (Definition 22). By Lemma 93,

$$z :^{\omega} A \vdash [\![z : \tau]\!] : ![\![\tau]\!]_{\mathsf{v}} \ .$$

Thus,

$$\emptyset \vdash !(\lambda x.\mathsf{let}\,!y = x \,\mathsf{in}\, \mathsf{let}\,!z = !(c\, y) \,\mathsf{in}\, \Lambda\alpha.\lambda k.k\, [\![z : \tau]\!]) : !(!\iota \multimap \forall\alpha.(![\![\tau]\!]_{\mathsf{v}} \multimap \alpha) \multimap \alpha) \ .$$

with Lemmas20 (2) and 91. Since $[\![ty^{\rightarrow}(c)]\!]_{\mathsf{v}} = [\![\iota \rightarrow \tau]\!]_{\mathsf{v}} = !\iota \multimap \forall\alpha.(![\![\tau]\!]_{\mathsf{v}} \multimap \alpha) \multimap \alpha$, we have the conclusion.

□

Lemma 95. *If $[\![\Theta \vdash e : \tau]\!] \Rightarrow R$, then $[\![\Theta]\!] \vdash R : [\![\tau]\!]$.*

Proof. By induction on the derivation of $[\![\Theta \vdash e : \tau]\!] \Rightarrow R$.

Case (C_Var): We are given $[\![\Theta \vdash x : \Theta(x)]\!] \Rightarrow \Lambda\alpha.\lambda k.k\, !x$ for some x, k, and α such that $e = x$ and $\tau = \Theta(x)$ and $R = \Lambda\alpha.\lambda k.k\, !x$. By inversion, $\vdash \Theta$, so $\vdash [\![\Theta]\!]$ by Lemma 86. Without loss of generality, we can suppose that $\alpha \notin \mathit{dom}(\Theta) = \mathit{dom}([\![\Theta]\!])$ (Lemma 85).

The conclusion we have to show is

$$[\![\Theta]\!] \vdash \Lambda\alpha.\lambda k.k\, !x : \forall\alpha.(![\![\Theta(x)]\!]_{\mathsf{v}} \multimap \alpha) \multimap \alpha \ .$$

By Lemma 92, it suffices to show that

$$[\![\Theta]\!], \alpha^{\mathbf{0}} \vdash \lambda k.k\, !x : (![\![\Theta(x)]\!]_{\mathsf{v}} \multimap \alpha) \multimap \alpha \ .$$

We have $[\![\Theta]\!] = \omega([\![\Theta]\!])$ by Lemma 87 and $\vdash [\![\Theta]\!], \alpha^{\mathbf{0}}, k :^{\pi} ![\![\Theta(x)]\!]_{\mathsf{v}} \multimap \alpha$ for any π by Lemmas 84 and 85. Thus, the typing rules (T_Var) and (T_Bang) can be applied, and it suffices to show that $[\![\Theta]\!](x) = [\![\Theta(x)]\!]_{\mathsf{v}}$, which is shown by Lemma 88.

Case (C_Const): We are given $[\![\Theta \vdash c : ty^{\rightarrow}(c)]\!] \Rightarrow \Lambda\alpha.\lambda k.k\, [\![c : ty^{\rightarrow}(c)]\!]$ for some c, k, and α. By inversion, $\vdash \Theta$.

The conclusion we have to show is

$$[\![\Theta]\!] \vdash \Lambda\alpha.\lambda k.k\, [\![c : ty^{\rightarrow}(c)]\!] : \forall\alpha.(![\![ty^{\rightarrow}(c)]\!]_{\mathsf{v}} \multimap \alpha) \multimap \alpha \ .$$

By Lemma 94,

$$\emptyset \vdash [\![c : ty^{\rightarrow}(c)]\!] : ![\![ty^{\rightarrow}(c)]\!]_{\mathsf{v}} \ .$$

Thus, we have the conclusion by Lemmas 86, 20 (2), 92, (T_Var), (T_App), and (T_Abs).

Case (C_Abs): We are given $[\![\Theta \vdash \lambda x.e' : \tau_1 \rightarrow \tau_2]\!] \Rightarrow \Lambda\alpha.\lambda k.k\, !(\lambda y.\mathsf{let}\,!x = y \,\mathsf{in}\, R')$ for some x, y, e', R', τ_1, τ_2, α, and k. By inversion, $[\![\Theta, x : \tau_1 \vdash e' : \tau_2]\!] \Rightarrow R'$ and y is fresh.

The conclusion we have to show is

$$[\![\Theta]\!] \vdash \Lambda\alpha.\lambda k.k\, !(\lambda y.\mathsf{let}\,!x = y \,\mathsf{in}\, R') : \forall\alpha.(![\![\tau_1 \rightarrow \tau_2]\!]_{\mathsf{v}} \multimap \alpha) \multimap \alpha \ .$$

By the IH,

$$[\![\Theta]\!], x :^{\omega} [\![\tau_1]\!]_{\mathsf{v}} \vdash R' : [\![\tau_2]\!] \ .$$

By Lemmas 17 and 20 (2),

$$[\![\Theta]\!], y :^{\mathbf{0}} ![\![\tau_1]\!]_{\mathsf{v}}, x :^{\omega} [\![\tau_1]\!]_{\mathsf{v}} \vdash R' : [\![\tau_2]\!] \ .$$

Thus, by (T_Var), (T_LetBang), (T_Abs), and (T_Bang) with $\omega([\![\Theta]\!]) = [\![\Theta]\!]$ by Lemma 87,

$$[\![\Theta]\!] \vdash !(\lambda y.\mathsf{let}\,!x = y \,\mathsf{in}\, R') : !(![\![\tau_1]\!]_{\mathsf{v}} \multimap [\![\tau_2]\!]) = ![\![\tau_1 \rightarrow \tau_2]\!]_{\mathsf{v}} \ .$$

Thus, we have the conclusion by (T_Var), (T_App), (T_Abs), and Lemma 92.

Case (C_App): We are given $\llbracket \Theta \vdash e_1\, e_2 \ :\ \tau \rrbracket \Rightarrow \Lambda\alpha.\lambda k.R_1\, \alpha\, (\lambda x.R_2\, \alpha\, (\lambda y.\mathsf{let}\,!z = x\,\mathsf{in}\, z\, y\, \alpha\, k))$ for some e_1, e_2, R_1, R_2, α, x, y, z, and k. By inversion, $\llbracket \Theta \vdash e_1 \ :\ \tau_0 \to \tau \rrbracket \Rightarrow R_1$ and $\llbracket \Theta \vdash e_2 \ :\ \tau_0 \rrbracket \Rightarrow R_2$ for some τ_0, and x is fresh.

The conclusion we have to show is

$$\llbracket \Theta \rrbracket \vdash \Lambda\alpha.\lambda k.R_1\, \alpha\, (\lambda x.R_2\, \alpha\, (\lambda y.\mathsf{let}\,!z = x\,\mathsf{in}\, z\, y\, \alpha\, k)) : \forall\alpha.(!\llbracket \tau \rrbracket_{\mathsf{v}} \multimap \alpha) \multimap \alpha\ .$$

By the IHs:

$$\llbracket \Theta \rrbracket \vdash R_1 : \forall\alpha.(!\llbracket \tau_0 \to \tau \rrbracket_{\mathsf{v}} \multimap \alpha) \multimap \alpha$$
$$\llbracket \Theta \rrbracket \vdash R_2 : \forall\alpha.(!\llbracket \tau_0 \rrbracket_{\mathsf{v}} \multimap \alpha) \multimap \alpha\ .$$

Since $\llbracket \Theta \rrbracket = \llbracket \Theta \rrbracket + \llbracket \Theta \rrbracket$ by Lemmas 87 and 5, it suffices to show that

$$\llbracket \Theta \rrbracket, \alpha^{\mathbf{0}}, k :^{\mathbf{1}} !\llbracket \tau \rrbracket_{\mathsf{v}} \multimap \alpha, x :^{\mathbf{1}} !\llbracket \tau_0 \to \tau \rrbracket_{\mathsf{v}}, y :^{\mathbf{1}} !\llbracket \tau_0 \rrbracket_{\mathsf{v}} \vdash \mathsf{let}\,!z = x\,\mathsf{in}\, z\, y\, \alpha\, k : \alpha$$

by (T_TApp), (T_Abs), (T_App), and Lemmas 20 (2) and 92. We have

$$\llbracket \tau_0 \to \tau \rrbracket_{\mathsf{v}} = !\llbracket \tau_0 \rrbracket_{\mathsf{v}} \multimap \forall\beta.(!\llbracket \tau \rrbracket_{\mathsf{v}} \multimap \beta) \multimap \beta$$

for some $\beta \notin \mathit{ftv}(\tau) = \mathit{ftv}(\llbracket \tau \rrbracket_{\mathsf{v}})$ (Lemma 84). Thus, we have the derivation of the judgment above by (T_Var), (T_App), (T_TApp), and (T_LetBang).

Case (C_TAbs): We are given $\llbracket \Theta \vdash e \ :\ \forall\beta.\tau_0 \rrbracket \Rightarrow \Lambda\alpha.\lambda k.\nu\beta.\, R'\, \alpha\, (\lambda x.k\, \Lambda^{\circ}\langle\, \beta, x\, \rangle)$ for some β, α, τ_0, k, R', and x. By inversion, $\llbracket \Theta, \beta \vdash e \ :\ \tau_0 \rrbracket \Rightarrow R'$.

The conclusion we have to show is

$$\llbracket \Theta \rrbracket \vdash \Lambda\alpha.\lambda k.\nu\beta.\, R'\, \alpha\, (\lambda x.k\, \Lambda^{\circ}\langle\, \beta, x\, \rangle) : \forall\alpha.(!(\forall\beta.\llbracket \tau_0 \rrbracket_{\mathsf{v}}) \multimap \alpha) \multimap \alpha\ .$$

By the IH, $\llbracket \Theta \rrbracket, \beta^{\mathbf{0}} \vdash R' : \forall\alpha.(!\llbracket \tau_0 \rrbracket_{\mathsf{v}} \multimap \alpha) \multimap \alpha$ where $\alpha \notin \mathit{ftv}(\tau_0) = \mathit{ftv}(\llbracket \tau_0 \rrbracket_{\mathsf{v}})$ (Lemma 84). Since $\llbracket \Theta \rrbracket = \llbracket \Theta \rrbracket + \llbracket \Theta \rrbracket$ by Lemmas 87 and 5, it suffices to show that

$$\llbracket \Theta \rrbracket, \alpha^{\mathbf{0}}, k :^{\mathbf{1}} !(\forall\beta.\llbracket \tau_0 \rrbracket_{\mathsf{v}}) \multimap \alpha, \beta^{\mathbf{1}}, x :^{\mathbf{1}} !\llbracket \tau_0 \rrbracket_{\mathsf{v}} \vdash k\, \Lambda^{\circ}\langle\, \beta, x\, \rangle : \alpha$$

by (T_Abs), (T_Nu), (T_TApp), (T_App), and 20 (2) and 92. In turn, it suffices to show that

$$\llbracket \Theta \rrbracket, \alpha^{\mathbf{0}}, k :^{\mathbf{0}} !(\forall\beta.\llbracket \tau_0 \rrbracket_{\mathsf{v}}) \multimap \alpha, \beta^{\mathbf{1}}, x :^{\mathbf{1}} !\llbracket \tau_0 \rrbracket_{\mathsf{v}} \vdash \Lambda^{\circ}\langle\, \beta, x\, \rangle : !(\forall\beta.\llbracket \tau_0 \rrbracket_{\mathsf{v}})$$

by (T_App). By (T_Var) and (T_Gen), we can derive this judgment.

Case (C_TApp): We are given $\llbracket \Theta \vdash e \ :\ \tau_2[\tau_1/\beta] \rrbracket \Rightarrow \Lambda\alpha.\lambda k.R'\, \alpha\, (\lambda x.\mathsf{let}\,!y = x\,\mathsf{in}\, k\, !(y\, \llbracket \tau_1 \rrbracket_{\mathsf{v}}))$ for some τ_1, τ_2, β, α, k, x, and R'. By inversion, $\llbracket \Theta \vdash e \ :\ \forall\beta.\tau_2 \rrbracket \Rightarrow R'$ and $\Theta \vdash \tau_1$.

The conclusion we have to show is

$$\llbracket \Theta \rrbracket \vdash \Lambda\alpha.\lambda k.R'\, \alpha\, (\lambda x.\mathsf{let}\,!y = x\,\mathsf{in}\, k\, !(y\, \llbracket \tau_1 \rrbracket_{\mathsf{v}})) : \forall\alpha.(!\llbracket \tau_2[\tau_1/\beta] \rrbracket_{\mathsf{v}} \multimap \alpha) \multimap \alpha\ .$$

By the IH, $\llbracket \Theta \rrbracket \vdash R' : \forall\alpha.(!(\forall\beta.\llbracket \tau_2 \rrbracket_{\mathsf{v}}) \multimap \alpha) \multimap \alpha$ where $\alpha \notin \mathit{ftv}(\tau_2) = \llbracket \tau_2 \rrbracket_{\mathsf{v}}$ (Lemma 84). We have $\omega(\llbracket \Theta \rrbracket) = \llbracket \Theta \rrbracket$ by Lemma 87 and $\llbracket \Theta \rrbracket = \llbracket \Theta \rrbracket + \llbracket \Theta \rrbracket$ by Lemma 5. Thus, with Lemma 92, it suffices to show that

$$\llbracket \Theta \rrbracket, \alpha^{\mathbf{0}}, k :^{\mathbf{0}} !\llbracket \tau_2[\tau_1/\beta] \rrbracket_{\mathsf{v}} \multimap \alpha, x :^{\mathbf{0}} !(\forall\beta.\llbracket \tau_2 \rrbracket_{\mathsf{v}}), y :^{\omega} \forall\beta.\llbracket \tau_2 \rrbracket_{\mathsf{v}} \vdash y\, \llbracket \tau_1 \rrbracket_{\mathsf{v}} : \llbracket \tau_2[\tau_1/\beta] \rrbracket_{\mathsf{v}}\ .$$

Since $\Theta \vdash \tau_1$, we have $\llbracket \Theta \rrbracket \vdash \llbracket \tau_1 \rrbracket_{\mathsf{v}}$ by Lemmas 85 and 84. Thus, by (T_Var) and (T_TApp),

$$\llbracket \Theta \rrbracket, \alpha^{\mathbf{0}}, k :^{\mathbf{0}} !\llbracket \tau_2[\tau_1/\beta] \rrbracket_{\mathsf{v}} \multimap \alpha, x :^{\mathbf{0}} !(\forall\beta.\llbracket \tau_2 \rrbracket_{\mathsf{v}}), y :^{\omega} \forall\beta.\llbracket \tau_2 \rrbracket_{\mathsf{v}} \vdash y\, \llbracket \tau_1 \rrbracket_{\mathsf{v}} : \llbracket \tau_2 \rrbracket_{\mathsf{v}}[\llbracket \tau_1 \rrbracket_{\mathsf{v}}/\beta]\ .$$

By Lemma 90, we finish.

□

Lemma 96. *If $\Theta \vdash e : \tau$, then $\llbracket \Theta \vdash e \ :\ \tau \rrbracket \Rightarrow R$ for some R.*

Proof. Straightforward by induction on the derivation of $\Theta \vdash e : \tau$. □

Theorem 3 (Type preservation of CPS transformation for $\lambda_v^{\forall}$)**.** *If $\Theta \vdash e : \tau$, then there exists some R such that $\llbracket \Theta \vdash e \ :\ \tau \rrbracket \Rightarrow R$ and $\llbracket \Theta \rrbracket \vdash R : \llbracket \tau \rrbracket$.*

Proof. By Lemmas 96 and 95. □

3.4.2 Meaning Preservation

Lemma 97. $\mathsf{erase}([\![\chi : \tau]\!]) \Longmapsto^{*}_{\eta_v} (\!|\, \chi : \tau \,|\!)$.

Proof. By induction on τ. There are two cases we have to consider for τ by case analysis on the definition of $[\![\chi : \tau]\!]$.

Case $\tau = \iota$: We have the conclusion by:

$$\mathsf{erase}([\![\chi : \tau]\!]) = \mathsf{erase}(!\chi) = \chi = (\!|\, \chi : \tau \,|\!) \ .$$

Case $\tau = \iota \to \tau'$: We have the conclusion by:

$$\begin{array}{rcl}
\mathsf{erase}([\![\chi : \tau]\!]) & = & \mathsf{erase}(!(\lambda x.\mathsf{let}\,!y = x\,\mathsf{in}\,\mathsf{let}\,!z = !(\chi\, y)\,\mathsf{in}\,\Lambda\alpha.\lambda k.k\,[\![z : \tau']\!])) \quad (k, x, y, z \notin fv\,(\chi)) \\
& = & \lambda x.(\lambda y.(\lambda z.\lambda k.k\,\mathsf{erase}([\![z : \tau']\!]))\,(\chi\, y))\, x \\
& \Longmapsto^{*}_{\eta_v} & \lambda x.(\lambda y.(\lambda z.\lambda k.k\,(\!|\, z : \tau' \,|\!))\,(\chi\, y))\, x \quad \text{(by the IH)} \\
& \Longmapsto_{\eta_v} & \lambda x.(\lambda z.\lambda k.k\,(\!|\, z : \tau' \,|\!))\,(\chi\, x) \\
& = & (\!|\, \chi : \iota \to \tau' \,|\!) \\
& = & (\!|\, \chi : \tau \,|\!) \ .
\end{array}$$

□

Lemma 98. *If* $[\![\Theta \vdash e \ : \ \tau]\!] \Rightarrow R$, *then* $\mathsf{erase}(R) \Longmapsto^{*}_{\beta_v \eta_v} (\!|\, e \,|\!)$.

Proof. By induction on the derivation of $[\![\Theta \vdash e \ : \ \tau]\!] \Rightarrow R$.

Case (C_Var): We are given $[\![\Theta \vdash x \ : \ \Theta(x)]\!] \Rightarrow \Lambda\alpha.\lambda k.k\,!x$ for some x, k, and α. We have the conclusion by:

$$\begin{array}{rcl}
\mathsf{erase}(R) & = & \mathsf{erase}(\Lambda\alpha.\lambda k.k\,!x) \\
& = & \lambda k.k\, x \\
& = & (\!|\, x \,|\!) \ .
\end{array}$$

Case (C_Const): We are given $[\![\Theta \vdash c \ : \ ty^{\to}(c)]\!] \Rightarrow \Lambda\alpha.\lambda k.k\,[\![c : ty^{\to}(c)]\!]$ for some c, α, and k. We have the conclusion by Lemma 97.

Case (C_Abs): We are given $[\![\Theta \vdash \lambda x.e' \ : \ \tau_1 \to \tau_2]\!] \Rightarrow \Lambda\alpha.\lambda k.k\,!(\lambda y.\mathsf{let}\,!x = y\,\mathsf{in}\,R')$ for some x, y, k, e', α, τ_1, τ_2, and R'. By inversion, $[\![\Theta, x : \tau_1 \vdash e' \ : \ \tau_2]\!] \Rightarrow R'$ and y is fresh. We have the conclusion by:

$$\begin{array}{rcl}
\mathsf{erase}(R) & = & \mathsf{erase}(\Lambda\alpha.\lambda k.k\,!(\lambda y.\mathsf{let}\,!x = y\,\mathsf{in}\,R')) \\
& = & \lambda k.k\,(\lambda y.(\lambda x.\mathsf{erase}(R'))\, y) \\
& \Longmapsto^{*}_{\beta_v \eta_v} & \lambda k.k\,(\lambda y.(\lambda x.(\!|\, e' \,|\!))\, y) \quad \text{(by the IH)} \\
& \Longmapsto_{\eta_v} & \lambda k.k\,(\lambda x.(\!|\, e' \,|\!)) \\
& = & (\!|\, \lambda x.e' \,|\!) \ .
\end{array}$$

Case (C_App): We are given $[\![\Theta \vdash e_1\, e_2 \ : \ \tau]\!] \Rightarrow \Lambda\alpha.\lambda k.R_1\,\alpha\,(\lambda x.R_2\,\alpha\,(\lambda y.\mathsf{let}\,!z = x\,\mathsf{in}\,z\,y\,\alpha\,k))$ for some e_1, e_2, k, x, y, z, α, R_1, and R_2. By inversion, $[\![\Theta \vdash e_1 \ : \ \tau_0 \to \tau]\!] \Rightarrow R_1$ and $[\![\Theta \vdash e_2 \ : \ \tau_0]\!] \Rightarrow R_2$ for some τ_0, and x is fresh. We have the conclusion by:

$$\begin{array}{rcl}
\mathsf{erase}(R) & = & \mathsf{erase}(\Lambda\alpha.\lambda k.R_1\,\alpha\,(\lambda x.R_2\,\alpha\,(\lambda y.\mathsf{let}\,!z = x\,\mathsf{in}\,z\,y\,\alpha\,k))) \\
& = & \lambda k.\mathsf{erase}(R_1)\,(\lambda x.\mathsf{erase}(R_2)\,(\lambda y.(\lambda z.z\,y\,k)\,x)) \\
& \Longmapsto^{*}_{\beta_v \eta_v} & \lambda k.(\!|\, e_1 \,|\!)\,(\lambda x.(\!|\, e_2 \,|\!)\,(\lambda y.(\lambda z.z\,y\,k)\,x)) \quad \text{(by the IHs)} \\
& \Longmapsto_{\beta_v} & \lambda k.(\!|\, e_1 \,|\!)\,(\lambda x.(\!|\, e_2 \,|\!)\,(\lambda y.x\,y\,k)) \\
& = & (\!|\, e_1\, e_2 \,|\!) \ .
\end{array}$$

Case (C_TAbs): We are given $[\![\Theta \vdash e \ : \ \forall\beta.\tau']\!] \Rightarrow \Lambda\alpha.\lambda k.\nu\beta.\,R'\,\alpha\,(\lambda x.k\,\Lambda^{\circ}\langle\, \beta, x \,\rangle)$ for some β, α, τ', k, x, and R'. By inversion, $[\![\Theta, \beta \vdash e \ : \ \tau']\!] \Rightarrow R'$. We have the conclusion by:

$$\begin{array}{rcl}
\mathsf{erase}(R) & = & \mathsf{erase}(\Lambda\alpha.\lambda k.\nu\beta.\,R'\,\alpha\,(\lambda x.k\,\Lambda^{\circ}\langle\, \beta, x \,\rangle)) \\
& = & \lambda k.\mathsf{erase}(R')\,(\lambda x.k\,x) \\
& \Longmapsto^{*}_{\beta_v \eta_v} & \lambda k.(\!|\, e \,|\!)\,(\lambda x.k\,x) \quad \text{(by the IH)} \\
& \Longmapsto_{\eta_v} & \lambda k.(\!|\, e \,|\!)\,k \\
& \Longmapsto_{\eta_v} & (\!|\, e \,|\!) \quad \text{(note that } e \text{ is a value)} \ .
\end{array}$$

Case (C_TAPP): We are given $[\![\Theta \vdash e : \tau_2[\tau_1/\beta]]\!] \Rightarrow \Lambda\alpha.\lambda k.R'\,\alpha\,(\lambda x.\mathsf{let}\,!y = x\,\mathsf{in}\,k\,!(y\,[\![\tau_1]\!]_{\mathsf{v}}))$ for some τ_1, τ_2, β, α, k, x, y, and R'. By inversion, $[\![\Theta \vdash e : \forall\beta.\tau_2]\!] \Rightarrow R'$. We have the conclusion by:

$$\begin{array}{rcl} \mathsf{erase}(R) & = & \mathsf{erase}(\Lambda\alpha.\lambda k.R'\,\alpha\,(\lambda x.\mathsf{let}\,!y = x\,\mathsf{in}\,k\,!(y\,[\![\tau_1]\!]_{\mathsf{v}}))) \\ & = & \lambda k.\mathsf{erase}(R')\,(\lambda x.(\lambda y.k\,y)\,x) \\ & \Longmapsto^*_{\beta_v\eta_v} & \lambda k.(\!|\,e\,|\!)\,(\lambda x.(\lambda y.k\,y)\,x) \quad \text{(by the IH)} \\ & \Longmapsto^*_{\eta_v} & (\!|\,e\,|\!) \quad \text{(note that } e \text{ is a value)}\ . \end{array}$$

□

Definition 31. *The function $\Psi(w)$ returns a value in $\lambda_v^{\forall}$, defined as follows:*

$$\begin{array}{rcl} \Psi(c) & \stackrel{\text{def}}{=} & (\!|\,c : ty^{\rightarrow}(c)\,|\!) \\ \Psi(\lambda x.e) & \stackrel{\text{def}}{=} & \lambda x.(\!|\,e\,|\!) \end{array}$$

We write $w \Rightarrow R$ if and only if $\mathsf{erase}(R) \Longmapsto^*_{\beta_v\eta_v} \Psi(w)$.

Corollary 1 (Meaning preservation of $(\!|\cdot|\!)$)**.** *For any e:*

1. *if $e \longrightarrow^*_F w$, then $(\!|\,e\,|\!)\,(\lambda x.x) \longrightarrow^*_F \Psi(w)$; and*
2. *if $(\!|\,e\,|\!)\,(\lambda x.x) \longrightarrow^*_F w'$, then $e \longrightarrow^*_F w$ for some w such that $w' = \Psi(w)$.*

Proof. By the indifference and simulation properties of $(\!|\cdot|\!)$, jointly with the equivalence of the small-step and big-step CBV semantics for $\lambda_v^{\forall}$, all of which have been proven by Plotkin [1]. □

Lemma 99. *If $[\![\Theta \vdash e : \tau]\!] \Rightarrow R$, then R is erasable.*

Proof. Straightforward by induction on the derivation of $[\![\Theta \vdash e : \tau]\!] \Rightarrow R$. □

Theorem 4 (Meaning preservation of CPS transformation for $\lambda_v^{\forall}$)**.** *Suppose that $[\![\emptyset \vdash e : \tau]\!] \Rightarrow R$.*

1. *If $e \longrightarrow^*_F w$, then $R\,![\![\tau]\!]_{\mathsf{v}}\,(\lambda x.x) \longrightarrow^* R'$ for some R' such that $w \Rightarrow R'$.*
2. *If $R\,![\![\tau]\!]_{\mathsf{v}}\,(\lambda x.x) \longrightarrow^* R'$, then $e \longrightarrow^*_F w$ for some w such that $w \Rightarrow R'$.*

Proof.

1. By Corollary 1, $(\!|\,e\,|\!)\,(\lambda x.x) \longrightarrow^*_F \Psi(w)$. By Lemmas 98 and 46,
$$\mathsf{erase}(R)\,(\lambda x.x) \Longmapsto^*_{\beta_v\eta_v} (\!|\,e\,|\!)\,(\lambda x.x)\ .$$
By Lemma 95, $\emptyset \vdash R : \forall\alpha.(![\![\tau]\!]_{\mathsf{v}} \multimap \alpha) \multimap \alpha$ for any α (note that $\emptyset \vdash ![\![\tau]\!]_{\mathsf{v}}$ by Lemmas 89 and 84). Thus, $\emptyset \vdash R\,![\![\tau]\!]_{\mathsf{v}}\,(\lambda x.x) : ![\![\tau]\!]_{\mathsf{v}}$. Further, we can find R erasable by Lemma 99. Thus, $\mathsf{erase}(R\,![\![\tau]\!]_{\mathsf{v}}\,(\lambda x.x)) = \mathsf{erase}(R)\,(\lambda x.x)$ does not get stuck by Lemma 83. By Lemma 63, there exists some e' such that $\mathsf{erase}(R)\,(\lambda x.x) \longrightarrow^*_F e'$ and $e' \Longmapsto^*_{\beta_v\eta_v} \Psi(w)$. Since $\mathsf{erase}(R)\,(\lambda x.x)$ does not get stuck, e' does not either. Thus, by Lemma 64, there exists some w' such that $e' \longrightarrow^*_F w'$ and $w' \Longmapsto^*_{\beta_v\eta_v} \Psi(w)$. That is, $\mathsf{erase}(R)\,(\lambda x.x) \longrightarrow^*_F w'$ and $w' \Longmapsto^*_{\beta_v\eta_v} \Psi(w)$. Since R is erasable, Theorem 2 implies $R\,![\![\tau]\!]_{\mathsf{v}}\,(\lambda x.x) \longrightarrow^* R'$ for some R' such that $\mathsf{erase}(R') = w'$. Since $\mathsf{erase}(R') = w' \Longmapsto^*_{\beta_v\eta_v} \Psi(w)$, we have $w \Rightarrow R'$.
2. We can find R erasable by Lemma 99. Thus, Theorem 2 implies $\mathsf{erase}(R)\,(\lambda x.x) = \mathsf{erase}(R\,![\![\tau]\!]_{\mathsf{v}}\,(\lambda x.x)) \longrightarrow^*_F \mathsf{erase}(R')$. Note that $\mathsf{erase}(R')$ is a value. By Lemmas 98 and 46, $\mathsf{erase}(R)\,(\lambda x.x) \Longmapsto^*_{\beta_v\eta_v} (\!|\,e\,|\!)\,(\lambda x.x)$. By Lemma 53, there exists some e' such that $(\!|\,e\,|\!)\,(\lambda x.x) \longrightarrow^*_F e'$ and $\mathsf{erase}(R') \Longmapsto^*_{\beta_v\eta_v} e'$. Since $\mathsf{erase}(R')$ is a value, $e' = w'$ for some w' by Lemma 54. Thus, $(\!|\,e\,|\!)\,(\lambda x.x) \longrightarrow^*_F w'$. By Corollary 1, $e \longrightarrow^*_F w$ for some w such that $w' = \Psi(w)$. Since $\mathsf{erase}(R') \Longmapsto^*_{\beta_v\eta_v} e' = w' = \Psi(w)$, we have $w \Rightarrow R'$.

□

3.5 Parametricity and Soundness of the Logical Relation with respect to Contextual Equivalence

Lemma 100. *If* $\vdash \Delta$*, then* $\omega\Delta \leq \Delta$*.*

Proof. Straightforward by induction on Δ. □

Lemma 101. *If* $\vdash \Delta$*, then* $\Delta + \omega\Delta \ = \ \Delta$*.*

Proof. Straightforward by induction on Δ. □

Lemma 102. *If* $\Delta_1 \bot \Delta_2$*, then* $\omega(\Delta_1 + \Delta_2) = \omega\Delta_1 = \omega\Delta_2$*.*

Proof. Straightforward by induction on Δ_1. □

Lemma 103. *If* $\Delta_1 \bot \Delta_2$*, then* $\Delta_1 \leq \Delta_1 + \Delta_2$ *and* $\Delta_2 \leq \Delta_1 + \Delta_2$*.*

Proof. Straightforward by induction on Δ_1. □

Lemma 104. *If* $\Delta_1 \leq \Delta_2$ *and* $\Delta_2 \bot \Delta$*, then* $\Delta_1 \bot \Delta$ *and, further,* $\Delta_1 + \Delta \leq \Delta_2 + \Delta$*.*

Proof. It suffices to show that, for any π_1, π_2, and π, if $\pi_1 \leq \pi_2$ and $\pi_2 + \pi \ \neq \ \omega$, then $\pi_1 + \pi \ \neq \ \omega$ and $\pi_1 + \pi \leq \pi_2 + \pi$.

To show the former, suppose that $\pi_1 + \pi \ = \ \omega$. Since $\pi_1 \leq \pi_2$, there exists some π' such that $\pi_1 + \pi' \ = \ \pi_2$. Since $\pi_2 + \pi \ \neq \ \omega$, we have $\pi_1 + \pi' + \pi \ \neq \ \omega$. Since $\pi_1 + \pi \ = \ \omega$, we have $\omega + \pi' \ \neq \ \omega$ with Lemma 1. This is contradictory with the definition of uses. Thus, we have $\pi_1 + \pi \neq \omega$.

Furthermore, since $\pi_1 + \pi' \ = \ \pi_2$ for some π', we have $\pi_1 + \pi \leq \pi_2 + \pi$. □

Lemma 105. *For any* n*,* Δ_1*,* Δ_2*, and* ρ *such that* $dom(\Delta_1) \ = \ dom(\Delta_2)$*,* $\vdash (n, \Delta_1, \rho)$ *if and only if* $\vdash (n, \Delta_2, \rho)$*. In particular, for any* W*,* $\vdash W$ *if and only if* $\vdash \omega W$*.*

Proof. It is straightforward to show the first property. The second property is shown by the first one and Lemma 9. □

Lemma 106. *For any* ρ_1*,* ρ_2*, and* ρ_3*,*

$$\rho_1 \circ (\rho_2 \circ \rho_3) \ = \ (\rho_1 \circ \rho_2) \circ \rho_3 \ .$$

Proof. By:

$$\begin{array}{rcl}
\rho_1 \circ (\rho_2 \circ \rho_3) & = & \rho_1 \uplus \rho_1(\rho_2 \uplus \rho_2(\rho_3)) \\
& = & \rho_1 \uplus \rho_1(\rho_2) \uplus \rho_1(\rho_2(\rho_3)) \\
& = & \rho_1 \uplus \rho_1(\rho_2) \uplus (\rho_1 \uplus \rho_1(\rho_2))(\rho_3) \\
& = & (\rho_1 \circ \rho_2) \uplus (\rho_1 \circ \rho_2)(\rho_3) \\
& = & (\rho_1 \circ \rho_2) \circ \rho_3 \ .
\end{array}$$

□

Lemma 107. $\Delta \gg \Delta$ *for any* Δ*.*

Proof. Because $\Delta + \omega\Delta \ = \ \Delta$ by Lemma 101. □

Lemma 108. *If* $\Delta_1 \gg \Delta_2$*, then* $\Delta_1, \Delta \gg \Delta_2, \Delta$*.*

Proof. Since $\Delta_1 \gg \Delta_2$, there exists some Δ'_1 and Δ'_2 such that $\Delta_1 \ = \ (\Delta_2 + \Delta'_1), \Delta'_2$. By Lemma 101, $\Delta + \omega\Delta \ = \ \Delta$. Thus, $\Delta_1, \Delta = (\Delta_2 + \Delta'_1), \Delta'_2, \Delta = ((\Delta_2, \Delta) + (\Delta'_1, \omega\Delta)), \Delta'_2$. This means that $\Delta_1, \Delta \gg \Delta_2, \Delta$ holds. □

Lemma 109. *If* $\Delta_1 \gg \Delta_2$ *and* $\Delta_2 \gg \Delta_3$*, then* $\Delta_1 \gg \Delta_3$*.*

Proof. Since $\Delta_1 \gg \Delta_2$, there exist some Δ'_1 and Δ'_2 such that $\Delta_1 \ = \ (\Delta_2 + \Delta'_1), \Delta'_2$. Since $\Delta_2 \gg \Delta_3$, there exist some Δ''_1 and Δ''_2 such that $\Delta_2 \ = \ (\Delta_3 + \Delta''_1), \Delta''_2$. Thus, we have

$$\Delta_1 = (\Delta_2 + \Delta'_1), \Delta'_2 = (((\Delta_3 + \Delta''_1), \Delta''_2) + \Delta'_1), \Delta'_2 \ .$$

Since $((\Delta_3 + \Delta''_1), \Delta''_2) \bot \Delta'_1$, there exist some Δ'_{11} and Δ'_{12} such that

- $\Delta'_1 = \Delta'_{11}, \Delta'_{12}$ and
- $((\Delta_3 + \Delta''_1), \Delta''_2) + \Delta'_1 = (\Delta_3 + \Delta''_1 + \Delta'_{11}), (\Delta''_2 + \Delta'_{12})$.

We also have

$$(((\Delta_3 + \Delta''_1), \Delta''_2) + \Delta'_1), \Delta'_2 = ((\Delta_3 + \Delta''_1 + \Delta'_{11}), (\Delta''_2 + \Delta'_{12})), \Delta'_2 = (\Delta_3 + (\Delta''_1 + \Delta'_{11})), (\Delta''_2 + \Delta'_{12}), \Delta'_2 \ .$$

Thus, we have $\Delta_1 \ggcurly \Delta_3$. □

Lemma 110. *If $W_1 \sqsupseteq W_2$ and $W_2 \sqsupseteq W_3$, then $W_1 \sqsupseteq W_3$.*

Proof. $W_1.n \leq W_3.n$ and $\vdash W_1$ and $\vdash W_3$ hold obviously. Since $W_1 \sqsupseteq W_2$, there exists some ρ_{12} such that

- $W_1.\Delta, \dagger(\rho_{12}) \ggcurly W_2.\Delta$,
- $W_1.\rho = \rho_{12} \circ W_2.\rho$, and
- $W_2.\Delta \succ \rho_{12}$.

Since $W_2 \sqsupseteq W_3$, there exists some ρ_{23} such that

- $W_2.\Delta, \dagger(\rho_{23}) \ggcurly W_3.\Delta$,
- $W_2.\rho = \rho_{23} \circ W_3.\rho$, and
- $W_3.\Delta \succ \rho_{23}$.

Let $\rho_{123} = \rho_{12} \circ \rho_{23}$. We have the conclusion by the following.

- Since $W_1.\rho = \rho_{12} \circ W_2.\rho$ and $W_2.\rho = \rho_{23} \circ W_3.\rho$, we have $W_1.\rho = \rho_{123} \circ W_3.\rho$ by Lemma 106.
- By Lemmas 108 and 109, $W_1.\Delta, \dagger(\rho_{123}) = W_1.\Delta, \dagger(\rho_{12}), \dagger(\rho_{23}) \ggcurly W_2.\Delta, \dagger(\rho_{23}) \ggcurly W_3.\Delta$
- We show that $W_3.\Delta \succ \rho_{123}$. Let $\alpha \in \mathit{ftv}(\rho_{123}|_{\mathit{dom}(W_3.\Delta)}) \cap \mathit{dom}(W_3.\Delta)$.

  We first show that $\alpha \in \mathit{dom}(W_2.\Delta)$. Since $W_2.\Delta, \dagger(\rho_{23}) \ggcurly W_3.\Delta$ and $\alpha \in \mathit{dom}(W_3.\Delta)$, we have $\alpha \in \mathit{dom}(W_2.\Delta) \cup \mathit{dom}(\rho_{23})$. To show $\alpha \in \mathit{dom}(W_2.\Delta)$, it suffices to prove that $\alpha \notin \mathit{dom}(\rho_{23})$. $\vdash W_1$ implies $\mathit{dom}(W_1.\Delta) \# \mathit{dom}(W_1.\rho)$. We have $\alpha \in \mathit{dom}(W_1.\Delta)$ since $\alpha \in \mathit{ftv}(\rho_{123}|_{\mathit{dom}(W_3.\Delta)})$ and $W_1.\rho = \rho_{123} \circ W_3.\rho$ and $\vdash W_1$. Thus, $\alpha \notin \mathit{dom}(W_1.\rho) \supseteq \mathit{dom}(\rho_{23})$.

  Next, we show that $\alpha \in \mathit{ftv}(\rho_{12}|_{\mathit{dom}(W_2.\Delta)}) \cup \mathit{ftv}(\rho_{23}|_{\mathit{dom}(W_3.\Delta)})$. Since $\alpha \in \mathit{ftv}(\rho_{123}|_{\mathit{dom}(W_3.\Delta)})$, we can find that $\alpha \in \mathit{ftv}(\rho_{12}|_{\mathit{dom}(W_3.\Delta)})$ or $\alpha \in \mathit{ftv}(\rho_{12}(\rho_{23})|_{\mathit{dom}(W_3.\Delta)})$.

  Case $\alpha \in \mathit{ftv}(\rho_{12}|_{\mathit{dom}(W_3.\Delta)})$: Since $W_2.\Delta, \dagger(\rho_{23}) \ggcurly W_3.\Delta$, we have $\mathit{dom}(W_3.\Delta) \subseteq \mathit{dom}(W_2.\Delta) \cup \mathit{dom}(\rho_{23})$. Thus, $\alpha \in \mathit{ftv}(\rho_{12}|_{\mathit{dom}(W_2.\Delta) \cup \mathit{dom}(\rho_{23})})$. Since $W_1.\rho = \rho_{12} \circ W_2.\rho = \rho_{12} \circ (\rho_{23} \circ W_3.\rho)$, we have $\mathit{dom}(\rho_{12}) \# \mathit{dom}(\rho_{23})$. Thus, $\alpha \in \mathit{ftv}(\rho_{12}|_{\mathit{dom}(W_2.\Delta)})$.

  Case $\alpha \in \mathit{ftv}(\rho_{12}(\rho_{23})|_{\mathit{dom}(W_3.\Delta)})$: We can find $\alpha \in \mathit{ftv}(\rho_{23}|_{\mathit{dom}(W_3.\Delta)})$ or $\alpha \in \mathit{ftv}(\rho_{12}|_{\mathit{ftv}(\rho_{23}|_{\mathit{dom}(W_3.\Delta)})})$.

  Case $\alpha \in \mathit{ftv}(\rho_{23}|_{\mathit{dom}(W_3.\Delta)})$: It is what we have to prove.

  Case $\alpha \in \mathit{ftv}(\rho_{12}|_{\mathit{ftv}(\rho_{23}|_{\mathit{dom}(W_3.\Delta)})})$: Since $\vdash W_2$, we have $\mathit{ftv}(\rho_{23}|_{\mathit{dom}(W_3.\Delta)}) \subseteq \mathit{ftv}(\rho_{23}) \subseteq \mathit{dom}(W_2.\Delta)$. Thus, $\alpha \in \mathit{ftv}(\rho_{12}|_{\mathit{dom}(W_2.\Delta)})$.

  We show that $\alpha^{\mathbf{0}} \in W_3.\Delta$.

  If $\alpha \in \mathit{ftv}(\rho_{12}|_{\mathit{dom}(W_2.\Delta)})$, then $W_2.\Delta \succ \rho_{12}$ and $\alpha \in \mathit{dom}(W_2.\Delta)$ implies $\alpha^{\mathbf{0}} \in W_2.\Delta$. Since $W_2.\Delta, \dagger(\rho_{23}) \ggcurly W_3.\Delta$ and $\alpha \in \mathit{dom}(W_3.\Delta)$, we have $\alpha^{\mathbf{0}} \in W_3.\Delta$.

  Otherwise, if $\alpha \in \mathit{ftv}(\rho_{23}|_{\mathit{dom}(W_3.\Delta)})$, then $W_3.\Delta \succ \rho_{23}$ and $\alpha \in \mathit{dom}(W_3.\Delta)$ implies $\alpha^{\mathbf{0}} \in W_3.\Delta$.

□

Lemma 111. *If $\vdash W$, then $W \sqsupseteq W$.*

Proof. Obvious by letting $\rho = \emptyset$; note that $W.\Delta \ggcurly W.\Delta$ by Lemma 107. □

Lemma 112. *If $W_1 \sqsupseteq W_2$, then $\omega W_1 \sqsupseteq \omega W_2$.*

Proof. Since $W_1 \sqsupseteq W_2$, there exist some ρ such that

- $W_1.\Delta, \dagger(\rho) \ggcurly W_2.\Delta$,
- $W_1.\rho = \rho \circ W_2.\rho$, and
- $W_2.\Delta \succ \rho$.

Since $\omega W_i = (W_i.n, \omega(W_i.\Delta), W_i.\rho)$ for $i \in \{1, 2\}$, $W_1 \sqsupseteq W_2$ implies:

- $\vdash \omega W_1$ and $\vdash \omega W_2$ by Lemma 105;
- $\omega W_1.n \leq \omega W_2.n$; and
- $\omega W_1.\rho = \rho \circ \omega W_2.\rho$

$\omega W_2.\Delta \succ \rho$ holds obviously.

Thus, it suffices to show that

$$\omega(W_1.\Delta), \dagger(\rho) \ggcurly \omega(W_2.\Delta) \ .$$

Since $(W_1.\Delta, \dagger(\rho)) \ggcurly W_2.\Delta$, there exist some Δ and Δ_0 such that $W_1.\Delta, \dagger(\rho) = (W_2.\Delta + \Delta), \Delta_0$. Since $\omega(W_1.\Delta)$ assigns the use $\mathbf{0}$ to all the type variables, $\omega(W_1.\Delta), \dagger(\rho) = \omega(W_1.\Delta, \dagger(\rho)) = \omega((W_2.\Delta + \Delta), \Delta_0) = (\omega(W_2.\Delta) + \omega\Delta) + \omega\Delta_0$. Thus, we have the conclusion. ☐

Lemma 113. *If $\vdash W$ and $(M_1, M_2) \in Atom[W, A]$, then $M_1 = W.\rho_{fst}(M_1)$ and $M_2 = W.\rho_{snd}(M_2)$.*

Proof. Since $(M_1, M_2) \in \mathrm{Atom}[W, A]$, we have $W.\Delta \vdash M_1 : W.\rho_{\mathrm{fst}}(A)$ and $W.\Delta \vdash M_2 : W.\rho_{\mathrm{snd}}(A)$. Since $\vdash W$, we have $dom(W.\Delta) \# dom(W.\rho)$. Thus, the type variables in $dom(W.\rho)$ do not occur free in M_1 and M_2, so we have the conclusion. ☐

Lemma 114. *Suppose that $W_1 \sqsupseteq W_2$.*

- *For any α, $W_1.\rho_{fst}(\alpha) = W_1.\rho_{fst}(W_2.\rho_{fst}(\alpha))$ and $W_1.\rho_{snd}(\alpha) = W_1.\rho_{snd}(W_2.\rho_{snd}(\alpha))$.*
- *$(M_1, M_2) \in Atom[W_2, A]$ implies $(M_1, M_2)_{W_1} \in Atom[W_1, A]$.*

Proof.

- We show only $W_1.\rho_{\mathrm{fst}}(\alpha) = W_1.\rho_{\mathrm{fst}}(W_2.\rho_{\mathrm{fst}}(\alpha))$; the other equation is shown similarly.

  Obvious if $\alpha \notin dom(W_2.\rho)$.

  Suppose that $\alpha \in dom(W_2.\rho)$. Since $W_1 \sqsupseteq W_2$, there exist some ρ such that $W_1.\rho = \rho \circ W_2.\rho$. Since $\alpha \in dom(W_2.\rho)$, we have $W_1.\rho_{\mathrm{fst}}(\alpha) = \rho_{\mathrm{fst}}(W_2.\rho_{\mathrm{fst}}(\alpha))$. Since $W_1 \sqsupseteq W_2$ implies $\vdash W_2$, we have $dom(W_2.\Delta) \# dom(W_2.\rho)$ and $W_2.\Delta \vdash W_2.\rho_{\mathrm{fst}}(\alpha)$. Thus, $W_2.\rho_{\mathrm{fst}}(\alpha) = W_2.\rho_{\mathrm{fst}}(W_2.\rho_{\mathrm{fst}}(\alpha))$. Hence

  $$W_1.\rho_{\mathrm{fst}}(\alpha) = \rho_{\mathrm{fst}}(W_2.\rho_{\mathrm{fst}}(\alpha)) = \rho_{\mathrm{fst}}(W_2.\rho_{\mathrm{fst}}(W_2.\rho_{\mathrm{fst}}(\alpha))) = W_1.\rho_{\mathrm{fst}}(W_2.\rho_{\mathrm{fst}}(\alpha)) \ .$$

- Suppose that $(M_1, M_2) \in \mathrm{Atom}[W_2, A]$. We show only $W_1.\Delta \vdash W_1.\rho_{\mathrm{fst}}(M_1) : W_1.\rho_{\mathrm{fst}}(A)$; the other judgment can be shown similarly.

  By definition, $W_2.\Delta \vdash M_1 : W_2.\rho_{\mathrm{fst}}(A)$. Since $W_1 \sqsupseteq W_2$, there exists some ρ such that:

  - $W_1.\Delta, \dagger(\rho) \ggcurly W_2.\Delta$;
  - $W_1.\rho = \rho \circ W_2.\rho$; and
  - $W_2.\Delta \succ \rho$.

  Since $W_1.\Delta, \dagger(\rho) \ggcurly W_2.\Delta$, there exist some Δ'_1 and Δ'_2 such that $W_1.\Delta, \dagger(\rho) = (W_2.\Delta + \Delta'_1), \Delta'_2$. Further, there exist some Δ_{21}, Δ_{22}, Δ'_{11}, Δ'_{21}, and Δ'_{22} such that

  - $W_2.\Delta = \Delta_{21}, \omega\Delta_{22}$,

– $\Delta'_1 = \Delta'_{11}, \omega\Delta_{22}$,
– $\Delta'_2 = \Delta'_{21}, \omega\Delta'_{22}$,
– $W_1.\Delta = (\Delta_{21} + \Delta'_{11}), \Delta'_{21}$, and
– $\dagger(\rho) = \omega\Delta_{22}, \omega\Delta'_{22}$.

Since $\dagger(\rho) = \omega\Delta_{22}, \omega\Delta'_{22}$, we can take ρ_1 and ρ_2 such that

– $\rho = \rho_1 \uplus \rho_2$,
– $dom(\rho_1) = dom(\omega\Delta_{22})$, and
– $dom(\rho_2) = dom(\omega\Delta'_{22})$.

Let Δ_{211} and Δ_{212} be typing contexts such that

– $\Delta_{21} = \Delta_{211}, \omega\Delta_{212}$ and
– $\forall\,\alpha \in dom(\Delta_{211}).\ \alpha^{\mathbf{1}} \in \Delta_{211}$.

Since $W_2.\Delta \vdash M_1 : W_2.\rho_{\mathrm{fst}}(A)$, we have $W_2.\Delta, \omega\Delta'_{21} \vdash M_1 : W_2.\rho_{\mathrm{fst}}(A)$ by Lemma 20. Since $W_2.\Delta = \Delta_{21}, \omega\Delta_{22} = \Delta_{211}, \omega\Delta_{212}, \omega\Delta_{22}$, we have

$$\Delta_{211}, \omega\Delta_{212}, \omega\Delta_{22}, \omega\Delta'_{21} \vdash M_1 : W_2.\rho_{\mathrm{fst}}(A)\ . \tag{3}$$

Since $W_1 \sqsupseteq W_2$ implies $\vdash W_1$, we have $\forall\,\alpha \in dom(\rho_1).\ W_1.\Delta \vdash {\rho_1}_{\mathrm{fst}}(\alpha)$. Since $W_1.\Delta = (\Delta_{21} + \Delta'_{11}), \Delta'_{21} = ((\Delta_{211}, \omega\Delta_{212}) + \Delta'_{11}), \Delta'_{21}$, we have

$$\forall\,\alpha \in dom(\rho_1).\ \Delta_{211}, \omega\Delta_{212}, \omega\Delta'_{21} \vdash {\rho_1}_{\mathrm{fst}}(\alpha)\ .$$

Since $W_2.\Delta \succ \rho$ and $dom(\rho_1) = dom(\omega\Delta_{22}) \subseteq dom(W_2.\Delta)$, we have $\forall\,\alpha \in ftv(\rho_1) \cap dom(W_2.\Delta).\ \alpha^{\mathbf{0}} \in W_2.\Delta$. Since

– $W_2.\Delta = \Delta_{211}, \omega\Delta_{212}, \omega\Delta_{22}$ and
– $\forall\,\alpha \in dom(\Delta_{211}).\ \alpha^{\mathbf{1}} \in \Delta_{211} \subseteq W_2.\Delta$,

we have $ftv(\rho_1) \cap dom(\Delta_{211}) = \emptyset$. Thus,

$$\forall\,\alpha \in dom(\rho_1).\ \omega\Delta_{212}, \omega\Delta'_{21} \vdash {\rho_1}_{\mathrm{fst}}(\alpha)\ . \tag{4}$$

By Lemma 34 with the judgments (3) and (4), noting $dom(\rho_1) = dom(\omega\Delta_{22})$, we have

$$\Delta_{211}, \omega\Delta_{212}, \omega\Delta'_{21} \vdash {\rho_1}_{\mathrm{fst}}(M_1) : {\rho_1}_{\mathrm{fst}}(W_2.\rho_{\mathrm{fst}}(A))\ .$$

Since $dom(\rho_2) = dom(\omega\Delta'_{22})$, we have $dom(\rho_2)\#(dom(\Delta_{211}, \omega\Delta_{212}, \omega\Delta'_{21}))$. Thus, noting $\Delta_{21} = \Delta_{211}, \omega\Delta_{212}$, we have

$$\Delta_{21}, \omega\Delta'_{21} \vdash \rho_{\mathrm{fst}}(M_1) : \rho_{\mathrm{fst}}(W_2.\rho_{\mathrm{fst}}(A))\ .$$

Since $W_1.\Delta = (\Delta_{21} + \Delta'_{11}), \Delta'_{21}$, we have

$$W_1.\Delta \vdash \rho_{\mathrm{fst}}(M_1) : \rho_{\mathrm{fst}}(W_2.\rho_{\mathrm{fst}}(A))$$

by Lemma 25. Since $(M_1, M_2) \in \mathrm{Atom}\,[W_2, A]$ and $\vdash W_2$ (implied by $W_1 \sqsupseteq W_2$), we have

$$W_1.\Delta \vdash \rho_{\mathrm{fst}}(W_2.\rho_{\mathrm{fst}}(M_1)) : \rho_{\mathrm{fst}}(W_2.\rho_{\mathrm{fst}}(A))$$

by Lemma 113. Since $W_1.\rho = \rho \circ W_2.\rho$, we have

$$W_1.\Delta \vdash W_1.\rho_{\mathrm{fst}}(M_1) : W_1.\rho_{\mathrm{fst}}(A)\ ,$$

which is what is required to show.

□

Lemma 115. *If $W_1 \sqsupseteq W_2$ and $(R_1, R_2) \in W_2.\rho[\alpha](\blacktriangleright W_2)$, then $(R_1, R_2)_{W_1} \in W_1.\rho[\alpha](\blacktriangleright W_1)$.*

Proof. Since $W_1 \sqsupseteq W_2$, there exists some ρ such that $W_1.\rho = \rho \circ W_2.\rho$. Let $(A_1, A_2, r) = W_2.\rho(\alpha)$, $B_1 = \rho_{\mathrm{fst}}(A_1)$, and $B_2 = \rho_{\mathrm{snd}}(A_2)$. By definition, $W_1.\rho(\alpha) = (B_1, B_2, r)$. $W_1 \sqsupseteq W_2$ implies $\blacktriangleright W_1 \sqsupseteq \blacktriangleright W_2$. Since $(R_1, R_2) \in r(\blacktriangleright W_2)$, monotonicity of r implies $(R_1, R_2)_{W_1} \in r(\blacktriangleright W_1)$. Thus, we have the conclusion. □

Lemma 116. *Suppose that $W_1 \sqsupseteq W_2$.*

1. *If $(R_1, R_2) \in \mathcal{R}[\![A]\!]\, W_2$, then $(R_1, R_2)_{W_1} \in \mathcal{R}[\![A]\!]\, W_1$.*
2. *If $(M_1, M_2) \in \mathcal{E}[\![A]\!]\, W_2$, then $(M_1, M_2)_{W_1} \in \mathcal{E}[\![A]\!]\, W_1$.*

Proof. By induction on A. Note that $(R_1, R_2) \in \mathrm{Atom}\,[W_2, A]$ implies $(R_1, R_2)_{W_1} \in \mathrm{Atom}\,[W_1, A]$ by Lemma 114. We first show the first property and then the second property by assuming that the first holds.

1. We first consider $(R_1, R_2) \in \mathcal{R}[\![A]\!]\, W_2$ implies $(R_1, R_2)_{W_1} \in \mathcal{R}[\![A]\!]\, W_1$. We proceed by case analysis on A.

   Case $A = \iota$: Obvious.

   Case $A = \alpha$: Let $(R_1, R_2) \in \mathcal{R}[\![\alpha]\!]\, W_2$. By definition, $(R_1, R_2) \in W_2.\rho[\alpha](\blacktriangleright W_2)$. By Lemma 115, we have $(R_1, R_2)_{W_1} \in W_1.\rho[\alpha](\blacktriangleright W_1)$. Thus, $(R_1, R_2)_{W_1} \in \mathcal{R}[\![\alpha]\!]\, W_1$.

   Case $A = B \multimap C$: Let $(R_1, R_2) \in \mathcal{R}[\![B \multimap C]\!]\, W_2$. It suffices to show that $(R_1, R_2)_{W_1} \in \mathcal{R}[\![B \multimap C]\!]\, W_1$, that is, for any $W' \sqsupseteq W_1$, $(W_1', W_2') \ni W'$, R_1', and R_2' such that

   - $W_1' \sqsupseteq W_1$ and
   - $(R_1', R_2') \in \mathcal{R}[\![B]\!]\, W_2'$,

   it suffices to show that
   $$(R_1\, R_1', R_2\, R_2')_{W'} \in \mathcal{E}[\![C]\!]\, W' \ .$$
   Since $W' \sqsupseteq W_1 \sqsupseteq W_2$ and $W_1' \sqsupseteq W_1 \sqsupseteq W_2$, we have $W' \sqsupseteq W_2$ and $W_1' \sqsupseteq W_2$ by Lemma 110. Since further

   - $(R_1, R_2) \in \mathcal{R}[\![B \multimap C]\!]\, W_2$,
   - $(W_1', W_2') \ni W'$, and
   - $(R_1', R_2') \in \mathcal{R}[\![B]\!]\, W_2'$,

   we have the conclusion.

   Case $A = \forall \alpha.B$: Let $(R_1, R_2) \in \mathcal{R}[\![\forall \alpha.B]\!]\, W_2$. It suffices to show that $(R_1, R_2)_{W_1} \in \mathcal{R}[\![\forall \alpha.B]\!]\, W_1$, that is, for any $W' \sqsupseteq W_1$, and C_1, C_2, and r such that $\omega\, W' \vdash (C_1, C_2, r)$ and $\{\alpha\} \# \omega\, W'$, it suffices to show that
   $$(R_1\, C_1, R_2\, C_2)_{\omega\, W'} \in \mathcal{E}[\![B]\!]\, \{\alpha \Mapsto (C_1, C_2, r)\} \uplus \omega\, W' \ .$$
   Since $W' \sqsupseteq W_1$ and $W_1 \sqsupseteq W_2$, we have $W' \sqsupseteq W_2$ by Lemma 110. Since further

   - $(R_1, R_2) \in \mathcal{R}[\![\forall \alpha.B]\!]\, W_2$,
   - $\omega\, W' \vdash (C_1, C_2, r)$, and
   - $\{\alpha\} \# \omega\, W'$,

   we have the conclusion.

   Case $A = !B$: By the IH with Lemma 112.

2. Let $(M_1, M_2) \in \mathcal{E}[\![A]\!]\, W_2$. We show $(M_1, M_2)_{W_1} \in \mathcal{E}[\![A]\!]\, W_1$. Suppose that $W' \sqsupseteq W_1$ and $W'.\rho_{\mathrm{fst}}(W_1.\rho_{\mathrm{fst}}(M_1)) \longrightarrow^n R_1$ for some W', $n < W'.n$, and R_1. Then, it suffices to show that there exist some R_2 such that

   - $W'.\rho_{\mathrm{snd}}(W_1.\rho_{\mathrm{snd}}(M_2)) \longrightarrow^* R_2$ and
   - $(R_1, R_2) \in \mathcal{E}[\![A]\!]\, (W' - n)$

   Since $W' \sqsupseteq W_1$, we have $W'.\rho_{\mathrm{fst}}(W_1.\rho_{\mathrm{fst}}(M_1)) = W'.\rho_{\mathrm{fst}}(M_1)$ and $W'.\rho_{\mathrm{snd}}(W_1.\rho_{\mathrm{snd}}(M_1)) = W'.\rho_{\mathrm{snd}}(M_1)$ by Lemma 114. Since $W_1 \sqsupseteq W_2$, we have $W' \sqsupseteq W_2$ by Lemma 110. Thus, since $(M_1, M_2) \in \mathcal{E}[\![A]\!]\, W_2$, we have the conclusion.

$\square$

Lemma 117. *For any W and Δ, if $\vdash W$ and $dom(\Delta) \# W$, then $(W.n, (W.\Delta, \Delta), W.\rho) \sqsupseteq W$.*

Proof. Obvious because $W.\Delta, \Delta \gg W.\Delta$ and $\vdash (W.n, (W.\Delta, \Delta), W.\rho)$ from $\vdash W$ and $dom(\Delta) \# W$. $\square$

Lemma 118. *For any W and Δ, if $\vdash W$ and $W.\Delta \leq \Delta$, then $(W.n, \Delta, W.\rho) \sqsupseteq W$.*

Proof. Obvious because $\Delta \gg W.\Delta$ from $W.\Delta \leq \Delta$, and $\vdash (W.n, \Delta, W.\rho)$ from $\vdash W$ with Lemma 14. $\square$

Lemma 119. *If $W_1 \sqsupseteq W_2$ and $(W_2, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$, then $(W_1, W_1.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma]\!]$. Furthermore, if $W_1.\rho = W_2.\rho$, then $\varsigma = W_1.\rho(\varsigma)$.*

Proof. Since $(W_2, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$, we have

- $\vdash W_2$,
- $\Gamma \succ W_2.\rho$, and
- there exist some Δ and $\prod_{x \in dom_{=\mathbf{1}}(\Gamma)} \Delta_x$ such that:
  - $W_2.\Delta = \Delta + \sum_{x \in dom_{=\mathbf{1}}(\Gamma)} \Delta_x$;
  - for any $\alpha^\pi \in \Gamma$, $\exists \pi' \geq \pi.\ \alpha^{\pi'} \in \Delta$ or $\pi = \mathbf{0} \wedge \alpha \in dom(W_2.\rho)$;
  - for any $x :^{\mathbf{1}} A \in \Gamma$, $(\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\,(W_2.n,\ \Delta_x,\ W_2.\rho)$; and
  - for any $x :^{\omega} A \in \Gamma$, $(\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\,\omega\, W_2$.

Since $W_1 \sqsupseteq W_2$, there exists some ρ such that

- $W_1.\rho = \rho \circ W_2.\rho$,
- $W_1.\Delta, \dagger(\rho) \gg W_2.\Delta$, and
- $W_2.\Delta \succ \rho$.

Thus, there exist some Δ' and Δ'_0 such that $W_1.\Delta, \dagger(\rho) = (W_2.\Delta + \Delta'), \Delta'_0$. Since $W_2.\Delta = \Delta + \sum_{x \in dom_{=\mathbf{1}}(\Gamma)} \Delta_x$, there exist some Δ_1, Δ_2, $\prod_{x \in dom_{=\mathbf{1}}(\Gamma)} \Delta_{1,x}$, Δ'_1, Δ'_{01}, and Δ'_{02} such that

- $\Delta = \Delta_1, \omega\Delta_2$,
- $\Delta_x = \Delta_{1,x}, \omega\Delta_2$ for any $x \in dom_{=\mathbf{1}}(\Gamma)$,
- $\Delta' = \Delta'_1, \omega\Delta_2$,
- $\Delta'_0 = \Delta'_{01}, \omega\Delta'_{02}$,
- $W_1.\Delta = (\Delta_1 + \sum_{x \in dom_{=\mathbf{1}}(\Gamma)} \Delta_{1,x} + \Delta'_1), \Delta'_{01}$, and
- $\dagger(\rho) = \omega\Delta_2, \omega\Delta'_{02}$.

Let $\Delta_3 = (\Delta_1 + \Delta'_1), \Delta'_{01}$ and $\Delta_{3,x} = \Delta_{1,x}, \omega\Delta'_{01}$ for $x \in dom_{=\mathbf{1}}(\Gamma)$. Further, let $\varsigma_1 = W_1.\rho(\varsigma)$. We show $(W_1, \varsigma_1) \in \mathcal{G}[\![\Gamma]\!]$ in what follows.

- We have $\vdash W_1$ from $W_1 \sqsupseteq W_2$.
- We have $W_1.\Delta = \Delta_3 + \sum_{x \in dom_{=\mathbf{1}}(\Gamma)} \Delta_{3,x}$.
- We show that $\Gamma \succ W_1.\rho$, i.e., let $\alpha \in ftv(W_1.\rho|_{dom(\Gamma)}) \cap dom(\Gamma)$ and then show that $\alpha^{\mathbf{0}} \in \Gamma$.

  We first show that $\alpha \in dom(W_2.\Delta)$ by contradiction. Suppose that $\alpha \notin dom(W_2.\Delta)$. Since $\alpha \in dom(\Gamma)$, $(W_2, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$ implies $\alpha \in dom(W_2.\rho)$. $\vdash W_1$ and $dom(W_1.\rho) = dom(\rho) \cup dom(W_2.\rho)$ implies $dom(W_1.\Delta) \# (dom(\rho) \cup dom(W_2.\rho))$. $\vdash W_1$ and $\alpha \in ftv(W_1.\rho)$ implies $\alpha \in dom(W_1.\Delta)$. Thus, there is a contradiction.

  Next, we show that $\alpha \in ftv(\rho|_{dom(W_2.\Delta)}) \cup ftv(W_2.\rho|_{dom(\Gamma)})$. Since $\alpha \in ftv(W_1.\rho|_{dom(\Gamma)})$ and $W_1.\rho = \rho \circ W_2.\rho$, we have $\alpha \in ftv(\rho|_{dom(\Gamma)}) \cup ftv(\rho(W_2.\rho)|_{dom(\Gamma)})$.

Case $\alpha \in ftv(\rho|_{dom(\Gamma)})$: $(W_2, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$ implies $\forall \alpha \in dom(\Gamma).\ \alpha \in dom(W_2.\Delta) \cup dom(W_2.\rho)$. Thus, $\alpha \in ftv(\rho|_{dom(W_2.\Delta) \cup dom(W_2.\rho)})$. Since $W_1.\rho = \rho \circ W_2.\rho$ is well defined, we have $dom(\rho) \# dom(W_2.\rho)$. Thus, $\alpha \in ftv(\rho|_{dom(W_2.\Delta)})$.

Case $\alpha \in ftv(\rho(W_2.\rho)|_{dom(\Gamma)})$: We have $\alpha \in ftv(W_2.\rho|_{dom(\Gamma)}) \cup ftv(\rho|_{ftv(W_2.\rho|_{dom(\Gamma)})})$.

Case $\alpha \in ftv(W_2.\rho|_{dom(\Gamma)})$: We have what is required to prove.

Case $\alpha \in ftv(\rho|_{ftv(W_2.\rho|_{dom(\Gamma)})})$: $\vdash W_2$ implies $ftv(W_2.\rho) \subseteq dom(W_2.\Delta)$. Thus, $\alpha \in ftv(\rho|_{dom(W_2.\Delta)})$.

We show that $\alpha^{\mathbf{0}} \in \Gamma$.

If $\alpha \in ftv(\rho|_{dom(W_2.\Delta)})$, then $W_2.\Delta \succ \rho$ and $\alpha \in dom(W_2.\Delta)$ implies $\alpha^{\mathbf{0}} \in W_2.\Delta$. $\vdash W_2$ implies $\alpha \notin dom(W_2.\rho)$. Let $\alpha^{\pi} \in \Gamma$. $(W_2, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$ implies $\alpha^{\pi'} \in \Delta$ for some $\pi' \geq \pi$. Since $\Delta \leq W_2.\Delta$ and $\alpha^{\mathbf{0}} \in W_2.\Delta$, we have $\pi' = \mathbf{0}$. Since $\mathbf{0} = \pi' \geq \pi$, we have $\pi = \mathbf{0}$.

Otherwise, if $\alpha \in ftv(W_2.\rho|_{dom(\Gamma)})$, then $\Gamma \succ W_2.\rho$ and $dom(\Gamma)$ implies $\alpha^{\mathbf{0}} \in \Gamma$.

- For $\alpha^{\pi} \in \Gamma$, suppose that $\forall \pi' \geq \pi.\ \alpha^{\pi'} \notin \Delta_3$. We show that $\pi = \mathbf{0} \wedge \alpha \in dom(W_1.\rho)$.

  Since $\alpha^{\pi} \in \Gamma$, we can perform case analysis on $\exists \pi' \geq \pi.\ \alpha^{\pi'} \in \Delta$ or $\pi = \mathbf{0} \wedge \alpha \in dom(W_2.\rho)$ (which is implied by $(W_2, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$).

  Case $\exists \pi' \geq \pi.\ \alpha^{\pi'} \in \Delta$: Since $\Delta = \Delta_1, \omega\Delta_2$, we proceed by case analysis on $\alpha^{\pi'} \in \Delta_1$ or $\alpha^{\pi'} \in \omega\Delta_2$.

  Case $\alpha^{\pi'} \in \Delta_1$: Since $\Delta_3 = (\Delta_1 + \Delta_1'), \Delta_{01}'$, we have $\alpha^{\pi''} \in \Delta_3$ for some $\pi'' \geq \pi'$. Since $\pi' \geq \pi$, we have $\pi'' \geq \pi$. However, we have assumed $\forall \pi' \geq \pi.\ \alpha^{\pi'} \notin \Delta_3$. Thus, there is a contradiction.

  Case $\alpha^{\pi'} \in \omega\Delta_2$: Since $\pi' \geq \pi$, We have $\pi' = \pi = \mathbf{0}$. Since $\dagger(\rho) = \omega\Delta_2, \omega\Delta_{02}'$, we have $\alpha \in dom(\rho)$. Since $W_1.\rho = \rho \circ W_2.\rho$, we have $\alpha \in dom(W_1.\rho)$.

  Case $\pi = \mathbf{0} \wedge \alpha \in dom(W_2.\rho)$: Since $W_1.\rho = \rho \circ W_2.\rho$, we have $\alpha \in dom(W_1.\rho)$.

- Let $x :^{\mathbf{1}} A \in \Gamma$.

  We first show that $(W_1.n,\ \Delta_{3,x},\ W_1.\rho) \sqsupseteq (W_2.n,\ \Delta_x,\ W_2.\rho)$.

  - $\vdash (W_1.n,\ \Delta_{3,x},\ W_1.\rho)$ and $\vdash (W_2.n,\ \Delta_x,\ W_2.\rho)$ by Lemma 105 with $\vdash W_1$ and $\vdash W_2$ and $dom(W_1.\Delta) = dom(\Delta_{3,x})$ and $dom(W_2.\Delta) = dom(\Delta_x)$.
  - We have $W_1.n \leq W_2.n$ by $W_1 \sqsupseteq W_2$.
  - We have $\Delta_{3,x}, \dagger(\rho) = \Delta_{1,x}, \omega\Delta_{01}', \omega\Delta_2, \omega\Delta_{02}' \gg \Delta_{1,x}, \omega\Delta_2 = \Delta_x$.
  - We have $W_1.\rho = \rho \circ W_2.\rho$.
  - We show that $\Delta_x \succ \rho$. Let $\alpha \in ftv(\rho|_{dom(\Delta_x)}) \cap dom(\Delta_x)$. Since $dom(\Delta_x) = dom(W_2.\Delta)$, we have $\alpha^{\mathbf{0}} \in W_2.\Delta$ by $W_2.\Delta \succ \rho$. Since $\Delta_x \leq W_2.\Delta$, we have $\alpha^{\mathbf{0}} \in \Delta_x$.

  Thus, since $(\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\,(W_2.n,\ \Delta_x,\ W_2.\rho)$, we have

  $$(\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x))_{W_1} \in \mathcal{R}[\![A]\!]\,(W_1.n,\ \Delta_{3,x},\ W_1.\rho)$$

  by Lemma 116. Thus,

  $$(\varsigma_{1\,\mathrm{fst}}(x), \varsigma_{1\,\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\,(W_1.n,\ \Delta_{3,x},\ W_1.\rho)\ .$$

- Let $x :^{\omega} A \in \Gamma$. We have had $(\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\,\omega\,W_2$. Since $\omega\,W_1 \sqsupseteq \omega\,W_2$ by Lemma 112, we have

  $$(\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x))_{\omega\,W_1} \in \mathcal{R}[\![A]\!]\,\omega\,W_1$$

  by Lemma 116. Thus,

  $$(\varsigma_{1\,\mathrm{fst}}(x), \varsigma_{1\,\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\,\omega\,W_1\ .$$

If $W_1.\rho = W_2.\rho$, then $\varsigma = W_1.\rho(\varsigma)$ by Lemma 113. □

Lemma 120. *If $\Gamma \vdash x : A$, then $\Gamma \vdash x \preceq x : A$.*

Proof. Let $(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$ and $W' \sqsupseteq W$ such that $0 < W'.n$. Then, it suffices to show that

$$(\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x))_{W'} \in \mathcal{R}[\![A]\!]\, W' \ .$$

By Lemma 119, $(W', W'.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma]\!]$. Since $\Gamma \vdash x : A$, we have $x :^{\pi} A \in \Gamma$ for some $\pi \neq \mathbf{0}$. By case analysis on π.

Case $\pi = \mathbf{0}$: Contradictory.

Case $\pi = \mathbf{1}$: Since $(W', W'.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma]\!]$, we have $(\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x))_{W'} \in \mathcal{R}[\![A]\!]\,(W'.n,\ \Delta,\ W'.\rho)$ for some $\Delta \leq W'.\Delta$. By Lemmas 118, 116, and 113, we have the conclusion.

Case $\pi = \omega$: Since $(W', W'.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma]\!]$, we have $(\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x))_{W'} \in \mathcal{R}[\![A]\!]\,\omega\, W'$. We also have $W \sqsupseteq \omega W$ by Lemmas 100 and 118. Thus, we have the conclusion by Lemmas 116 and 113.

∎

Lemma 121. *If $\Gamma \vdash c : ty(c)$, then $\Gamma \vdash \nu\overline{\alpha_1}.\, c \preceq \nu\overline{\alpha_2}.\, c : ty(c)$ for any $\overline{\alpha_1}$ and $\overline{\alpha_2}$.*

Proof. By structural induction on $ty(c)$. Let $(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$ and $W' \sqsupseteq W$ such that $0 < W'.n$. It suffices to show that

$$(\nu\overline{\alpha_1}.\, c, \nu\overline{\alpha_2}.\, c) \in \mathcal{R}[\![ty(c)]\!]\, W' \ .$$

If $ty(c) = \iota$ for some ι, we have $(\nu\overline{\alpha_1}.\, c, \nu\overline{\alpha_2}.\, c) \in \mathcal{R}[\![\iota]\!]\, W'$ by definition. Thus, we have the conclusion.
If $ty(c) = \iota \multimap A$ for some ι and A, then it suffices to show that, for any W'', W_1, W_2, R_1', and R_2', if

- $W'' \sqsupseteq W'$,
- $(W_1, W_2) \Supset W''$,
- $W_1 \sqsupseteq W'$, and
- $(R_1', R_2') \in \mathcal{R}[\![\iota]\!]\, W_2$,

then

$$((\nu\overline{\alpha_1}.\, c)\, R_1', (\nu\overline{\alpha_2}.\, c)\, R_2') \in \mathcal{E}[\![A]\!]\, W'' \ .$$

Since $(R_1', R_2') \in \mathcal{R}[\![\iota]\!]\, W_2$, we have $R_1' = \nu\overline{\beta_1}.\, c'$ and $R_2' = \nu\overline{\beta_2}.\, c'$ for some $\overline{\beta_1}$, $\overline{\beta_2}$, and c' such that $ty(c') = \iota$. By Assumption 1, for $i \in \{1, 2\}$, there exists some n_i such that $(\nu\overline{\alpha_i}.\, c)\, R_i' \longrightarrow^{n_i} \nu\overline{\alpha_i}.\, \nu\overline{\beta_i}.\, \zeta(c, c')$, and $\Gamma \vdash \zeta(c, c') : A$. Let $W''' \sqsupseteq W''$ and $n_1 < W'''.n$. Then it suffices to show that

$$(\nu\overline{\alpha_1}.\, \nu\overline{\beta_1}.\, \zeta(c, c'), \nu\overline{\alpha_2}.\, \nu\overline{\beta_2}.\, \zeta(c, c')) \in \mathcal{R}[\![A]\!]\,(W''' - n_1) \ .$$

By the IH,

$$\Gamma \vdash \nu\overline{\alpha_1}.\, \nu\overline{\beta_1}.\, \zeta(c, c') \preceq \nu\overline{\alpha_2}.\, \nu\overline{\beta_2}.\, \zeta(c, c') : A \ .$$

Since $(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$ and $W''' - n_1 \sqsupseteq W''' \sqsupseteq W'' \sqsupseteq W' \sqsupseteq W$, we have $(W''' - n_1, W'''.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma]\!]$ by Lemmas 110 and 119. Thus, we have

$$(\nu\overline{\alpha_1}.\, \nu\overline{\beta_1}.\, \zeta(c, c'), \nu\overline{\alpha_2}.\, \nu\overline{\beta_2}.\, \zeta(c, c')) \in \mathcal{E}[\![A]\!]\,(W''' - n_1) \ .$$

Since $n_1 < W'''.n$, we have $0 < W'''.n - n_1$. Thus, we have the conclusion. ∎

Lemma 122. *If $(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$ and $(R_1, R_2) \in \mathcal{R}[\![A]\!]\,(W.n,\ \Delta,\ W.\rho)$ and $W.\Delta \perp \Delta$, then*

$$((W.n,\ W.\Delta + \Delta,\ W.\rho), \varsigma \uplus \{x \mapsto R_1, R_2\}) \in \mathcal{G}[\![\Gamma, x :^{\mathbf{1}} A]\!] \ .$$

Proof. $(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$ implies $\vdash W$. By Lemma 102, $dom(W.\Delta) = dom(\Delta)$. Thus, by Lemma 105, $\vdash (W.n,\ W.\Delta + \Delta,\ W.\rho)$. The remaining part is obvious by definition. ∎

Lemma 123. *If $\Gamma, x :^{\mathbf{1}} A \vdash M_1 \preceq M_2 : B$, then $\Gamma \vdash \lambda x.M_1 \preceq \lambda x.M_2 : A \multimap B$.*

Proof. Let $(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$. By definition and Lemma 114, it suffices to show that, for any W', W'', W_1, W_2, R'_1, and R'_2, if

- $W' \sqsupseteq W$,
- $0 < W'.n$,
- $W'' \sqsupseteq W'$,
- $(W_1, W_2) \Supset W''$,
- $W_1 \sqsupseteq W'$, and
- $(R'_1, R'_2) \in \mathcal{R}[\![A]\!]\, W_2$,

then

$$(\varsigma_{\mathrm{fst}}(\lambda x.M_1)\, R'_1, \varsigma_{\mathrm{snd}}(\lambda x.M_2)\, R'_2)_{W''} \in \mathcal{E}[\![B]\!]\, W'' \,.$$

Let $W''' \sqsupseteq W''$ such that $W'''.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(\lambda x.M_1)\, R'_1) \longrightarrow^n R_1$ for some $n < W'''.n$ and R_1. Then, it suffices to show that there exists some R'_2 such that

- $W'''.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(\lambda x.M_2)\, R'_2) \longrightarrow^* R_2$ and
- $(R_1, R_2) \in \mathcal{R}[\![B]\!]\,(W''' - n)$.

Since $(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$ and $W_1 \sqsupseteq W' \sqsupseteq W$ and $W_1.\rho = W''.\rho$, we have $(W_1, W''.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma]\!]$ by Lemmas 110 and 119. Since $(W_1, W_2) \Supset W''$ and $(R'_1, R'_2) \in \mathcal{R}[\![A]\!]\, W_2$, we have $(W'', W''.\rho(\varsigma) \uplus \{x \Mapsto R'_1, R'_2\}) \in \mathcal{G}[\![\Gamma, x :^{\mathbf{1}} A]\!]$ by Lemma 122. Since $\Gamma, x :^{\mathbf{1}} A \vdash M_1 \preceq M_2 : B$, we have

$$(\varsigma_{\mathrm{fst}}(M_1[R'_1/x]), \varsigma_{\mathrm{snd}}(M_2[R'_2/x]))_{W''} \in \mathcal{E}[\![B]\!]\, W'' \,.$$

Since $W'''.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(\lambda x.M_1)\, R'_1) \longrightarrow^n R_1$, we can find $W'''.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(M_1[R'_1/x])) \longrightarrow^{n_1} R_1$ for some $n_1 < n$. Since $W''' \sqsupseteq W''$ and $n_1 < n < W'''.n$, there exists some R_2 such that

- $W'''.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(M_2[R'_2/x])) \longrightarrow^* R_2$ and
- $(R_1, R_2) \in \mathcal{R}[\![B]\!]\,(W''' - n_1)$.

We have $W'''.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(\lambda x.M_2)\, R'_2) \longrightarrow W'''.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(M_2[R'_2/x])) \longrightarrow^* R_2$. Since $W''' - n \sqsupseteq W''' - n_1$, we have $(R_1, R_2) \in \mathcal{R}[\![B]\!]\,(W''' - n)$ by Lemmas 116 and 113. ☐

Lemma 124. *If $(W, \varsigma) \in \mathcal{G}[\![\Gamma_1 + \Gamma_2]\!]$, then there exist some W_1 and W_2 such that*

- $(W_1, \varsigma) \in \mathcal{G}[\![\Gamma_1]\!]$,
- $(W_2, \varsigma) \in \mathcal{G}[\![\Gamma_2]\!]$, *and*
- $(W_1, W_2) \Supset W$.

Proof. Since $(W, \varsigma) \in \mathcal{G}[\![\Gamma_1 + \Gamma_2]\!]$, we have

- $\vdash W$,
- $\Gamma_1 + \Gamma_2 \succ W.\rho$, and
- there exist some Δ' and $\prod_{x \in dom_{=\mathbf{1}}(\Gamma_1 + \Gamma_2)} \Delta_x$ such that
  - $W.\Delta = \Delta' + \sum_{x \in dom_{=\mathbf{1}}(\Gamma_1+\Gamma_2)} \Delta_x$,
  - $\forall \alpha^\pi \in \Gamma.\ (\exists \pi' \geq \pi.\ \alpha^{\pi'} \in \Delta') \vee (\pi = \mathbf{0} \wedge \alpha \in dom(W.\rho))$,
  - $\forall x :^{\mathbf{1}} A \in \Gamma.\ (\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\,(W.n,\, \Delta_x,\, W.\rho)$, and
  - $\forall x :^{\omega} A \in \Gamma.\ (\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\, \omega\, W$.

For $i \in \{1,2\}$ and $x \in dom_{=\mathbf{1}}(\Gamma_i)$, let $\Delta_{i,x} = \Delta_x$ if $x \in dom_{=\mathbf{1}}(\Gamma_1 + \Gamma_2)$, and otherwise $\Delta_{i,x} = \omega\Delta'$. We also build Δ'_i for $i \in \{1,2\}$ as follows:

- if $\alpha^{\mathbf{1}} \in \Delta'$ and $\alpha^{\mathbf{1}} \in \Gamma_i$, then $\alpha^{\mathbf{1}} \in \Delta'_i$;
- if $\alpha^{\mathbf{1}} \in \Delta'$ and $\alpha^{\mathbf{0}} \in \Gamma_i$ and $\alpha^{\mathbf{1}} \in \Gamma_{2-i}$, then $\alpha^{\mathbf{0}} \in \Delta'_i$;
- if $\alpha^{\mathbf{1}} \in \Delta'$ and $\alpha^{\mathbf{0}} \in \Gamma_1 + \Gamma_2 \vee \alpha \notin dom(\Gamma_1 + \Gamma_2)$, then $\alpha^{\mathbf{0}} \in \Delta'_1$ and $\alpha^{\mathbf{1}} \in \Delta'_2$; and
- if $\alpha^{\mathbf{0}} \in \Delta'$, then $\alpha^{\mathbf{0}} \in \Delta'_i$.

Then, we can find $\Delta'_i \perp \sum_{x \in dom_{=\mathbf{1}}(\Gamma_i)} \Delta_{i,x}$ for $i \in \{1,2\}$: for any α, if $\alpha^{\mathbf{1}} \in \Delta'_i$, then $\alpha^{\mathbf{1}} \in \Delta'$; since $\Delta' \perp \sum_{x \in dom_{=\mathbf{1}}(\Gamma_1+\Gamma_2)} \Delta_x$, we have $\alpha^{\mathbf{0}} \in \Delta_x$.

Let $i \in \{1,2\}$ and $W_i = (W.n,\ \Delta'_i + \sum_{x \in dom_{=\mathbf{1}}(\Gamma_i)} \Delta_{i,x},\ W.\rho)$. We show that $(W_1, W_2) \ni W$, i.e., $W_1.\Delta + W_2.\Delta = W.\Delta$. First, we have the following.

- $\Delta' = \Delta'_1 + \Delta'_2$. If $\alpha^{\mathbf{0}} \in \Delta'$, then $\alpha^{\mathbf{0}} \in \Delta'_1$ and $\alpha^{\mathbf{0}} \in \Delta'_2$; If $\alpha^{\mathbf{1}} \in \Delta'$, then only either of Δ'_1 and Δ'_2 has $\alpha^{\mathbf{1}}$.
- We have

$$\begin{array}{ll} & \omega\Delta' + \sum_{x \in dom_{=\mathbf{1}}(\Gamma_1+\Gamma_2)} \Delta_x \\ = & \omega\Delta' + \sum_{x \in dom_{=\mathbf{1}}(\Gamma_1) \cap dom_{=\mathbf{1}}(\Gamma_1+\Gamma_2)} \Delta_{1,x} + \sum_{x \in dom_{=\mathbf{1}}(\Gamma_2) \cap dom_{=\mathbf{1}}(\Gamma_1+\Gamma_2)} \Delta_{2,x} \\ = & \omega\Delta' + \sum_{x \in dom_{=\mathbf{1}}(\Gamma_1)} \Delta_{1,x} + \sum_{x \in dom_{=\mathbf{1}}(\Gamma_2)} \Delta_{2,x} \end{array}$$

Thus,

$$\begin{array}{lll} W.\Delta & = & \Delta' + \sum_{x \in dom_{=\mathbf{1}}(\Gamma_1+\Gamma_2)} \Delta_x \\ & = & \Delta'_1 + \Delta'_2 + \omega\Delta' + \sum_{x \in dom_{=\mathbf{1}}(\Gamma_1+\Gamma_2)} \Delta_x \\ & = & \Delta'_1 + \Delta'_2 + \omega\Delta' + \sum_{x \in dom_{=\mathbf{1}}(\Gamma_1)} \Delta_{1,x} + \sum_{x \in dom_{=\mathbf{1}}(\Gamma_2)} \Delta_{2,x} \\ & = & \Delta'_1 + \sum_{x \in dom_{=\mathbf{1}}(\Gamma_1)} \Delta_{1,x} + \Delta'_2 + \sum_{x \in dom_{=\mathbf{1}}(\Gamma_2)} \Delta_{2,x} \\ & = & W_1.\Delta + W_2.\Delta\ . \end{array}$$

Finally, we show that $(W_i, \varsigma) \in \mathcal{G}[\![\Gamma_i]\!]$.

- We have $\vdash W_i$ by Lemma 105 with $\vdash W$ and $dom(W.\Delta) = dom(W_i.\Delta)$ (which is shown by Lemma 102).
- We show that $\Gamma_i \succ W_i.\rho$. Let $\alpha \in ftv(W_i.\rho|_{dom(\Gamma_i)}) \cap dom(\Gamma_i)$. Since $W_i.\rho = W.\rho$ and $dom(\Gamma_i) = dom(\Gamma_1 + \Gamma_2)$, $\Gamma_1 + \Gamma_2 \succ W.\rho$ implies $\alpha^{\mathbf{0}} \in \Gamma_1 + \Gamma_2$. Thus, $\alpha^{\mathbf{0}} \in \Gamma_i$.
- Let $\alpha^{\mathbf{1}} \in \Gamma_i$. Since $\alpha^{\mathbf{1}} \in \Gamma_1 + \Gamma_2$, we have $\alpha^{\mathbf{1}} \in \Delta'$ from $(W, \varsigma) \in \mathcal{G}[\![\Gamma_1 + \Gamma_2]\!]$. By the definition of Δ'_i, $\alpha^{\mathbf{1}} \in \Delta'_i$.
- Let $\alpha^{\mathbf{0}} \in \Gamma_i$.

  If $\alpha \in dom(\Delta')$, then $\alpha \in dom(\Delta'_i)$. Thus, there exists some $\pi' \geq \pi$ such that $\alpha^{\pi'} \in \Delta'_i$.

  Otherwise, if $\alpha \notin dom(\Delta')$, then, since $(W, \varsigma) \in \mathcal{G}[\![\Gamma_1 + \Gamma_2]\!]$, $\alpha \in dom(W.\rho) = dom(W_i.\rho)$.
- Let $x :^{\mathbf{1}} A \in \Gamma_i$. We show $(\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\,(W.n,\ \Delta_{i,x},\ W.\rho)$.

  If $x :^{\mathbf{1}} A \in \Gamma_1 + \Gamma_2$, then $\Delta_{i,x} = \Delta_x$. Thus, $(\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\,(W.n,\ \Delta_{i,x},\ W.\rho)$ from $(W, \varsigma) \in \mathcal{G}[\![\Gamma_1 + \Gamma_2]\!]$.

  Otherwise, if $x :^{\omega} A \in \Gamma_1 + \Gamma_2$, then $(\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\,\omega W$ from $(W, \varsigma) \in \mathcal{G}[\![\Gamma_1 + \Gamma_2]\!]$. Since $\omega\Delta' = \Delta_{i,x}$ by definition, we have the conclusion.
- Let $x :^{\omega} A \in \Gamma_i$. Since $\omega W_1 = \omega W_2 = \omega W$, it suffices to show that $(\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\,\omega W$, which is shown by $(W, \varsigma) \in \mathcal{G}[\![\Gamma_1 + \Gamma_2]\!]$.

□

Lemma 125. *If $\Gamma_1 \vdash M_{11} \preceq M_{21} : A \multimap B$ and $\Gamma_2 \vdash M_{12} \preceq M_{22} : A$ and $\Gamma_1 \perp \Gamma_2$, then $\Gamma_1 + \Gamma_2 \vdash M_{11}\, M_{12} \preceq M_{21}\, M_{22} : B$.*

Proof. Let $(W, \varsigma) \in \mathcal{G}[\![\Gamma_1 + \Gamma_2]\!]$. By the definition and Lemma 114, suppose that

- $W' \sqsupseteq W$,
- $n < W'.n$, and
- $W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(M_{11}\, M_{12})) \longrightarrow^{n} R_1$

for some W', n, and R_1, and then it suffices to show that there exists some R_2 such that

- $W'.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(M_{21}\, M_{22})) \longrightarrow^{*} R_2$ and
- $(R_1, R_2) \in \mathcal{R}[\![B]\!]\,(W' - n)$.

By Lemma 119, $(W', W'.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma_1 + \Gamma_2]\!]$. By Lemma 124, there exist some W_1 and W_2 such that

- $(W_1, W'.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma_1]\!]$,
- $(W_2, W'.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma_2]\!]$, and
- $(W_1, W_2) \Supset W'$.

Since $\Gamma_1 \vdash M_{11} \preceq M_{21} : A \multimap B$ and $W_1.\rho = W'.\rho$ (from $(W_1, W_2) \Supset W'$), we have

$$(\varsigma_{\mathrm{fst}}(M_{11}), \varsigma_{\mathrm{snd}}(M_{21}))_{W_1} \in \mathcal{E}[\![A \multimap B]\!]\, W_1 \ .$$

Since $\Gamma_2 \vdash M_{12} \preceq M_{22} : A$ and $W_2.\rho = W'.\rho$, we have

$$(\varsigma_{\mathrm{fst}}(M_{12}), \varsigma_{\mathrm{snd}}(M_{22}))_{W_2} \in \mathcal{E}[\![A]\!]\, W_2 \ .$$

Since $W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(M_{11}\, M_{12})) \longrightarrow^{n} R_1$, we can find $W_1.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(M_{11})) \longrightarrow^{n_1} R_{11}$ and $W_2.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(M_{12})) \longrightarrow^{n_2} R_{12}$ for some R_{11}, R_{12}, n_1, and n_2 such that $n_1 + n_2 < n$; note that $W_1.\rho = W_2.\rho = W'.\rho$. Since $W_1.n = W_2.n = n$, there exist some R_{21} and R_{22} such that

- $W_1.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(M_{21})) \longrightarrow^{*} R_{21}$,
- $W_2.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(M_{22})) \longrightarrow^{*} R_{22}$,
- $(R_{11}, R_{21}) \in \mathcal{R}[\![A \multimap B]\!]\,(W_1 - n_1)$, and
- $(R_{12}, R_{22}) \in \mathcal{R}[\![A]\!]\,(W_2 - n_2)$.

Since $W_2 - n_1 - n_2 \sqsupseteq W_2 - n_2$, we have $(R_{12}, R_{22}) \in \mathcal{R}[\![A]\!]\,(W_2 - n_1 - n_2)$ by Lemmas 116 and 113. Since $(W_1, W_2) \Supset W'$, we have $(W_1 - n_1 - n_2, W_2 - n_1 - n_2) \Supset W' - n_1 - n_2$. Further, by Lemma 118, $W' - n_1 - n_2 \sqsupseteq W_1 - n_1 - n_2$. Since $W_1 - n_1 - n_2 \sqsupseteq W_1 - n_1$, we have $W' - n_1 - n_2 \sqsupseteq W_1 - n_1$ by Lemma 110. Now, we have

- $(R_{11}, R_{21}) \in \mathcal{R}[\![A \multimap B]\!]\,(W_1 - n_1)$,
- $(R_{12}, R_{22}) \in \mathcal{R}[\![A]\!]\,(W_2 - n_1 - n_2)$,
- $W' - n_1 - n_2 \sqsupseteq W_1 - n_1$.
- $(W_1 - n_1 - n_2, W_2 - n_1 - n_2) \Supset W' - n_1 - n_2$, and
- $W_1 - n_1 - n_2 \sqsupseteq W_1 - n_1$.

Thus, by the definition of $\mathcal{R}$,

$$(R_{11}\, R_{12}, R_{21}\, R_{22}) \in \mathcal{E}[\![B]\!]\, W' - n_1 - n_2 \ .$$

Since

$$\begin{array}{rll} W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(M_{11}\, M_{12})) & \longrightarrow^{n_1} & R_{11}\; W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(M_{12})) \\ & \longrightarrow^{n_2} & R_{11}\, R_{12} \\ & \longrightarrow^{n_3} & R_1 \end{array}$$

for some $n_3 = n - n_1 - n_2$, there exists some R_2 such that

- $R_{21}\, R_{22} \longrightarrow^* R_2$ and
- $(R_1, R_2) \in \mathcal{R}[\![B]\!]\,(W' - n)$.

Note that $n_3 < W'.n - n_1 - n_2$. We have

$$\begin{array}{rcl} W'.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(M_{21}\, M_{22})) & \longrightarrow^* & R_{21}\; W'.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(M_{22})) \\ & \longrightarrow^* & R_{21}\, R_{22} \\ & \longrightarrow^* & R_2\ . \end{array}$$

Thus, we have the conclusion. □

Lemma 126. *If $(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$, then $(\omega\, W, \varsigma) \in \mathcal{G}[\![\omega\Gamma]\!]$.*

Proof. By induction on Γ. Note that: since $(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$, we have $\vdash W$, which implies $\vdash \omega\, W$ by Lemma 105; and $\omega\Gamma \succ \omega\, W.\rho$.

Case $\Gamma = \emptyset$: Obvious.

Case $\Gamma = \Gamma', \alpha^\pi$: We have $(W, \varsigma) \in \mathcal{G}[\![\Gamma']\!]$. By the IH, $(\omega\, W, \varsigma) \in \mathcal{G}[\![\omega\Gamma']\!]$. By the definition of $\mathcal{G}$, we have $(\omega\, W).\Delta = \Delta' + \sum_{x \in dom_{=\mathbf{1}}(\omega\Gamma')} \Delta'_x$ for some Δ' and $\prod_{x \in dom_{=\mathbf{1}}(\omega\Gamma')} \Delta'_x$.

Then, it suffices to show that, for any α, if $\alpha \notin dom(\Delta')$, then $\alpha \in dom(W.\rho)$.

Suppose that $\alpha \notin dom(\Delta')$. By Lemmas 12 and 9, $\alpha \notin dom(W.\Delta)$. Since $(W, \varsigma) \in \mathcal{G}[\![\Gamma', \alpha^\pi]\!]$, we can find $\pi = \mathbf{0}$ (if $\pi \neq \mathbf{0}$, then $\alpha \in dom(W.\Delta)$) and $\alpha \in dom(W.\rho)$.

Case $\Gamma = \Gamma', x :^\pi A$: We have $(W, \varsigma) \in \mathcal{G}[\![\Gamma']\!]$. By the IH, $(\omega\, W, \varsigma) \in \mathcal{G}[\![\omega\Gamma']\!]$.

If $\pi = \mathbf{1}$ or $\pi = \mathbf{0}$, then we have $(\omega\, W, \varsigma) \in \mathcal{G}[\![\omega\Gamma', x :^{\mathbf{0}} A]\!] = \mathcal{G}[\![\omega(\Gamma', x :^\pi A)]\!]$ by the definition of $\mathcal{G}$ and $(\omega\, W, \varsigma) \in \mathcal{G}[\![\omega\Gamma']\!]$.

Otherwise, suppose that $\pi = \omega$. Then, it suffices to show that

$$(\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\,\omega\omega\, W\ .$$

Since $(W, \varsigma) \in \mathcal{G}[\![\Gamma', x :^\omega A]\!]$, we can find $(\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\,\omega\, W$. Since $\omega\omega\, W = \omega\, W$ by Lemma 5, we have the conclusion.

□

Lemma 127. *If $\vdash \Gamma$ and $\omega\Gamma \vdash M_1 \preceq M_2 : A$, then $\Gamma \vdash\, !M_1 \preceq\, !M_2 :\, !A$.*

Proof. Let $(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$. It suffices to show that

$$(\varsigma_{\mathrm{fst}}(!M_1), \varsigma_{\mathrm{snd}}(!M_2))_W \in \mathcal{E}[\![!A]\!]\; W\ .$$

Suppose that

- $W' \sqsupseteq W$,
- $n < W'.n$, and
- $W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(!M_1)) \longrightarrow^n R_1$

for some W', n, and R_1, and then it suffices to show that there exists some R_2 such that

- $W'.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(!M_2)) \longrightarrow^* R_2$ and
- $(R_1, R_2) \in \mathcal{R}[\![!A]\!]\,(W' - n)$.

Since $(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$, Lemmas 119 and 126 imply $(\omega\, W', W'.\rho(\varsigma)) \in \mathcal{G}[\![\omega\Gamma]\!]$. Since $\omega\Gamma \vdash M_1 \preceq M_2 : A$, we have

$$(\varsigma_{\mathrm{fst}}(M_1), \varsigma_{\mathrm{snd}}(M_2))_{W'} \in \mathcal{E}[\![A]\!]\,\omega\, W'\ .$$

Since $W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(!M_1)) \longrightarrow^n R_1$ and $W'.\rho = \omega\, W'.\rho$, we can find $\omega\, W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(M_1)) \longrightarrow^n R'_1$ for some R'_1 such $R_1 =\, !R'_1$. Thus, by the definition of $\mathcal{E}$, there exists some R'_2 such that

- $\omega W'.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(M_2)) \longrightarrow^* R'_2$ and
- $(R'_1, R'_2) \in \mathcal{R}[\![A]\!](\omega W' - n)$.

Let $R_2 = !R'_2$. Now, it suffices to show that $(!R'_1, !R'_2) \in \mathcal{R}[\![!A]\!](W' - n)$. By definition, it suffices to show that

$$(\mathsf{let}\,!x = !R'_1\,\mathsf{in}\,x, \mathsf{let}\,!x = !R'_2\,\mathsf{in}\,x) \in \mathcal{E}[\![A]\!]\,\omega(W' - n)\ .$$

Let $W'' \sqsupseteq \omega(W' - n)$ and suppose that $1 < W''.n$. Then, it suffices to show that

$$(R'_1, R'_2)_{W''} \in \mathcal{R}[\![A]\!](W'' - 1)\ .$$

Since $(R'_1, R'_2) \in \mathcal{R}[\![A]\!](\omega W' - n)$ and $W'' - 1 \sqsupseteq W'' \sqsupseteq \omega(W' - n) = \omega W' - n$, we have the conclusion by Lemmas 110 and 116. □

Lemma 128. *If $\Gamma_1 \vdash M_{11} \preceq M_{21} : !B$ and $\Gamma_2, x :^{\omega} B \vdash M_{12} \preceq M_{22} : A$, then $\Gamma_1 + \Gamma_2 \vdash \mathsf{let}\,!x = M_{11}\,\mathsf{in}\,M_{12} \preceq \mathsf{let}\,!x = M_{21}\,\mathsf{in}\,M_{22} : A$.*

Proof. Let $(W, \varsigma) \in \mathcal{G}[\![\Gamma_1 + \Gamma_2]\!]$. It suffices to show that

$$(\varsigma_{\mathrm{fst}}(\mathsf{let}\,!x = M_{11}\,\mathsf{in}\,M_{12}), \varsigma_{\mathrm{snd}}(\mathsf{let}\,!x = M_{21}\,\mathsf{in}\,M_{22}))_W \in \mathcal{E}[\![A]\!]\,W\ .$$

Suppose that

- $W' \sqsupseteq W$,
- $n < W'.n$, and
- $W.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(\mathsf{let}\,!x = M_{11}\,\mathsf{in}\,M_{12})) \longrightarrow^n R_1$

for some W', n, and R_1, and then it suffices to show that there exists some R_2 such that

- $W'.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(\mathsf{let}\,!x = M_{21}\,\mathsf{in}\,M_{22})) \longrightarrow^* R_2$ and
- $(R_1, R_2) \in \mathcal{R}[\![A]\!](W' - n)$.

Since $(W, \varsigma) \in \mathcal{G}[\![\Gamma_1 + \Gamma_2]\!]$ and $W' \sqsupseteq W$, we have $(W', W'.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma_1 + \Gamma_2]\!]$ by Lemma 119. Lemma 124 implies that there exist some W_1 and W_2 such that

- $(W_1, W'.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma_1]\!]$,
- $(W_2, W'.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma_2]\!]$, and
- $(W_1, W_2) \Supset W'$.

Note that $W'.\rho = W_1.\rho = W_2.\rho$. Since $\Gamma_1 \vdash M_{11} \preceq M_{21} : !B$, we have

$$(\varsigma_{\mathrm{fst}}(M_{11}), \varsigma_{\mathrm{snd}}(M_{21}))_{W_1} \in \mathcal{E}[\![!B]\!]\,W_1\ .$$

Since $W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(\mathsf{let}\,!x = M_{11}\,\mathsf{in}\,M_{12})) \longrightarrow^* R_1$, we can find that

- $W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(M_{11})) \longrightarrow^{n_1} \nu\overline{\alpha_1}.\,!R'_{11}$,
- $W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(\mathsf{let}\,!x = M_{11}\,\mathsf{in}\,M_{12})) \longrightarrow^{n_1} W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(\mathsf{let}\,!x = \nu\overline{\alpha_1}.\,!R'_{11}\,\mathsf{in}\,M_{12})) \longrightarrow^1 W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(M_{12}[\nu\overline{\alpha_1}.\,R'_{11}/x]))$

for some $\overline{\alpha_1}$, R'_{11}, and n_1. Thus, by the definitions of $\mathcal{E}$ and $\mathcal{R}$, there exist some $\overline{\alpha_2}$ and R'_{21} such that

- $W_1.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(M_{21})) \longrightarrow^* \nu\overline{\alpha_2}.\,!R'_{21}$ and
- $(\nu\overline{\alpha_1}.\,R'_{11}, \nu\overline{\alpha_2}.\,R'_{21}) \in \mathcal{R}[\![B]\!]\,\omega(W_1 - n_1 - 1)$.

Note that $n_1 + 1 \leq n < W'.n = W_1.n$. Since $(W_1, W_2) \Supset W'$, we have $\omega W_1 = \omega W_2$. Thus, $(\nu\overline{\alpha_1}.\, R'_{11}, \nu\overline{\alpha_2}.\, R'_{21}) \in \mathcal{R}[\![B]\!]\, \omega(W_2 - n_1 - 1)$. Since $(W_2, W'.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma_2]\!]$, we have $(W_2 - n_1 - 1, W'.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma_2]\!]$ by Lemmas 119 and 113. Thus, by the definition of $\mathcal{G}$,

$$(W_2 - n_1 - 1, W'.\rho(\varsigma) \uplus \{x \Mapsto \nu\overline{\alpha_1}.\, R'_{11}, \nu\overline{\alpha_2}.\, R'_{21}\}) \in \mathcal{G}[\![\Gamma_2, x :^{\omega} B]\!] .$$

Since $\Gamma_2, x :^{\omega} B \vdash M_{12} \preceq M_{22} : A$ and $W_2.\rho = W'.\rho$, we have

$$(\varsigma_{\mathrm{fst}}(M_{12}[\nu\overline{\alpha_1}.\, R'_{11}/x]), \varsigma_{\mathrm{snd}}(M_{22}[\nu\overline{\alpha_2}.\, R'_{21}/x]))_{W_2 - n_1 - 1} \in \mathcal{E}[\![A]\!]\,(W_2 - n_1 - 1) .$$

Since $W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(\mathsf{let}\,!x = M_{11}\ \mathsf{in}\ M_{12})) \longrightarrow^{n_1+1} W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(M_{12}[\nu\overline{\alpha_1}.\, R'_{11}/x])) \longrightarrow^{n-n_1-1} R_1$, there exists some R_2 such that

- $W'.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(M_{22}[\nu\overline{\alpha_2}.\, R'_{21}/x])) \longrightarrow^{*} R_2$ and
- $(R_1, R_2) \in \mathcal{R}[\![A]\!]\,(W_2 - n)$.

Now, we have the conclusion because:

- $$\begin{array}{rll} W'.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(\mathsf{let}\,!x = M_{21}\ \mathsf{in}\ M_{22})) & \longrightarrow^{*} & W'.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(\mathsf{let}\,!x = \nu\overline{\alpha_2}.\,!R'_{21}\ \mathsf{in}\ M_{22})) \\ & \longrightarrow & W'.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(M_{22}[\nu\overline{\alpha_2}.\, R'_{21}/x])) \\ & \longrightarrow^{*} & R_2 \end{array}$$

  and
- since $(R_1, R_2) \in \mathcal{R}[\![A]\!]\,(W_2 - n)$ and $W' - n \sqsupseteq W_2 - n$ by Lemma 118, we have $(R_1, R_2) \in \mathcal{R}[\![A]\!]\,(W' - n)$ by Lemmas 116 and 113.

□

Lemma 129. *For any $\pi \leq \mathbf{1}$, If $\vdash W$ and $\{\alpha\} \# W$, then $\vdash (W.n, (W.\Delta, \alpha^{\pi}), W.\rho)$.*

Proof. The conclusion is shown by the following.

- $dom(W.\rho) \# dom(W.\Delta, \alpha^{\pi})$ because $\{\alpha\} \# W$.
- Let $\beta \in dom(W.\rho)$. Since $W.\Delta \vdash W.\rho_{\mathrm{fst}}(\beta)$ and $W.\Delta \vdash W.\rho_{\mathrm{snd}}(\beta)$, we have $W.\Delta, \alpha^{\pi} \vdash W.\rho_{\mathrm{fst}}(\beta)$ and $W.\Delta, \alpha^{\pi} \vdash W.\rho_{\mathrm{snd}}(\beta)$.

□

Lemma 130. *Suppose that $\{\alpha\} \# W_1$ and $\{\alpha\} \# W_2$. If $W_1 \sqsupseteq W_2$ and $\pi \leq \mathbf{1}$, then $(W_1.n, (W_1.\Delta, \alpha^{\pi}), W_1.\rho) \sqsupseteq (W_2.n, (W_2.\Delta, \alpha^{\pi}), W_2.\rho)$.*

Proof. Let $W'_1 = (W_1.n, (W_1.\Delta, \alpha^{\pi}), W_1.\rho)$ and $W'_2 = (W_2.n, (W_2.\Delta, \alpha^{\pi}), W_2.\rho)$. Since $W_1 \sqsupseteq W_2$, we have

- $\vdash W_1$ and $\vdash W_2$,
- $W_1.n \leq W_2.n$,
- there exists some ρ such that
  - $W_1.\Delta, \dagger(\rho) \gg W_2.\Delta$,
  - $W_1.\rho = \rho \circ W_2.\rho$,
  - $W_2.\Delta \succ \rho$.

We have the conclusion by the following.

- $\vdash W'_1$ and W'_2 by Lemma 129 with $\vdash W_1$ and $\vdash W_2$.
- $W_1.\Delta, \alpha^{\pi}, \dagger(\rho) \gg W_2.\Delta, \alpha^{\pi}$ since $W_1.\Delta, \dagger(\rho) \gg W_2.\Delta$.

- We show that $W_2'.\Delta \succ \rho$. Let $\beta \in \mathit{ftv}(\rho|_{\mathit{dom}(W_2'.\Delta)}) \cap \mathit{dom}(W_2'.\Delta)$. Since $\{\alpha\} \# W_1$ and $\vdash W_1$, we have $\alpha \notin \mathit{ftv}(\rho) \cup \mathit{dom}(\rho)$. Thus, $\beta \in \mathit{ftv}(\rho|_{\mathit{dom}(W_2.\Delta)}) \cap \mathit{dom}(W_2.\Delta)$. Thus, $W_2.\Delta \succ \rho$ implies $\beta^{\mathbf{0}} \in W_2.\Delta$, so $\beta^{\mathbf{0}} \in W_2.\Delta, \alpha^{\pi} = W_2'.\Delta$.

□

Lemma 131. *If $\vdash W$ and $\{\alpha\} \# \mathit{ftv}(A)$, then:*

- $(R_1, R_2) \in \mathcal{R}[\![A]\!]\, W@\alpha$ *implies* $(\nu\alpha.\, R_1, \nu\alpha.\, R_2) \in \mathcal{R}[\![A]\!]\, W$*; and*
- $(M_1, M_2) \in \mathcal{E}[\![A]\!]\, W@\alpha$ *implies* $(\nu\alpha.\, M_1, \nu\alpha.\, M_2) \in \mathcal{E}[\![A]\!]\, W$*.*

Proof. By induction on A. We first consider the first property on $\mathcal{R}$ and then the second one on $\mathcal{E}$ with the first property.

- Let $(R_1, R_2) \in \mathcal{R}[\![A]\!]\, W@\alpha$. We show $(\nu\alpha.\, R_1, \nu\alpha.\, R_2) \in \mathcal{R}[\![A]\!]\, W$ by case analysis on A.

  Case $A = \iota$: By definition.

  Case $A = \beta$: Let $(B_1, B_2, r) = W.\rho(\beta)$. Since $(R_1, R_2) \in \mathcal{R}[\![\beta]\!]\, W@\alpha$, we have $(R_1, R_2) \in r(\blacktriangleright(W@\alpha))$ by definition. Since $\beta \in \mathit{dom}(W.\rho)$ and $\vdash W$, we have $W.\Delta \vdash B_1$ and $W.\Delta \vdash B_2$ and $r \in \mathrm{Rel}_{W.n}[B_1, B_2]$. Since $W@\alpha$ is well defined, we have $\{\alpha\} \# W$, so $\{\alpha\} \# \mathit{ftv}(B_1)$ and $\{\alpha\} \# \mathit{ftv}(B_2)$. We also have $\vdash \blacktriangleright W$ from $\vdash W$. Thus, by the third condition on $r \in \mathrm{Rel}_{W.n}[B_1, B_2]$ about extension with fresh type variables, $(\nu\alpha.\, R_1, \nu\alpha.\, R_2) \in r(\blacktriangleright W)$. Thus, we have $(\nu\alpha.\, R_1, \nu\alpha.\, R_2) \in \mathcal{R}[\![\beta]\!]\, W$.

  Case $A = B \multimap C$: Suppose that

  - $W' \sqsupseteq W$,
  - $(W_1, W_2) \Supset W'$,
  - $W_1 \sqsupseteq W$, and
  - $(R_1', R_2') \in \mathcal{R}[\![B]\!]\, W_2$

  for some W', W_1, W_2, R_1', and R_2', and then it suffices to show that

  $$((\nu\alpha.\, R_1)\, R_1', (\nu\alpha.\, R_2)\, R_2')_{W'} \in \mathcal{E}[\![C]\!]\, W' \;.$$

  Since $W'.\rho_{\mathrm{fst}}((\nu\alpha.\, R_1)\, R_1') \longrightarrow W'.\rho_{\mathrm{fst}}(\nu\alpha.\, (R_1\, R_1'))$ and $W'.\rho_{\mathrm{snd}}((\nu\alpha.\, R_2)\, R_2') \longrightarrow W'.\rho_{\mathrm{snd}}(\nu\alpha.\, (R_2\, R_2'))$, it suffices to show that

  $$((\nu\alpha.\, R_1\, R_1'), (\nu\alpha.\, R_2\, R_2'))_{W'} \in \mathcal{E}[\![C]\!]\, W'$$

  by Lemmas 116 and 113. Since $\{\alpha\} \# W$, we can suppose that $\{\alpha\} \# W'$ without loss of generality. Since

  - $(R_1, R_2) \in \mathcal{R}[\![B \multimap C]\!]\, W@\alpha$
  - $W'@\alpha \sqsupseteq W@\alpha$ by Lemma 130 with $W' \sqsupseteq W$,
  - $W_1@\alpha \sqsupseteq W@\alpha$ by Lemma 130 with $W_1 \sqsupseteq W$,
  - $(W_1@\alpha, (W_2.n,\, (W_2.\Delta, \alpha^{\mathbf{0}}),\, W_2.\rho)) \Supset W'@\alpha$ (from $(W_1, W_2) \Supset W'$), and
  - $(R_1', R_2') \in \mathcal{R}[\![B]\!]\, (W_2.n,\, (W_2.\Delta, \alpha^{\mathbf{0}}),\, W_2.\rho)$ by Lemmas 117, 116, and 113,

  we have

  $$(R_1\, R_1', R_2\, R_2')_{W'@\alpha} \in \mathcal{E}[\![C]\!]\, W'@\alpha \;.$$

  By the IH with $\vdash W'$ implied by $W' \sqsupseteq W$, we have the conclusion

  $$(\nu\alpha.\, R_1\, R_1', \nu\alpha.\, R_2\, R_2')_{W'} \in \mathcal{E}[\![C]\!]\, W' \;.$$

  Case $A = \forall\beta.B$: Suppose that

  - $W' \sqsupseteq W$,
  - $\omega\, W' \vdash (C_1, C_2, r)$, and
  - $\{\beta\} \# \omega\, W'$

for some W', C_1, C_2, and r, and then it suffices to show that

$$((\nu\alpha.\, R_1)\; C_1, (\nu\alpha.\, R_2)\; C_2)_{\omega\, W'} \;\in\; \mathcal{E}[\![B]\!]\,\{\beta \Mapsto (C_1, C_2, r)\} \uplus \omega\, W' \;.$$

Since $\omega\, W'.\rho_{\mathrm{fst}}((\nu\alpha.\, R_1)\; C_1) \longrightarrow \omega\, W'.\rho_{\mathrm{fst}}(\nu\alpha.\, (R_1\; C_1))$ and $\omega\, W'.\rho_{\mathrm{snd}}((\nu\alpha.\, R_2)\; C_2) \longrightarrow \omega\, W'.\rho_{\mathrm{snd}}(\nu\alpha.\, (R_2\; C_2))$, it suffices to show that

$$(\nu\alpha.\, R_1\; C_1, \nu\alpha.\, R_2\; C_2)_{\omega\, W'} \;\in\; \mathcal{E}[\![B]\!]\,\{\beta \Mapsto (C_1, C_2, r)\} \uplus \omega\, W'$$

by Lemmas 116 and 113. Since $\{\alpha\} \# W$, we can suppose that $\{\alpha\} \# \omega\, W'$ without loss of generality. Since

- $(R_1, R_2) \in \mathcal{R}[\![\forall\beta.B]\!]\; W@\alpha$
- $W'@\alpha \sqsupseteq W@\alpha$ by Lemma 130 with $W' \sqsupseteq W$, and
- $\omega(W'@\alpha) \vdash (C_1, C_2, r)$ from $W' \vdash (C_1, C_2, r)$,

we have

$$(R_1\; C_1, R_2\; C_2)_{\omega(W'@\alpha)} \;\in\; \mathcal{E}[\![B]\!]\,\{\beta \Mapsto (C_1, C_2, r)\} \uplus \omega(W'@\alpha) \;.$$

By Lemma 118, $\{\beta \Mapsto (C_1, C_2, r)\} \uplus (\omega\, W')@\alpha \sqsupseteq \{\beta \Mapsto (C_1, C_2, r)\} \uplus \omega(W'@\alpha)$. By Lemmas 116 and 113, we have

$$(R_1\; C_1, R_2\; C_2)_{(\omega\, W')@\alpha} \;\in\; \mathcal{E}[\![B]\!]\,\{\beta \Mapsto (C_1, C_2, r)\} \uplus (\omega\, W')@\alpha \;.$$

Since we can suppose that $\alpha \neq \beta$ without loss of generality, we have $\{\alpha\} \# \mathit{ftv}(B)$. We also have $\vdash \{\beta \Mapsto (C_1, C_2, r)\} \uplus \omega\, W'$ by Lemma 105 with $\vdash W'$ and $\omega\, W' \vdash (C_1, C_2, r)$. Thus, by the IH, we have the conclusion

$$(\nu\alpha.\, R_1\; C_1, \nu\alpha.\, R_2\; C_2)_{\omega\, W'} \;\in\; \mathcal{E}[\![B]\!]\,\{\beta \Mapsto (C_1, C_2, r)\} \uplus \omega\, W' \;.$$

Case $A = {!B}$: It suffices to show that $(\nu\alpha.\, R_1, \nu\alpha.\, R_2) \in \mathcal{R}[\![!B]\!]\; W$, that is,

$$(\mathsf{let}\, !x = \nu\alpha.\, R_1\, \mathsf{in}\, x, \mathsf{let}\, !x = \nu\alpha.\, R_2\, \mathsf{in}\, x) \;\in\; \mathcal{E}[\![B]\!]\, \omega\, W \;.$$

By Lemma 35, there exist some $\overline{\beta_1}$, $\overline{\beta_2}$, R'_1, and R'_2 such that

- $R_1 = \nu\overline{\beta_1}.\, !R'_1$ and
- $R_2 = \nu\overline{\beta_2}.\, !R'_2$.

Suppose that

- $W' \sqsupseteq \omega\, W$,
- $1 < W'.n$, and
- $W'.\rho_{\mathrm{fst}}(\mathsf{let}\, !x = \nu\alpha.\, \nu\overline{\beta_1}.\, !R'_1\, \mathsf{in}\, x) \longrightarrow W'.\rho_{\mathrm{fst}}(\nu\alpha.\, \nu\overline{\beta_1}.\, R'_1)$

for some W' and n, and then it suffices to show that

$$(\nu\alpha.\, \nu\overline{\beta_1}.\, R'_1, \nu\alpha.\, \nu\overline{\beta_2}.\, R'_2)_{W'} \;\in\; \mathcal{E}[\![B]\!]\, (W' - 1) \;.$$

Since $(R_1, R_2) \in \mathcal{R}[\![!B]\!]\; W@\alpha$, we have $(\nu\overline{\beta_1}.\, R'_1, \nu\overline{\beta_2}.\, R'_2) \in \mathcal{R}[\![B]\!]\, (\omega(W@\alpha) - 1)$. Since $\vdash W$, we have $\vdash \omega\, W - 1$ by Lemma 105. Thus, by the IH, we have

$$(\nu\alpha.\, \nu\overline{\beta_1}.\, R'_1, \nu\alpha.\, \nu\overline{\beta_2}.\, R'_2) \;\in\; \mathcal{R}[\![B]\!]\, (\omega\, W - 1) \;.$$

Since $W' \sqsupseteq \omega\, W$, we have $W' - 1 \sqsupseteq \omega\, W - 1$. Thus, we have the conclusion by Lemma 116.

- Let $(M_1, M_2) \in \mathcal{E}[\![A]\!]\; W@\alpha$. We show $(\nu\alpha.\, M_1, \nu\alpha.\, M_2) \in \mathcal{E}[\![A]\!]\; W$ with the first property. Suppose that

  - $W' \sqsupseteq W$,
  - $n < W'.n$, and
  - $W'.\rho_{\mathrm{fst}}(\nu\alpha.\, M_1) \longrightarrow^n R_1$

  for some W', n, and R_1, and the it suffices to show that there exists some R_2 such that

  - $W'.\rho_{\mathrm{snd}}(\nu\alpha.\, M_2) \longrightarrow^* R_2$ and

- $(R_1, R_2) \in \mathcal{R}[\![A]\!]\,(W' - n)$.

By the semantics, $R_1 = \nu\alpha.\, R'_1$ for some R'_1 such that $W'.\rho_{\mathrm{fst}}(M_1) \longrightarrow^n R'_1$. Since $\{\alpha\}\# W$, we can suppose that $\{\alpha\}\# W'$ without loss of generality. Thus, $W'@\alpha \sqsupseteq W@\alpha$ by Lemma 130 with $W' \sqsupseteq W$. Since $(M_1, M_2) \in \mathcal{E}[\![A]\!]\, W@\alpha$, Lemma 116 implies that there exists some R'_2 such that

- $W'.\rho_{\mathrm{snd}}(M_2) \longrightarrow^* R'_2$ and
- $(R'_1, R'_2) \in \mathcal{R}[\![A]\!]\, W'@\alpha - n$.

By the first property on $\mathcal{R}$ with $\vdash W' - n$ implied by $W' \sqsupseteq W$, we have the conclusion $(\nu\alpha.\, R'_1, \nu\alpha.\, R'_2) \in \mathcal{R}[\![A]\!]\,(W' - n)$ where let $R_2 = \nu\alpha.\, R'_2$.

□

Lemma 132. *If $\{\alpha\}\# W$ and $(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$, then $(W@\alpha, \varsigma) \in \mathcal{G}[\![\Gamma, \alpha^{\mathbf{1}}]\!]$.*

Proof. Since $(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$, we have

- $\vdash W$,
- $\Gamma \succ W.\rho$,
- there exist some Δ and $\prod_{x \in dom_{=\mathbf{1}}(\Gamma)} \Delta_x$ such that
  - $W.\Delta = \Delta + \sum_{x \in dom_{=\mathbf{1}}(\Gamma)} \Delta_x$,
  - $\forall \beta^\pi \in \Gamma.\ (\exists \pi' \geq \pi.\ \beta^{\pi'} \in \Delta) \vee (\pi = \mathbf{0} \wedge \beta \in dom(W.\rho))$,
  - $\forall x :^{\mathbf{1}} A \in \Gamma.\ (\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\,(W.n,\ \Delta_x,\ W.\rho)$, and
  - $\forall x :^{\omega} A \in \Gamma.\ (\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\,\omega W$.

Let $\Delta_0 = \Delta, \alpha^{\mathbf{1}}$ and $\Delta_{0,x} = \Delta_x, \alpha^{\mathbf{0}}$. We have $W.\Delta, \alpha^{\mathbf{1}} = \Delta_0 + \sum_{x \in dom_{=\mathbf{1}}(\Gamma)} \Delta_{0,x}$.

We have $(W@\alpha, \varsigma) \in \mathcal{G}[\![\Gamma, \alpha^{\mathbf{1}}]\!]$ by the following.

- $\vdash W@\alpha$ by Lemma 129.
- We show that $\Gamma, \alpha^{\mathbf{1}} \succ (W@\alpha).\rho$. Let $\beta \in ftv((W@\alpha).\rho|_{dom(\Gamma, \alpha^{\mathbf{1}})}) \cap dom(\Gamma, \alpha^{\mathbf{1}})$. Since $\{\alpha\}\# W$ and $\vdash W$ and $(W@\alpha).\rho = W.\rho$, we have $\alpha \notin dom((W@\alpha).\rho) \cup ftv((W@\alpha).\rho)$. Thus, $\beta \in ftv(W.\rho|_{dom(\Gamma)}) \cap dom(\Gamma)$, and so $\Gamma \succ W.\rho$ implies $\beta^{\mathbf{0}} \in \Gamma$.
- $\alpha^{\mathbf{1}} \in \Delta_0$.
- $\forall x :^{\mathbf{1}} A \in \Gamma, \alpha^{\mathbf{1}}.\ (\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\,(W.n,\ \Delta_{0,x},\ W.\rho)$ by Lemmas 117, 116, and 113.
- $\forall x :^{\omega} A \in \Gamma, \alpha^{\mathbf{1}}.\ (\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\,\omega(W@\alpha)$ because, since $\omega(W@\alpha) \sqsupseteq \omega W$ by Lemma 117 with $\vdash \omega W$, it is proven by Lemmas 116 and 113.

□

Lemma 133. *If $\Gamma, \alpha^{\mathbf{1}} \vdash M_1 \preceq M_2 : A$ and $\Gamma \vdash A$, then $\Gamma \vdash \nu\alpha.\, M_1 \preceq \nu\alpha.\, M_2 : A$.*

Proof. Let $(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$. It suffices to show that

$$(\varsigma_{\mathrm{fst}}(\nu\alpha.\, M_1), \varsigma_{\mathrm{snd}}(\nu\alpha.\, M_2))_W \in \mathcal{E}[\![A]\!]\, W\ .$$

Suppose that

- $W' \sqsupseteq W$,
- $n < W'.n$, and
- $W.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(\nu\alpha.\, M_1)) \longrightarrow^n R_1$

for some W', n, and R_1, and then it suffices to show that there exists some R_2 such that

- $W'.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(\nu\alpha.\, M_2)) \longrightarrow^* R_2$ and
- $(R_1, R_2) \in \mathcal{R}[\![A]\!]\,(W' - n)$.

Without loss of generality, we can suppose that $\{\alpha\} \# W'$. Since $(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$ and $W' \sqsupseteq W$, we have $(W', W'.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma]\!]$ by Lemma 119. Thus, by Lemma 132,

$$(W'@\alpha, W'.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma, \alpha^{\mathbf{1}}]\!] .$$

Since $\Gamma, \alpha^{\mathbf{1}} \vdash M_1 \preceq M_2 : A$, we have

$$(\varsigma_{\mathrm{fst}}(M_1), \varsigma_{\mathrm{snd}}(M_2))_{W'} \in \mathcal{E}[\![A]\!]\; W'@\alpha .$$

Since $W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(\nu\alpha.\, M_1)) \longrightarrow^n R_1$, there exists some R'_1 such that

- $W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(M_1)) \longrightarrow^n R'_1$ and
- $R_1 = \nu\alpha.\, R'_1$.

Thus, there exists some R'_2 such that

- $W'.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(M_2)) \longrightarrow^* R'_2$ and
- $(R'_1, R'_2) \in \mathcal{R}[\![A]\!]\,(W'@\alpha - n)$.

Since $\Gamma \vdash A$, we have $\alpha \notin \mathit{ftv}(A)$. We also have $\vdash W' - n$ from $W' \sqsupseteq W$. Thus, by Lemma 131, $(\nu\alpha.\, R'_1, \nu\alpha.\, R'_2) \in \mathcal{R}[\![A]\!]\,(W' - n)$. We have the conclusion by letting $R_2 = \nu\alpha.\, R'_2$. since $W'.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(\nu\alpha.\, M_2)) \longrightarrow^* \nu\alpha.\, R'_2$. ☐

Lemma 134. *If $(W, \varsigma) \in \mathcal{G}[\![\Gamma_1, \alpha^{\mathbf{1}}, \Gamma_2]\!]$, then there exist some Δ_1 and Δ_2 such that*

- $W.\Delta = \Delta_1, \alpha^{\mathbf{1}}, \Delta_2$ *and*
- $((W.n, (\Delta_1, \alpha^{\mathbf{0}}, \Delta_2), W.\rho), \varsigma) \in \mathcal{G}[\![\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2]\!]$.

Proof. Let $\Gamma = \Gamma_1, \alpha^{\mathbf{1}}, \Gamma_2$. Since $(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$, we have

- $\vdash W$,
- $\Gamma \succ W.\rho$,
- there exist some Δ and $\prod_{x \in \mathit{dom}_{=\mathbf{1}}(\Gamma)} \Delta_x$ such that
  - $W.\Delta = \Delta + \sum_{x \in \mathit{dom}_{=\mathbf{1}}(\Gamma)} \Delta_x$,
  - $\forall \beta^{\pi} \in \Gamma.\ (\exists \pi' \geq \pi.\ \beta^{\pi'} \in \Delta) \vee (\pi = \mathbf{0} \wedge \beta \in \mathit{dom}(W.\rho))$,
  - $\forall x :^{\mathbf{1}} A \in \Gamma.\ (\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\,(W.n, \Delta_x, W.\rho)$, and
  - $\forall x :^{\omega} A \in \Gamma.\ (\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\,\omega W$.

Since $\alpha^{\mathbf{1}} \in \Gamma$, we have $\alpha^{\mathbf{1}} \in \Delta$. Let Δ'_1 and Δ'_2 such that $\Delta = \Delta'_1, \alpha^{\mathbf{1}}, \Delta'_2$.

Let $\Delta' = \Delta'_1, \alpha^{\mathbf{0}}, \Delta'$ and $W' = (W.n, \Delta' + \sum_{x \in \mathit{dom}_{=\mathbf{1}}(\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2)} \Delta_x, W.\rho)$. Since $\Delta \bot \Delta_x$, we have $\alpha^{\mathbf{0}} \in \Delta_x$. Thus, $\alpha^{\mathbf{0}} \in W'.\Delta$.

Finally, $(W', \varsigma) \in \mathcal{G}[\![\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2]\!]$ is shown by the following.

- $\vdash W'$ by Lemma 105 with $\vdash W$.
- $\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2 \succ W.\rho$ holds obviously because $\mathit{dom}(\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2) = \mathit{dom}(\Gamma)$ and $\Gamma \succ W.\rho$.
- $\alpha^{\mathbf{0}} \in \Delta'$.

☐

Lemma 135. *If* $\vdash W$ *and* $\{\alpha\}\#\omega W$ *and* $\omega W \vdash (A_1, A_2, r)$*, then* $\{\alpha \Mapsto (A_1, A_2, r)\} \uplus \omega W \sqsupseteq \omega(W@\alpha)$*.*

Proof. We have the conclusion by the following.

- We show that $\vdash \omega(W@\alpha)$. By Lemmas 9 and 105 with $\vdash W$, we have $\vdash \omega W$. By Lemma 129 with $\{\alpha\}\#\omega W$, we have $\vdash \omega(W@\alpha)$.
- $\vdash \{\alpha \Mapsto (A_1, A_2, r)\} \uplus \omega W$ by definition with $\vdash \omega W$ and $\omega W \vdash (A_1, A_2, r)$ and $\{\alpha\}\#\omega W$.
- We have $\omega W.\Delta, \dagger(\{\alpha \Mapsto (A_1, A_2, r)\}) = \omega W.\Delta, \alpha^{\mathbf{0}} \ggcurly \omega W.\Delta, \alpha^{\mathbf{0}} = \omega(W@\alpha).\Delta$ by Lemma 5.
- We have $\{\alpha \Mapsto (A_1, A_2, r)\} \uplus \omega W.\rho = \{\alpha \Mapsto (A_1, A_2, r)\} \circ \omega W.\rho$ because $\{\alpha\}\#\omega W$ and $\vdash W$.
- $\omega(W@\alpha).\Delta \succ \{\alpha \Mapsto (A_1, A_2, r)\}$ holds obviously because $\omega(W@\alpha).\Delta$ assigns the use $\mathbf{0}$ to all the bound type variables.

□

Lemma 136. *If* $W_1 \sqsupseteq W_2$ *and* $\forall\, \alpha \in dom(\rho).\ W_2 \vdash \rho(\alpha)$*, then* $\forall\, \alpha \in dom(W_1.\rho(\rho)).\ W_1 \vdash W_1.\rho(\rho)(\alpha)$*.*

Proof. Let $\alpha \in dom(\rho) = dom(W_1.\rho(\rho))$, $(A_1, A_2, r) = \rho(\alpha)$, $B_1 = W_1.\rho_{\mathrm{fst}}(A_1)$, and $B_2 = W_1.\rho_{\mathrm{snd}}(A_2)$. We have $W_1.\rho(\rho)(\alpha) = (B_1, B_2, r)$. It suffices to show that $W_1 \vdash (B_1, B_2, r)$, which is proven below.

- We show that $W_1.\Delta \vdash B_1$; $W_1.\Delta \vdash B_2$ can be proven similarly.

  Since $B_1 = W_1.\rho_{\mathrm{fst}}(A_1) = W_1.\rho_{\mathrm{fst}}(\rho_{\mathrm{fst}}(\alpha))$, it suffices to show that $ftv(W_1.\rho) \subseteq dom(W_1.\Delta)$ and $ftv(\rho) \setminus dom(W_1.\rho) \subseteq dom(W_1.\Delta)$.

  We have $ftv(W_1.\rho) \subseteq dom(W_1.\Delta)$ since $\vdash W_1$ implied by $W_1 \sqsupseteq W_2$.

  We show that $ftv(\rho) \setminus dom(W_1.\rho) \subseteq dom(W_1.\Delta)$. Let $\beta \in ftv(\rho) \setminus dom(W_1.\rho)$. By the assumption $\forall\, \alpha \in dom(\rho).\ W_2 \vdash \rho(\alpha)$, we have $ftv(\rho) \subseteq dom(W_2.\Delta)$. Thus, $\beta \in dom(W_2.\Delta) \setminus dom(W_1.\rho)$. Since $W_1 \sqsupseteq W_2$, there exists some ρ' such that

  - $W_1.\Delta, \dagger(\rho') \ggcurly W_2.\Delta$ and
  - $W_1.\rho = \rho' \circ W_2.\rho$.

  $W_1.\Delta, \dagger(\rho') \ggcurly W_2.\Delta$ implies $dom(W_2.\Delta) \subseteq dom(W_1.\Delta) \cup dom(\rho')$. Thus, $\beta \in (dom(W_1.\Delta) \cup dom(\rho')) \setminus dom(W_1.\rho)$. $W_1.\rho = \rho' \circ W_2.\rho$ implies $dom(\rho') \subseteq dom(W_1.\rho)$. Thus, $\beta \in dom(W_1.\Delta)$.
- We show that $r \in \mathrm{Rel}_{W_1.n}[B_1, B_2]$. By the assumption $\forall\, \alpha \in dom(\rho).\ W_2 \vdash \rho(\alpha)$, we have $r \in \mathrm{Rel}_{W_2.n}[A_1, A_2]$. Thus, $r \in \mathrm{Rel}_{W_2.n}[B_1, B_2]$ by definition. Since $W_1.n \leq W_2.n$, we can view $r \in \mathrm{Rel}_{W_1.n}[B_1, B_2]$.

□

Lemma 137. *If* $W_1 \sqsupseteq W_2$ *and* $dom(\rho)\# W_1$ *and* $\vdash \rho \uplus W_2$*, then* $W_1.\rho(\rho) \uplus W_1 \sqsupseteq \rho \uplus W_2$*.*

Proof. Since $W_1 \sqsupseteq W_2$, we have the following.

- $\vdash W_1$ and $\vdash W_2$,
- $W_1.n \leq W_2.n$, and
- there exists some ρ' such that
  - $W_1.\Delta, \dagger(\rho') \ggcurly W_2.\Delta$,
  - $W_1.\rho = \rho' \circ W_2.\rho$, and
  - $W_2.\Delta \succ \rho'$.

Let $W_1' = W_1.\rho(\rho) \uplus W_1$. The conclusion $W_1' \sqsupseteq \rho \uplus W_2$ is shown in what follows, where ρ' is the intermediate interpretation mapping.

- We show that $\vdash W_1'$.

- We show that $dom(W_1'.\Delta) \# dom(W_1'.\rho)$. Since $\vdash W_1$, we have $dom(W_1.\rho) \# dom(W_1.\Delta)$. By the assumption $dom(\rho) \# W_1$, we have $dom(\rho) \# dom(W_1.\rho)$ and $dom(\rho) \# dom(W_1.\Delta)$. The first property implies that W_1' is well defined. The second property implies $dom(W_1'.\rho) = (dom(\rho) \cup dom(W_1.\rho)) \# dom(W_1.\Delta) = dom(W_1'.\Delta)$.
- Let $\alpha \in dom(W_1'.\rho)$. We show that $W_1' \vdash W_1'.\rho(\alpha)$.
  Since $\alpha \in dom(W_1'.\rho)$, we have $\alpha \in dom(\rho)$ or $\alpha \in dom(W_1.\rho)$.
  Case $\alpha \in dom(\rho)$: By $\vdash W_1$.
  Case $\alpha \in dom(\rho)$: We have $W_1 \vdash W_1.\rho(\rho)(\alpha)$ by Lemma 136 with $W_1 \sqsupseteq W_2$ and $\vdash \rho \uplus W_2$. Thus, $W_1' \vdash W_1.\rho(\rho)(\alpha) = W_1'.\rho(\alpha)$.

- $\vdash \rho \uplus W_2$ by the assumption.
- $W_1'.n = W_1.n \leq W_2.n = (\rho \uplus W_2).n$.
- $W_1'.\Delta, \dagger(\rho') = W_1.\Delta, \dagger(\rho') \gg W_2.\Delta = (\rho \uplus W_2).\Delta$.
- We show that $W_1'.\rho = \rho' \circ (\rho \uplus W_2.\rho)$. We have $ftv(\rho) \# dom(W_2.\rho)$ because $ftv(\rho) \subseteq dom(W_2.\Delta)$ by $\vdash \rho \uplus W_2$, and $dom(W_2.\Delta) \# dom(W_2.\rho)$ by $\vdash W_2$. Thus:

$$
\begin{array}{rcll}
W_1'.\rho & = & W_1.\rho(\rho) \uplus W_1.\rho & \text{(by definition)} \\
& = & (\rho' \circ W_2.\rho)(\rho) \uplus (\rho' \circ W_2.\rho) & \text{(since } W_1.\rho = \rho' \circ W_2.\rho) \\
& = & (\rho' \uplus \rho'(W_2.\rho))(\rho) \uplus \rho' \uplus \rho'(W_2.\rho) & \\
& = & \rho'(\rho) \uplus \rho' \uplus \rho'(W_2.\rho) & \text{(since } ftv(\rho) \# dom(W_2.\rho)) \\
& = & \rho' \uplus \rho'(\rho \uplus W_2.\rho) & \\
& = & \rho' \circ (\rho \uplus W_2.\rho)\ . &
\end{array}
$$

- We have $(\rho \uplus W_2).\Delta \succ \rho'$ because $W_2.\Delta \succ \rho'$ and $(\rho \uplus W_2).\Delta = W_2.\Delta$.

□

Lemma 138. *If $W_1 \sqsupseteq \rho \uplus W_2$, then there exists some W_1' such that $W_1' \sqsupseteq W_2$ and $W_1 = W_1'.\rho(\rho) \uplus W_1'$.*

Proof. Since $W_1 \sqsupseteq \rho \uplus W_2$, we have

- $\vdash W_1$ and $\vdash \rho \uplus W_2$,
- $W_1.n \leq (\rho \uplus W_2).n = W_2.n$, and
- there exists some ρ' such that (note that $(\rho \uplus W_2).\Delta = W_2.\Delta$):
  - $W_1.\Delta, \dagger(\rho') \gg W_2.\Delta$;
  - $W_1.\rho = \rho' \circ (\rho \uplus W_2.\rho)$; and
  - $(\rho \uplus W_2).\Delta = W_2.\Delta \succ \rho'$.

Let $W_1' = (W_1.n,\ W_1.\Delta,\ \rho' \circ W_2.\rho)$.
We first show that $W_1' \sqsupseteq W_2$ with ρ' as the intermediate interpretation mapping.

- We show that $\vdash W_1'$.
  - We show that $dom(W_1'.\Delta) \# dom(W_1'.\rho)$. By the definition of W_1', it suffices to show that $dom(W_1.\Delta) \# (dom(\rho') \cup dom(W_2.\rho))$. Since $\vdash W_1$ and $W_1.\rho = \rho' \circ (\rho \uplus W_2.\rho)$, we have $dom(W_1.\Delta) \# (dom(\rho') \cup dom(\rho) \cup dom(W_2.\rho))$.
  - We show that $\forall \alpha \in dom(W_1'.\rho).\ W_1' \vdash W_1'.\rho(\alpha)$. This is proven by $\vdash W_1$ and $W_1.\rho = \rho' \circ (\rho \uplus W_2.\rho)$.
- We have $\vdash W_2$ by $\vdash \rho \uplus W_2$.
- $W_1'.n = W_1.n \leq W_2.n$.
- $W_1'.\Delta, \dagger(\rho') = W_1.\Delta, \dagger(\rho') \gg W_2.\Delta$.

- $W_1'.\rho = \rho' \circ W_2.\rho$ by definition.
- We have $W_2.\Delta \succ \rho'$.

Next, we show that $W_1 = W_1'.\rho(\rho) \uplus W_1'$. It suffices to show that $W_1.\rho = W_1'.\rho(\rho) \uplus W_1'.\rho$. Noting that $ftv(\rho) \# dom(W_2.\rho)$ because $ftv(\rho) \subseteq dom(W_2.\Delta)$ and $dom(W_2.\Delta) \# dom(W_2.\rho)$ by $\vdash \rho \uplus W_2$, we have:

$$\begin{array}{rcll} W_1.\rho & = & \rho' \circ (\rho \uplus W_2.\rho) \\ & = & \rho' \uplus \rho'(\rho) \uplus \rho'(W_2.\rho) \\ & = & \rho'(\rho) \uplus \rho' \circ W_2.\rho \\ & = & (\rho' \uplus \rho'(W_2.\rho))(\rho) \uplus \rho' \circ W_2.\rho & \text{(since } ftv(\rho) \# dom(W_2.\rho)\text{)} \\ & = & (\rho' \circ W_2.\rho)(\rho) \uplus \rho' \circ W_2.\rho \\ & = & W_1'.\rho(\rho) \uplus W_1'.\rho\ . \end{array}$$

□

Lemma 139. *If $\vdash W$ and $dom(\rho) \# W$ and $\forall \alpha \in dom(\rho).\ W \vdash \rho(\alpha)$, then $\rho \uplus W \sqsupseteq W$.*

Proof. We have the conclusion by the following, where ρ is used as the intermediate interpretation mapping.

- We have $\vdash W$ by the assumption.
- We show that $\vdash \rho \uplus W$.

  Since $\vdash W$ and $dom(\rho) \# W$, we have $dom((\rho \uplus W).\Delta) = dom(W.\Delta) \# (dom(\rho) \cup dom(W.\rho)) = dom((\rho \uplus W).\rho)$.

  Let $\alpha \in dom((\rho \uplus W).\rho)$. If $\alpha \in dom(\rho)$, then we have $W \vdash \rho(\alpha)$ by the assumption. Otherwise, if $\alpha \in dom(W.\rho)$, then $\vdash W$ implies $W \vdash W.\rho(\alpha)$. Thus, in either case, $\rho \uplus W \vdash (\rho \uplus W).\rho(\alpha)$.
- We have $W_1.n = (\rho \uplus W).n$.
- We have $(\rho \uplus W).\Delta, \dagger(\rho) = W.\Delta, \dagger(\rho) \gg W.\Delta$ by Lemma 9.
- We have $(\rho \uplus W).\rho = \rho \uplus W.\rho$.
- We show that $W.\Delta \succ \rho$. It suffices to show that $dom(\rho) \# dom(W.\Delta)$, which is implied by $dom(\rho) \# W$.

□

Lemma 140. *If $dom(\rho) \# ftv(A)$,*

1. *$\mathcal{R}[\![A]\!]\, \rho \uplus W \subseteq \mathcal{R}[\![A]\!]\, W$ and*
2. *$\mathcal{E}[\![A]\!]\, \rho \uplus W \subseteq \mathcal{E}[\![A]\!]\, W$.*

Proof. By induction on A. We first consider the first case and then show the second case with the first property.

1. Let $(R_1, R_2) \in \mathcal{R}[\![A]\!]\, \rho \uplus W$. We show that $(R_1, R_2) \in \mathcal{R}[\![A]\!]\, W$. By case analysis on A.

   Case $A = \iota$: Obvious.

   Case $A = \alpha$: Since $(R_1, R_2) \in \mathcal{R}[\![\alpha]\!]\, \rho \uplus W$, we have $(R_1, R_2) \in (\rho \uplus W).\rho[\alpha](\blacktriangleright(\rho \uplus W))$. Since $dom(\rho) \# \{\alpha\}$, we have $(R_1, R_2) \in W.\rho[\alpha](\blacktriangleright(\rho \uplus W))$. Let $(B_1, B_2, r) = W.\rho(\alpha)$. Since $\rho \uplus W$ is well defined, we have $dom(\rho) \# W$. Since $(\rho \uplus W).\Delta \vdash B_1$ and $(\rho \uplus W).\Delta \vdash B_2$, we have $dom(\rho) \# ftv(B_1)$ and $dom(\rho) \# ftv(B_2)$. Thus, by the irrelevance condition on $W.\rho[\alpha] = r \in \mathrm{Rel}_{W.n}[B_1, B_2]$, we have $(R_1, R_2) \in W.\rho[\alpha](\blacktriangleright W)$. Thus, $(R_1, R_2) \in \mathcal{R}[\![\alpha]\!]\, W$.

   Case $A = B \multimap C$: Suppose that

   - $W' \sqsupseteq W$,
   - $(W_1, W_2) \Supset W'$,
   - $W_1 \sqsupseteq W$, and
   - $(R_1', R_2') \in \mathcal{R}[\![B]\!]\, W_2$

for some W', W_1, W_2, R'_1, and R'_2, and then it suffices to show that

$$(R_1\, R'_1, R_2\, R'_2)_{W'} \in \mathcal{E}[\![C]\!]\; W' \;.$$

Without loss of generality, we can suppose that $dom(\rho) \# W'$. Since $W' \sqsupseteq W$ and $W_1 \sqsupseteq W$, Lemma 137 implies

- $W'.\rho(\rho) \uplus W' \sqsupseteq \rho \uplus W$ and
- $W_1.\rho(\rho) \uplus W_1 \sqsupseteq \rho \uplus W$

Since $W_1.\rho = W'.\rho$ from $(W_1, W_2) \Supset W'$, we have

$$W'.\rho(\rho) \uplus W_1 \sqsupseteq \rho \uplus W \;.$$

Since $(W_1, W_2) \Supset W'$, we have

$$(W'.\rho(\rho) \uplus W_1, W'.\rho(\rho) \uplus W_2) \Supset W'.\rho(\rho) \uplus W' \;.$$

We have the following.

- $\vdash W_2$ by Lemma 105 with $\vdash W'$, which is implied by $W' \sqsupseteq W$.
- $dom(W'.\rho(\rho)) \# W_2$ since $dom(W'.\rho(\rho)) \# W'$, which is implied by well-definedness of $W'.\rho(\rho) \uplus W'$.
- $\forall \alpha \in dom(W'.\rho(\rho)).\ W_2 \vdash W'.\rho(\rho)(\alpha)$ since $\vdash W'.\rho(\rho) \uplus W'$, which is implied by $W'.\rho(\rho) \uplus W' \sqsupseteq \rho \uplus W$.

Thus, by Lemma 139,

$$W'.\rho(\rho) \uplus W_2 \sqsupseteq W_2 \;.$$

Since $(R'_1, R'_2) \in \mathcal{R}[\![B]\!]\; W_2$, we have

$$(R'_1, R'_2) \in \mathcal{R}[\![B]\!]\; W'.\rho(\rho) \uplus W_2$$

by Lemmas 116 and 113. Note that $dom(\rho) \# dom(W_2.\Delta) \supseteq ftv(R'_1) \cup ftv(R'_1)$. Since

- $(R_1, R_2) \in \mathcal{R}[\![B \multimap C]\!]\; \rho \uplus W$ (which further implies $dom(\rho) \# (ftv(R_1) \cup ftv(R_2))$),
- $W'.\rho(\rho) \uplus W' \sqsupseteq \rho \uplus W$,
- $(W'.\rho(\rho) \uplus W_1, W'.\rho(\rho) \uplus W_2) \Supset W'.\rho(\rho) \uplus W'$,
- $W'.\rho(\rho) \uplus W_1 \sqsupseteq \rho \uplus W$, and
- $(R'_1, R'_2) \in \mathcal{R}[\![B]\!]\; W'.\rho(\rho) \uplus W_2$,

we have

$$(R_1\, R'_1, R_2\, R'_2)_{W'.\rho(\rho) \uplus W'} \in \mathcal{E}[\![C]\!]\; W'.\rho(\rho) \uplus W' \;.$$

Since $dom(\rho) \# (ftv(R_1) \cup ftv(R'_1) \cup ftv(R_2) \cup ftv(R'_2))$, we have

$$(R_1\, R'_1, R_2\, R'_2)_{W'} \in \mathcal{E}[\![C]\!]\; W'.\rho(\rho) \uplus W' \;.$$

Since $dom(\rho) \# ftv(B \multimap C)$ implies $dom(W'.\rho(\rho)) \# ftv(C)$, we have the conclusion

$$(R_1\, R'_1, R_2\, R'_2)_{W'} \in \mathcal{E}[\![C]\!]\; W'$$

by the IH.

Case $A = \forall \alpha.B$: Suppose that

- $W' \sqsupseteq W$,
- $\omega W' \vdash (C_1, C_2, r)$, and
- $\{\alpha\} \# \omega W'$

for some W', C_1, C_2, and r, and then it suffices to show that

$$(R_1\, C_1, R_2\, C_2)_{\omega W'} \in \mathcal{E}[\![B]\!]\; \{\alpha \Mapsto (C_1, C_2, r)\} \uplus \omega W' \;.$$

Without loss of generality, we can suppose that $dom(\rho) \# W'$ and $dom(\rho) \# \{\alpha\}$. We have the following.

- $(R_1, R_2) \in \mathcal{R}[\![\forall\alpha.B]\!]\,\rho \uplus W$.
- $W'.\rho(\rho) \uplus W' \sqsupseteq \rho \uplus W$ by Lemma 137 with $W' \sqsupseteq W$.
- $\omega(W'.\rho(\rho) \uplus W') \vdash (C_1, C_2, r)$ from $\omega\, W' \vdash (C_1, C_2, r)$.
- $\{\alpha\}\#\omega(W'.\rho(\rho) \uplus W')$ from $\{\alpha\}\#\omega\, W'$ and $dom(\rho) \# \{\alpha\}$.

Thus, we have

$$(R_1\ C_1, R_2\ C_2)_{\omega(W'.\rho(\rho)\uplus W')} \in \mathcal{E}[\![B]\!]\,\{\alpha \Mapsto (C_1, C_2, r)\} \uplus \omega(W'.\rho(\rho) \uplus W')$$

Since $dom(\rho) \# (ftv(R_1) \cup ftv(R_2) \cup ftv(C_1) \cup ftv(C_2))$, we have

$$(R_1\ C_1, R_2\ C_2)_{\omega\, W'} \in \mathcal{E}[\![B]\!]\ W'.\rho(\rho) \uplus (\{\alpha \Mapsto (C_1, C_2, r)\} \uplus \omega\, W')\ .$$

Since $dom(\rho) \# ftv(\forall\alpha.B)$ and $dom(\rho) \# \{\alpha\}$ implies $dom(W'.\rho(\rho)) = dom(\rho) \# ftv(B)$, we have the conclusion

$$(R_1\ C_1, R_2\ C_2)_{\omega\, W'} \in \mathcal{E}[\![B]\!]\,\{\alpha \Mapsto (C_1, C_2, r)\} \uplus \omega\, W'$$

by the IH.

Case $A = {!B}$: By the IH.

2. Let $(M_1, M_2) \in \mathcal{E}[\![A]\!]\,\rho \uplus W$. We show that $(M_1, M_2) \in \mathcal{E}[\![A]\!]\,W$. Suppose that

- $W' \sqsupseteq W$,
- $n < W'.n$,
- $W'.\rho_{\mathrm{fst}}(M_1) \longrightarrow^n R_1$

for some W', n, and R_1, and then it suffices to show that there exists some R_2 such that

- $W'.\rho_{\mathrm{snd}}(M_2) \longrightarrow^* R_2$ and
- $(R_1, R_2) \in \mathcal{R}[\![A]\!]\,(W' - n)$.

Without loss of generality, we can suppose that $dom(\rho)\# W'$. Since $W' \sqsupseteq W$, we have

$$W'.\rho(\rho) \uplus W' \sqsupseteq \rho \uplus W$$

by Lemma 137. Since $(M_1, M_2) \in \mathcal{E}[\![A]\!]\,\rho \uplus W$, we have $dom(\rho) \# dom(W.\Delta) \supseteq ftv(M_1) \cup ftv(M_2)$. Thus,

- $(W'.\rho(\rho) \uplus W').\rho_{\mathrm{fst}}(M_1) = W'.\rho_{\mathrm{fst}}(M_1)$ and
- $(W'.\rho(\rho) \uplus W').\rho_{\mathrm{snd}}(M_2) = W'.\rho_{\mathrm{snd}}(M_2)$.

We have the following.

- $(M_1, M_2) \in \mathcal{E}[\![A]\!]\,\rho \uplus W$.
- $W'.\rho(\rho) \uplus W' \sqsupseteq \rho \uplus W$.
- $n < W'.n = (W'.\rho(\rho) \uplus W').n$.
- $(W'.\rho(\rho) \uplus W').\rho_{\mathrm{fst}}(M_1) = W'.\rho_{\mathrm{fst}}(M_1) \longrightarrow^n R_1$.

Thus, there exists some R_2 such that

- $(W'.\rho(\rho) \uplus W').\rho_{\mathrm{snd}}(M_2) = W'.\rho_{\mathrm{snd}}(M_2) \longrightarrow^* R_2$ and
- $(R_1, R_2) \in \mathcal{R}[\![A]\!]\,(W'.\rho(\rho) \uplus W') - n$.

Since $dom(\rho) \# ftv(A)$ implies $dom(W'.\rho(\rho)) \# ftv(A)$, we have the conclusion

$$(R_1, R_2) \in \mathcal{R}[\![A]\!]\,(W' - n)$$

by the first property on $\mathcal{R}$.

□

Lemma 141. *If* $\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2 \vdash M_1 \preceq M_2 : !A$*, then* $\Gamma_1, \alpha^{\mathbf{1}}, \Gamma_2 \vdash \Lambda^\circ\langle \alpha, M_1 \rangle \preceq \Lambda^\circ\langle \alpha, M_2 \rangle : !\forall\alpha.A$*.*

Proof. Let $(W, \varsigma) \in \mathcal{G}[\![\Gamma_1, \alpha^{\mathbf{1}}, \Gamma_2]\!]$. It suffices to show that

$$(\varsigma_{\mathrm{fst}}(\Lambda^\circ\langle \alpha, M_1 \rangle), \varsigma_{\mathrm{snd}}(\Lambda^\circ\langle \alpha, M_2 \rangle))_W \in \mathcal{E}[\![!\forall\alpha.A]\!]\, W\ .$$

Suppose that

- $W_1 \sqsupseteq W$,
- $n < W_1.n$, and
- $W_1.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(\Lambda^\circ\langle \alpha, M_1 \rangle)) \longrightarrow^n R_1$

for some W_1, n, and R_1, and then it suffices to show that there exists some R_2 such that

- $W_1.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(\Lambda^\circ\langle \alpha, M_2 \rangle)) \longrightarrow^* R_2$ and
- $(R_1, R_2) \in \mathcal{R}[\![!\forall\alpha.A]\!]\,(W_1 - n)$.

Since $(W, \varsigma) \in \mathcal{G}[\![\Gamma_1, \alpha^{\mathbf{1}}, \Gamma_2]\!]$ and $W_1 \sqsupseteq W$, we have $(W_1, W_1.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma_1, \alpha^{\mathbf{1}}, \Gamma_2]\!]$ by Lemma 119. Lemma 134 implies that there exist some W_1', Δ_1, and Δ_2 such that

- $W_1.\Delta = \Delta_1, \alpha^{\mathbf{1}}, \Delta_2$, and
- $W_1' = (W_1.n, (\Delta_1, \alpha^{\mathbf{0}}, \Delta_2), W_1.\rho)$,
- $(W_1', W_1.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2]\!]$.

Note that $W_1.\rho = W_1'.\rho$ and $W_1.n = W_1'.n$. Since $\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2 \vdash M_1 \preceq M_2 : !A$, we have

$$(\varsigma_{\mathrm{fst}}(M_1), \varsigma_{\mathrm{snd}}(M_2))_{W_1'} \in \mathcal{E}[\![!A]\!]\, W_1'\ .$$

Since $W_1.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(\Lambda^\circ\langle \alpha, M_1 \rangle)) \longrightarrow^n R_1$, there exist some $\overline{\beta_1}$, R_1', and $n_1 < n$ such that

- $W_1.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(M_1)) \longrightarrow^{n_1} \nu\overline{\beta_1}.\,!R_1'$ and
- $W_1.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(\Lambda^\circ\langle \alpha, M_1 \rangle)) \longrightarrow^{n_1} \Lambda^\circ\langle \alpha, \nu\overline{\beta_1}.\,!R_1' \rangle \longrightarrow^{n-n_1} \nu\overline{\beta_1}.\,!\Lambda\alpha.R_1' = R_1$.

Since $W_1'.\rho = W_1.\rho$ and $n < W_1.n = W_1'.n$, we can find that there exist some $\overline{\beta_2}$ and R_2' such that

- $W_1'.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(M_2)) \longrightarrow^* \nu\overline{\beta_2}.\,!R_2'$ and
- $(\nu\overline{\beta_1}.\,!R_1', \nu\overline{\beta_2}.\,!R_2') \in \mathcal{R}[\![!A]\!]\,(W_1' - n_1)$.

Thus, we have

$$W_1'.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(\Lambda^\circ\langle \alpha, M_2 \rangle)) \longrightarrow^* \Lambda^\circ\langle \alpha, \nu\overline{\beta_2}.\,!R_2' \rangle \longrightarrow^* \nu\overline{\beta_2}.\,!\Lambda\alpha.R_2'\ .$$

We let $R_2 = \nu\overline{\beta_2}.\,!\Lambda\alpha.R_2'$. Then, it suffices to show that

$$(\nu\overline{\beta_1}.\,!\Lambda\alpha.R_1', \nu\overline{\beta_2}.\,!\Lambda\alpha.R_2') \in \mathcal{R}[\![!\forall\alpha.A]\!]\,(W_1 - n)\ .$$

By definition, it suffices to show that, for any W_2 such that $W_2 \sqsupseteq \omega(W_1 - n)$ and $1 < W_2.n$,

$$(\nu\overline{\beta_1}.\,\Lambda\alpha.R_1', \nu\overline{\beta_2}.\,\Lambda\alpha.R_2')_{W_2} \in \mathcal{R}[\![\forall\alpha.A]\!]\,(W_2 - 1)\ .$$

By alpha-renaming the type variable α bound in A to a fresh type variable γ, it suffices to show that

$$(\nu\overline{\beta_1}.\,\Lambda\alpha.R_1', \nu\overline{\beta_2}.\,\Lambda\alpha.R_2')_{W_2} \in \mathcal{R}[\![\forall\gamma.A[\gamma/\alpha]]\!]\,(W_2 - 1)\ .$$

Let $W_3 \sqsupseteq W_2 - 1$ and suppose that $\omega\, W_3 \vdash \overline{(B_1, B_2, r)}$ for some B_1, B_2, and r. Without loss of generality, we can suppose that $\{\gamma\}\#\omega\, W_3$ and that the type variables $\overline{\beta_1}$ and $\overline{\beta_2}$ do not occur free in B_1 and B_2, respectively. Then, it suffices to show that

$$((\nu\overline{\beta_1}.\, \Lambda\alpha.R_1')\, B_1, (\nu\overline{\beta_2}.\, \Lambda\alpha.R_2')\, B_2)_{\omega\, W_3} \in \mathcal{E}[\![A[\gamma/\alpha]]\!]\, \{\gamma \Mapsto (B_1, B_2, r)\} \uplus \omega\, W_3 \ .$$

Let $W_4 \sqsupseteq \{\gamma \Mapsto (B_1, B_2, r)\} \uplus \omega\, W_3$ and suppose that $1 < W_4.n$, and then it suffices to show that

$$(\nu\overline{\beta_1}.\, R_1'[B_1/\alpha], \nu\overline{\beta_2}.\, R_2'[B_2/\alpha])_{W_4} \in \mathcal{R}[\![A[\gamma/\alpha]]\!]\, (W_4 - 1) \ .$$

Since $(\nu\overline{\beta_1}.\, !R_1', \nu\overline{\beta_2}.\, !R_2') \in \mathcal{R}[\![!A]\!]\, (W_1' - n_1)$ and $1 < W_2.n \leq \omega\, W_1.n - n < W_1'.n - n_1$, we have

$$(\nu\overline{\beta_1}.\, R_1', \nu\overline{\beta_2}.\, R_2') \in \mathcal{R}[\![A]\!]\, \omega(W_1' - n_1 - 1) \ .$$

Let

- $\rho_1 = W_1.\rho|_{dom(\Gamma_1, \alpha^{\mathbf{1}}, \Gamma_2)}$,
- $\rho_2 = W_1.\rho|_{dom(W_1.\rho) \setminus dom(\Gamma_1, \alpha^{\mathbf{1}}, \Gamma_2)}$, and
- $W_1'' = (W_1.n,\ W_1'.\Delta,\ \rho_1)$.

Since $W_1'.n = W_1.n$ and $W_1'.\rho = W_1.\rho = \rho_1 \uplus \rho_2$, we have $\rho_2 \uplus W_1'' = W_1'$. Thus, we have

$$(\nu\overline{\beta_1}.\, R_1', \nu\overline{\beta_2}.\, R_2') \in \mathcal{R}[\![A]\!]\, \rho_2 \uplus \omega(W_1'' - n_1 - 1) \ .$$

Since $\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2 \vdash M_1 \preceq M_2 : !A$ implies $\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2 \vdash M_1 : !A$, we have $\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2 \vdash A$ by Lemma 18. Thus, $dom(\rho_2) \# dom(\Gamma_1, \alpha^{\mathbf{0}}, \Gamma_2) \supseteq ftv(A)$. Hence, by Lemma 140,

$$(\nu\overline{\beta_1}.\, R_1', \nu\overline{\beta_2}.\, R_2') \in \mathcal{R}[\![A]\!]\, \omega(W_1'' - n_1 - 1) \ .$$

Since $(W_1, W_1.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma_1, \alpha^{\mathbf{1}}, \Gamma_2]\!]$, we have $\Gamma_1, \alpha^{\mathbf{1}}, \Gamma_2 \succ W_1.\rho$. Thus, $\alpha \notin ftv(W_1.\rho|_{dom(\Gamma_1, \alpha^{\mathbf{1}}, \Gamma_2)}) = ftv(\rho_1) = ftv(W_1''.\rho)$. Hence, noting that we can suppose $\{\gamma\} \# dom(W_1''.\Delta) = dom(W_1'.\Delta) = dom(\Delta_1, \alpha^{\mathbf{0}}, \Delta_2)$ without loss of generality, by alpha-renaming the type variable α in the above formula to γ, we have

$$(\nu\overline{\beta_1}.\, R_1'[\gamma/\alpha], \nu\overline{\beta_2}.\, R_2'[\gamma/\alpha]) \in \mathcal{R}[\![A[\gamma/\alpha]]\!]\, \omega((W_1.n,\ (\Delta_1, \gamma^{\mathbf{0}}, \Delta_2),\ \rho_1) - n_1 - 1) \ .$$

Let $W_1''' = (W_1.n,\ (W_1'.\Delta, \gamma^{\mathbf{0}}),\ W_1.\rho)$. By applying Lemmas 110, 112, 116, and 113 with

$$\begin{array}{rcll} W_1''' & = & (W_1.n,\ (W_1'.\Delta, \gamma^{\mathbf{0}}),\ \rho_1 \uplus \rho_2) & \\ & \sqsupseteq & (W_1.n,\ (W_1'.\Delta, \dagger(\rho_2), \gamma^{\mathbf{0}}),\ \rho_1) & \text{(by Lemma 135)} \\ & \sqsupseteq & (W_1.n,\ (W_1'.\Delta, \gamma^{\mathbf{0}}),\ \rho_1) & \text{(by Lemma 117)} \\ & = & (W_1.n,\ (\Delta_1, \alpha^{\mathbf{0}}, \Delta_2, \gamma^{\mathbf{0}}),\ \rho_1) & \\ & \sqsupseteq & (W_1.n,\ (\Delta_1, \gamma^{\mathbf{0}}, \Delta_2),\ \rho_1) & \text{(by Lemma 117)}\ , \end{array}$$

we have

$$(\nu\overline{\beta_1}.\, R_1'[\gamma/\alpha], \nu\overline{\beta_2}.\, R_2'[\gamma/\alpha]) \in \mathcal{R}[\![A[\gamma/\alpha]]\!]\, \omega(W_1''' - n_1 - 1) \ .$$

Since $n_1 < n$, we have

$$W_3 \sqsupseteq W_2 - 1 \sqsupseteq W_2 \sqsupseteq \omega(W_1 - n) \sqsupseteq \omega(W_1 - n_1 - 1) = \omega(W_1' - n_1 - 1) \ .$$

Thus, by Lemmas 110, 112, and 5,

$$\omega\, W_3 \sqsupseteq \omega(W_1' - n_1 - 1) \ .$$

By Lemma 130,

$$\omega(W_3@\gamma) \sqsupseteq \omega(W_1''' - n_1 - 1) \ .$$

Since $\vdash W_3$ and $\{\gamma\}\#\omega\, W_3$ and $\omega\, W_3 \vdash (B_1, B_2, r)$, we have

$$\{\gamma \Mapsto (B_1, B_2, r)\} \uplus \omega\, W_3 \sqsupseteq \omega(W_3@\gamma)$$

by Lemma 135. Thus, by Lemma 110,

$$\{\gamma \mapsto (B_1, B_2, r)\} \uplus \omega W_3 \sqsupseteq \omega(W_1''' - n_1 - 1) .$$

Thus, by Lemma 116,

$$(\nu\overline{\beta_1}.\, R_1'[B_1/\alpha], \nu\overline{\beta_2}.\, R_2'[B_2/\alpha])_{\omega W_3} \in \mathcal{R}[\![A[\gamma/\alpha]]\!]\, \{\gamma \mapsto (B_1, B_2, r)\} \uplus \omega W_3 .$$

Since $W_4 - 1 \sqsupseteq W_4 \sqsupseteq \{\gamma \mapsto (B_1, B_2, r)\} \uplus \omega W_3$, we have the conclusion by Lemmas 110, 116, and 114.

□

Lemma 142. *If $W_1 \sqsupseteq W_2$, then $W_1.\rho(W_2.\rho(\rho)) = W_1.\rho(\rho)$ for any ρ.*

Proof. Let $\alpha \in dom(\rho)$ and $(A_1, A_2, r) = \rho(\alpha)$. Since $W_1 \sqsupseteq W_2$, there exists some ρ' such that

- $W_1.\Delta, \dagger(\rho') \gg W_2.\Delta$ and
- $W_1.\rho = \rho' \circ W_2.\rho$.

It suffices to show that $W_1.\rho_{\mathrm{fst}}(W_2.\rho_{\mathrm{fst}}(A_1)) = W_1.\rho_{\mathrm{fst}}(A_1)$; $W_1.\rho_{\mathrm{snd}}(W_2.\rho_{\mathrm{snd}}(A_2)) = W_1.\rho_{\mathrm{snd}}(A_2)$ is proven similarly. Noting that $W_1 \sqsupseteq W_2$ implies $\vdash W_2$, we have:

$$\begin{array}{rcll} W_1.\rho_{\mathrm{fst}}(A_1) & = & {\rho'}_{\mathrm{fst}}(W_2.\rho_{\mathrm{fst}}(A_1)) & (\text{since } W_1.\rho = \rho' \circ W_2.\rho) \\ & & {\rho'}_{\mathrm{fst}}(W_2.\rho_{\mathrm{fst}}(W_2.\rho_{\mathrm{fst}}(A_1))) & (\text{since } dom(W_2.\rho) \,\#\, dom(W_2.\Delta) \text{ implied by } \vdash W_2) \\ & & W_1.\rho_{\mathrm{fst}}(W_2.\rho_{\mathrm{fst}}(A_1)) . & \end{array}$$

□

Lemma 143. *Let α be a type variable, A be a type, and r be a function that, given a world W, returns $\mathcal{R}[\![A]\!]\,(W.n+1,\ W.\Delta,\ W.\rho)$, and $\rho = \{\alpha \mapsto (A, A, r)\}$. Suppose that $\{\alpha\} \,\#\, ftv(A)$.*

For any W and A', if $\vdash W$ and $\{\alpha\}\# W$ and $W \vdash W.\rho(\rho)(\alpha)$, then:

- *$\mathcal{R}[\![A']\!]\; W.\rho(\rho) \uplus W = \mathcal{R}[\![A'[A/\alpha]]\!]\; W$; and*
- *$\mathcal{E}[\![A']\!]\; W.\rho(\rho) \uplus W = \mathcal{E}[\![A'[A/\alpha]]\!]\; W$.*

Proof. By induction on A'.

- We first show that $\mathcal{R}[\![A']\!]\; W.\rho(\rho) \uplus W = \mathcal{R}[\![A'[A/\alpha]]\!]\; W$.

  Case $A' = \iota$: Obvious since $A' = A'[A/\alpha] = \iota$.

  Case $A' = \alpha$: We first show that $\mathcal{R}[\![A]\!]\; W.\rho(\rho) \uplus W = \mathcal{R}[\![A]\!]\; W$. Since $\{\alpha\} \,\#\, ftv(A)$, we have $\mathcal{R}[\![A]\!]\; W.\rho(\rho) \uplus W \subseteq \mathcal{R}[\![A]\!]\; W$ by Lemma 140. By Lemmas 139, 116, and 113 and $\{\alpha\}\# W$, we have $\mathcal{R}[\![A]\!]\; W \subseteq \mathcal{R}[\![A]\!]\; W.\rho(\rho) \uplus W$.
  Thus, we have

$$\begin{array}{rcl} \mathcal{R}[\![A']\!]\; W.\rho(\rho) \uplus W & = & r(\blacktriangleright(W.\rho(\rho) \uplus W)) \\ & = & \mathcal{R}[\![A]\!]\; W.\rho(\rho) \uplus W \\ & = & \mathcal{R}[\![A]\!]\; W \\ & = & \mathcal{R}[\![A'[A/\alpha]]\!]\; W . \end{array}$$

  Case $A' = \beta$ for some $\beta \neq \alpha$: We have

$$\mathcal{R}[\![A']\!]\; W.\rho(\rho) \uplus W = \mathcal{R}[\![\beta]\!]\; W.\rho(\rho) \uplus W = (W.\rho(\rho) \uplus W).\rho[\beta](\blacktriangleright(W.\rho(\rho) \uplus W)) = W.\rho[\beta](\blacktriangleright(W.\rho(\rho) \uplus W)) .$$

  Let $(B_1, B_2, r') = W.\rho(\beta)$. Since $\vdash W$, we have
  - $W.\Delta \vdash B_1$,
  - $W.\Delta \vdash B_2$, and
  - $r' \in \mathrm{Rel}_{W.n}[B_1, B_2]$.

Since $\{\alpha\}\#W$, we have $\{\alpha\}\,\#\,(\mathit{ftv}(B_1) \cup \mathit{ftv}(B_2))$. Thus, the irrelevance condition on $r' \in \mathrm{Rel}_{W.n}[B_1, B_2]$ implies $r'(\blacktriangleright(W.\rho(\rho) \uplus W)) \subseteq r'(\blacktriangleright W)$. Since $\blacktriangleright(W.\rho(\rho) \uplus W) \sqsupseteq \blacktriangleright W$ by Lemma 139, we have $r'(\blacktriangleright W) \subseteq r'(\blacktriangleright(W.\rho(\rho) \uplus W))$ by monotonicity of r', Lemma 113, and $\{\alpha\}\#W$. Thus,

$$W.\rho[\beta](\blacktriangleright(W.\rho(\rho) \uplus W)) = r'(\blacktriangleright(W.\rho(\rho) \uplus W)) = r'(\blacktriangleright W) \ .$$

Since

$$r'(\blacktriangleright W) = W.\rho[\beta](\blacktriangleright W) = \mathcal{R}[\![\beta]\!]\ W = \mathcal{R}[\![A'[A/\alpha]]\!]\ W \ ,$$

we have the conclusion $\mathcal{R}[\![A']\!]\ W.\rho(\rho) \uplus W \ = \ \mathcal{R}[\![A'[A/\alpha]]\!]\ W$.

Case $A' = B' \multimap C'$:

- We show that $\mathcal{R}[\![B' \multimap C']\!]\ W.\rho(\rho) \uplus W \subseteq \mathcal{R}[\![(B' \multimap C')[A/\alpha]]\!]\ W$.
  Let $(R_1, R_2) \in \mathcal{R}[\![B' \multimap C']\!]\ W.\rho(\rho) \uplus W$. To prove $(R_1, R_2) \in \mathcal{R}[\![(B' \multimap C')[A/\alpha]]\!]\ W$, suppose that
  * $W_0 \sqsupseteq W$,
  * $(W_1, W_2) \Supset W_0$, and
  * $W_1 \sqsupseteq W$, and
  * $(R'_1, R'_2) \in \mathcal{R}[\![B'[A/\alpha]]\!]\ W_2$.

  for some W_0, W_1, W_2, R'_1, and R'_2, and then it suffices to show that

  $$(R_1\, R'_1, R_2\, R'_2)_{W_0} \in \mathcal{E}[\![C'[A/\alpha]]\!]\ W_0 \ .$$

  We can suppose that $\{\alpha\}\#W_0$ without loss of generality.
  By Lemma 137 with $W_0 \sqsupseteq W$, $W_1 \sqsupseteq W$, and $\vdash W.\rho(\rho) \uplus W$, noting $W_0.\rho = W_1.\rho$ by $(W_1, W_2) \Supset W_0$, we have
  * $W_0.\rho(\rho) \uplus W_0 \sqsupseteq W.\rho(\rho) \uplus W$ and
  * $W_0.\rho(\rho) \uplus W_1 \sqsupseteq W.\rho(\rho) \uplus W$.

  Since $(W_1, W_2) \Supset W_0$, we have
  * $(W_0.\rho(\rho) \uplus W_1, W_0.\rho(\rho) \uplus W_2) \Supset W_0.\rho(\rho) \uplus W_0$.

  We have the following.
  * $(R'_1, R'_2) \in \mathcal{R}[\![B'[A/\alpha]]\!]\ W_2$.
  * $\{\alpha\}\#W_2$ since $\{\alpha\}\#W_0$ and $(W_1, W_2) \Supset W_0$.
  * $\vdash W_2$ by Lemma 105 with $\vdash W_0$ and $(W_1, W_2) \Supset W_0$.
  * $W_2 \vdash W_2.\rho(\rho)(\alpha)$ because $W_0 \vdash W_0.\rho(\rho)(\alpha)$, which is implied by $\vdash W_0.\rho(\rho) \uplus W_0$ from $W_0.\rho(\rho) \uplus W_0 \sqsupseteq W.\rho(\rho) \uplus W$.

  Thus, by the IH, we have

  $$(R'_1, R'_2) \in \mathcal{R}[\![B']\!]\ W_2.\rho(\rho) \uplus W_2 \ .$$

  Since
  * $(R_1, R_2) \in \mathcal{R}[\![B' \multimap C']\!]\ W.\rho(\rho) \uplus W$,
  * $W_0.\rho(\rho) \uplus W_0 \sqsupseteq W.\rho(\rho) \uplus W$,
  * $W_0.\rho(\rho) \uplus W_1 \sqsupseteq W.\rho(\rho) \uplus W$,
  * $(W_0.\rho(\rho) \uplus W_1, W_0.\rho(\rho) \uplus W_2) \Supset W_0.\rho(\rho) \uplus W_0$, and
  * $(R'_1, R'_2) \in \mathcal{R}[\![B']\!]\ W_0.\rho(\rho) \uplus W_2$ (note that $W_2.\rho = W_0.\rho$),

  noting that $\{\alpha\}\#W_0$, we have

  $$(R_1\, R'_1, R_2\, R'_2)_{W_0} \in \mathcal{E}[\![C']\!]\ W_0.\rho(\rho) \uplus W_0 \ .$$

  By the IH, we have the conclusion

  $$(R_1\, R'_1, R_2\, R'_2)_{W_0} \in \mathcal{E}[\![C'[A/\alpha]]\!]\ W_0 \ .$$

- We show that $\mathcal{R}[\![(B' \multimap C')[A/\alpha]]\!]\ W \subseteq \mathcal{R}[\![B' \multimap C']\!]\ W.\rho(\rho) \uplus W$.
  Let $(R_1, R_2) \in \mathcal{R}[\![(B' \multimap C')[A/\alpha]]\!]\ W$. To prove $(R_1, R_2) \in \mathcal{R}[\![B' \multimap C']\!]\ W.\rho(\rho) \uplus W$, suppose that

* $W_0 \sqsupseteq W.\rho(\rho) \uplus W$,
* $(W_1, W_2) \Supset W_0$, and
* $W_1 \sqsupseteq W.\rho(\rho) \uplus W$, and
* $(R'_1, R'_2) \in \mathcal{R}[\![B']\!]\, W_2$.

for some W_0, W_1, W_2, R'_1, and R'_2, and then it suffices to show that

$$(R_1\, R'_1, R_2\, R'_2)_{W_0} \in \mathcal{E}[\![C']\!]\, W_0 \;.$$

By Lemma 138 with $W_0 \sqsupseteq W.\rho(\rho) \uplus W$ and $W_1 \sqsupseteq W.\rho(\rho) \uplus W$, there exist some W'_0 and W'_1 such that

* $W'_0 \sqsupseteq W$,
* $W_0 = W'_0.\rho(\rho) \uplus W'_0$,
* $W'_1 \sqsupseteq W$, and
* $W_1 = W'_1.\rho(\rho) \uplus W'_1$.

Since $(W_1, W_2) \Supset W_0$, there exists some W'_2 such that

* $W_2 = W'_0.\rho(\rho) \uplus W'_2$ and
* $(W'_1, W'_2) \Supset W'_0$.

Note that $W'_1.\rho = W'_2.\rho = W'_0.\rho$. We have the following.

* $(R'_1, R'_2) \in \mathcal{R}[\![B']\!]\, W'_2.\rho(\rho) \uplus W'_2$ since $(R'_1, R'_2) \in \mathcal{R}[\![B']\!]\, W_2$ and $W_2 = W'_0.\rho(\rho) \uplus W'_2$ and $(W'_1, W'_2) \Supset W'_0$.
* $\vdash W'_2$ by Lemma 105 with $\vdash W'_0$ (from $W'_0 \sqsupseteq W$) and $(W'_1, W'_2) \Supset W'_0$.
* $\{\alpha\} \# W'_2$ because $\{\alpha\} \# W'_0$ (from $\vdash W_0$) and $(W'_1, W'_2) \Supset W'_0$.
* $W'_2 \vdash W'_2.\rho(\rho)(\alpha)$ because $\vdash W'_0.\rho(\rho) \uplus W'_0$ and $W'_0.\rho = W'_2.\rho$.

Thus, by the IH, we have

$$(R'_1, R'_2) \in \mathcal{R}[\![B'[A/\alpha]]\!]\, W'_2 \;.$$

We also have the following.

* $(R_1, R_2) \in \mathcal{R}[\![(B' \multimap C')[A/\alpha]]\!]\, W$.
* $W'_0 \sqsupseteq W$,
* $W'_1 \sqsupseteq W$, and
* $(W'_1, W'_2) \Supset W'_0$.

Thus,

$$(R_1\, R'_1, R_2\, R'_2)_{W'_0} \in \mathcal{E}[\![C'[A/\alpha]]\!]\, W'_0 \;.$$

Noting α does not occur free in R_1, R'_1, R_2, nor R'_2, since $\vdash W'_0$ and $\{\alpha\} \# W'_0$ and $W'_0 \vdash W'_0.\rho(\rho)(\alpha)$, the IH implies the conclusion

$$(R_1\, R'_1, R_2\, R'_2)_{W_0} \in \mathcal{E}[\![C']\!]\, W_0 \;.$$

Case $A' = \forall\beta.B'$:

– We show that $\mathcal{R}[\![\forall\beta.B']\!]\, W.\rho(\rho) \uplus W \subseteq \mathcal{R}[\![(\forall\beta.B')[A/\alpha]]\!]\, W$.
Let $(R_1, R_2) \in \mathcal{R}[\![\forall\beta.B']\!]\, W.\rho(\rho) \uplus W$. We show that $(R_1, R_2) \in \mathcal{R}[\![(\forall\beta.B')[A/\alpha]]\!]\, W$. Without loss of generality, we can suppose that $\{\beta\} \# (\mathit{ftv}(A) \cup \{\alpha\} \cup \mathit{ftv}(\rho))$ and $\{\beta\} \# W$. Suppose that

* $W_0 \sqsupseteq W$,
* $\omega\, W_0 \vdash (B_1, B_2, r_0)$, and
* $\{\beta\} \# W_0$

for some W_0, B_1, B_2, and r_0, and then it suffices to show that

$$(R_1\, B_1, R_2\, B_2)_{\omega\, W_0} \in \mathcal{E}[\![B'[A/\alpha]]\!]\, W'_0$$

where $W'_0 = \{\beta \Mapsto (B_1, B_2, r_0)\} \uplus \omega\, W_0$. Without loss of generality, we can suppose that $\{\alpha\} \# W_0$. Since $W_0 \sqsupseteq W$, we have $W_0.\rho(\rho) \uplus W_0 \sqsupseteq W.\rho(\rho) \uplus W$ by Lemma 137. Since $W_0 \vdash (B_1, B_2, r_0)$, we have $W_0.\rho(\rho) \uplus W_0 \vdash (B_1, B_2, r_0)$. Since $(R_1, R_2) \in \mathcal{R}[\![\forall\beta.B']\!]\, W.\rho(\rho) \uplus W$, we have

$$(R_1\, B_1, R_2\, B_2)_{W_0.\rho(\rho) \uplus W_0} \in \mathcal{E}[\![B']\!]\, \{\beta \Mapsto (B_1, B_2, r_0)\} \uplus (\omega(W_0.\rho(\rho) \uplus W_0)) \;.$$

Since $\{\beta\} \# \mathit{ftv}(\rho)$, we have

$$\{\beta \Mapsto (B_1, B_2, r_0)\} \uplus (\omega(W_0.\rho(\rho) \uplus W_0)) = W_0'.\rho(\rho) \uplus W_0' \ .$$

Since α and β do not occur in R_1, R_2, B_1, nor B_2, we have

$$(R_1\, B_1, R_2\, B_2)_{\omega\, W_0} \in \mathcal{E}[\![B']\!]\, W_0'.\rho(\rho) \uplus W_0' \ .$$

We have the following.

* $\vdash W_0'$ since $\vdash \omega\, W_0$ by Lemma 105 with $\vdash W_0$ (from $W_0 \sqsupseteq W$) and $\omega\, W_0 \vdash (B_1, B_2, r_0)$.
* $\{\alpha\} \# W_0'$ since $\alpha \neq \beta$ and $\{\alpha\} \# W_0$.
* $W_0' \vdash W_0'.\rho(\alpha)$ because $\vdash W_0.\rho(\rho) \uplus W_0$ from $W_0.\rho(\rho) \uplus W_0 \sqsupseteq W.\rho(\rho) \uplus W$.

Thus, by the IH, we have the conclusion

$$(R_1\, B_1, R_2\, B_2)_{\omega\, W_0} \in \mathcal{E}[\![B'[A/\alpha]]\!]\, W_0' \ .$$

– We show that $\mathcal{R}[\![(\forall\beta.B')[A/\alpha]]\!]\, W \subseteq \mathcal{R}[\![\forall\beta.B']\!]\, W.\rho(\rho) \uplus W$.
Let $(R_1, R_2) \in \mathcal{R}[\![(\forall\beta.B')[A/\alpha]]\!]\, W$. We show that $(R_1, R_2) \in \mathcal{R}[\![\forall\beta.B']\!]\, W.\rho(\rho) \uplus W$. Suppose that

* $W_0 \sqsupseteq W.\rho(\rho) \uplus W$,
* $\omega\, W_0 \vdash (B_1, B_2, r_0)$, and
* $\{\beta\} \# \omega\, W_0$

for some W_0, B_1, B_2, and r_0, and then it suffices to show that

$$(R_1\, B_1, R_2\, B_2)_{\omega\, W_0} \in \mathcal{E}[\![B']\!]\, \{\beta \Mapsto (B_1, B_2, r_0)\} \uplus \omega\, W_0 \ .$$

Since $W_0 \sqsupseteq W.\rho(\rho) \uplus W$, there exists some W_0' such that

* $W_0' \sqsupseteq W$ and
* $W_0 = W_0'.\rho(\rho) \uplus W_0'$

by Lemma 138. Since $\omega\, W_0 \vdash (B_1, B_2, r_0)$, we have $\omega\, W_0' \vdash (B_1, B_2, r_0)$. Since $(R_1, R_2) \in \mathcal{R}[\![(\forall\beta.B')[A/\alpha]]\!]\, W$ and $\{\beta\} \# \omega\, W_0'$ (from $\{\beta\} \# \omega\, W_0$), we have

$$(R_1\, B_1, R_2\, B_2)_{\omega\, W_0'} \in \mathcal{E}[\![B'[A/\alpha]]\!]\, \{\beta \Mapsto (B_1, B_2, r_0)\} \uplus \omega\, W_0' \ .$$

Let $W' = \{\beta \Mapsto (B_1, B_2, r_0)\} \uplus \omega\, W_0'$. We have the following.

* $\vdash W'$ since $\omega\, W_0'$ by Lemma 105 with $\vdash W_0'$ and $\omega\, W_0' \vdash (B_1, B_2, r_0)$.
* $\{\alpha\} \# W'$,
* $W' \vdash W'.\rho(\rho)(\alpha)$ because $\vdash W_0'.\rho(\rho) \uplus W_0'$.

Thus, by the IH,

$$(R_1\, B_1, R_2\, B_2)_{\omega\, W_0'} \in \mathcal{E}[\![B']\!]\, W'.\rho(\rho) \uplus W' \ .$$

Since $\{\alpha\} \# (\mathit{ftv}(R_1) \cup \mathit{ftv}(R_2) \cup \mathit{ftv}(B_1) \cup \mathit{ftv}(B_2))$ and we can suppose that $\{\beta\} \# \mathit{ftv}(\rho)$ without loss of generality, we have the conclusion

$$(R_1\, B_1, R_2\, B_2)_{\omega\, W_0} \in \mathcal{E}[\![B']\!]\, \{\beta \Mapsto (B_1, B_2, r_0)\} \uplus \omega\, W_0'.\rho(\rho) \uplus \omega\, W_0' \ .$$

Case $A' = {!B'}$: By the IH with Lemma 105.

- Next, we consider $\mathcal{E}[\![A']\!]\, W.\rho(\rho) \uplus W = \mathcal{E}[\![A'[A/\alpha]]\!]\, W$.

  – We show that $\mathcal{E}[\![A']\!]\, W.\rho(\rho) \uplus W \subseteq \mathcal{E}[\![A'[A/\alpha]]\!]\, W$
  Let $(M_1, M_2) \in \mathcal{E}[\![A']\!]\, W.\rho(\rho) \uplus W$. We show that $(M_1, M_2) \in \mathcal{E}[\![A'[A/\alpha]]\!]\, W$. Suppose that

  * $W' \sqsupseteq W$,
  * $n < W'.n$, and
  * $W'.\rho_{\mathrm{fst}}(M_1) \longrightarrow^n R_1$

  for some W', n, and R_1, and then it suffices to show that there exists some R_2 such that

* $W'.\rho_{\mathrm{snd}}(M_2) \longrightarrow^* R_2$ and
* $(R_1, R_2) \in \mathcal{R}[\![A'[A/\alpha]]\!]\,(W' - n)$.

Without loss of generality, we can suppose that $\{\alpha\}\# W'$. Thus, we can suppose that α does not occur in R_1. Further, since $W' \sqsupseteq W$, we have

$$W'.\rho(\rho) \uplus W' \sqsupseteq W.\rho(\rho) \uplus W$$

by Lemma 137. Since $(M_1, M_2) \in \mathcal{E}[\![A']\!]\, W.\rho(\rho) \uplus W$, there exists some R_2 such that

* $W'.\rho_{\mathrm{snd}}(M_2) \longrightarrow^* R_2$ and
* $(R_1, R_2) \in \mathcal{R}[\![A']\!]\,((W'.\rho(\rho) \uplus W') - n)$

(note that α does not occur in M_2). We have the following.

* $\vdash W' - n$ since $W' \sqsupseteq W$.
* $\{\alpha\}\#(W' - n)$.
* $W' - n \vdash (W' - n).\rho(\rho)(\alpha)$ since $\vdash W'.\rho(\rho) \uplus W'$ which is implied by $W'.\rho(\rho) \uplus W' \sqsupseteq W.\rho(\rho) \uplus W$.

Thus, by the first property on $\mathcal{R}$, we have the conclusion $(R_1, R_2) \in \mathcal{R}[\![A'[A/\alpha]]\!]\,(W' - n)$.

– We show that $\mathcal{E}[\![A'[A/\alpha]]\!]\, W \subseteq \mathcal{E}[\![A']\!]\, W.\rho(\rho) \uplus W$
Let $(M_1, M_2) \in \mathcal{E}[\![A'[A/\alpha]]\!]\, W$. We show that $(M_1, M_2) \in \mathcal{E}[\![A']\!]\, W.\rho(\rho) \uplus W$.
Suppose that

* $W_0 \sqsupseteq W.\rho(\rho) \uplus W$,
* $n < W_0.n$, and
* $W_0.\rho_{\mathrm{fst}}(M_1) \longrightarrow^n R_1$

for some W_0, n, and R_1, and then it suffices to show that there exists some R_2 such that

* $W_0.\rho_{\mathrm{snd}}(M_2) \longrightarrow^* R_2$ and
* $(R_1, R_2) \in \mathcal{R}[\![A']\!]\,(W_0 - n)$.

Since $W_0 \sqsupseteq W.\rho(\rho) \uplus W$, Lemma 138 implies that there exists some W'_0 such that

* $W'_0 \sqsupseteq W$ and
* $W_0 = W'_0.\rho(\rho) \uplus W'_0$.

Since $(M_1, M_2) \in \mathcal{E}[\![A'[A/\alpha]]\!]\, W$ and $n < W_0.n = W'_0.n$ and we can suppose that α does not occur free in M_1 and M_2, there exists some R_2 such that

* $W_0.\rho_{\mathrm{snd}}(M_2) \longrightarrow^* R_2$ and
* $(R_1, R_2) \in \mathcal{R}[\![A'[A/\alpha]]\!]\,(W'_0 - n)$.

We have the following.

* $\vdash W'_0 - n$ since $W'_0 \sqsupseteq W$.
* $\{\alpha\}\#(W'_0 - n)$.
* $W'_0 - n \vdash (W'_0 - n).\rho(\rho)(\alpha)$ since $\vdash W'_0.\rho(\rho) \uplus W'_0$ which is implied by $W'_0.\rho(\rho) \uplus W'_0 \sqsupseteq W.\rho(\rho) \uplus W$.

Thus, by the first property on $\mathcal{R}$, we have the conclusion

$$(R_1, R_2) \in \mathcal{R}[\![A']\!]\,((W'_0.\rho(\rho) \uplus W'_0) - n)\ .$$

□

Lemma 144. *If $\{\alpha\}\#\omega W$ and $(\omega W, \varsigma) \in \mathcal{G}[\![\omega\Gamma]\!]$ and $\omega W \vdash (A_1, A_2, r)$, then $(\{\alpha \Mapsto (A_1, A_2, r)\} \uplus \omega W, \varsigma) \in \mathcal{G}[\![\omega\Gamma, \alpha^{\mathbf{0}}]\!]$.*

Proof. Since $(\omega W, \varsigma) \in \mathcal{G}[\![\omega\Gamma]\!]$, we have the following.

- $\vdash \omega W$.
- $\omega\Gamma \succ \omega W.\rho$.

- $\forall\, \beta \in \mathit{dom}(\omega\Gamma).\ (\exists\, \pi.\ \beta^{\pi} \in \omega\, W.\Delta) \vee \beta \in \mathit{dom}(\omega\, W.\rho)$, and
- $\forall\, x :^{\omega} A \in \omega\Gamma.\ (\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\, \omega\omega\, W = \mathcal{R}[\![A]\!]\, \omega\, W$ (by Lemma 5).

Let $W' = \{\alpha \mapsto (A_1, A_2, r)\} \uplus \omega\, W$. We have the conclusion $(W', \varsigma) \in \mathcal{G}[\![\omega\Gamma, \alpha^{\mathbf{0}}]\!]$ by the following.

- $\vdash W'$ since $\vdash \omega\, W$ and $\omega\, W \vdash (A_1, A_2, r)$.
- $\omega\Gamma, \alpha^{\mathbf{0}} \succ W'.\rho$ because $\omega\Gamma, \alpha^{\mathbf{0}}$ assigns the use $\mathbf{0}$ to all the bound type variables.
- $\forall\, \beta \in \mathit{dom}(\omega\Gamma, \alpha^{\mathbf{0}}).\ (\exists\, \pi.\ \beta^{\pi} \in W'.\Delta) \vee (\beta \in \mathit{dom}(W'.\rho))$.
- $\forall\, x :^{\omega} A \in \omega\Gamma, \alpha^{\mathbf{0}}.\ (\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\, \omega\, W'$ by Lemmas 116 and 113 with $(\varsigma_{\mathrm{fst}}(x), \varsigma_{\mathrm{snd}}(x)) \in \mathcal{R}[\![A]\!]\, \omega\, W$ and $\omega\, W' \sqsupseteq \omega\omega\, W = \omega\, W$, which is obtained by Lemma 112 with $W' \sqsupseteq \omega\, W$. $W' \sqsupseteq \omega\, W$ is proven by Lemma 139.

□

Lemma 145. *If* $\vdash \Gamma$ *and* $\omega\Gamma, \alpha^{\mathbf{0}} \vdash M_1 \preceq M_2 : A$, *then* $\Gamma \vdash \Lambda\alpha.M_1 \preceq \Lambda\alpha.M_2 : \forall\alpha.A$.

Proof. Let $(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$. It suffices to show that

$$(\varsigma_{\mathrm{fst}}(\Lambda\alpha.M_1), \varsigma_{\mathrm{snd}}(\Lambda\alpha.M_2))_{W} \in \mathcal{E}[\![\forall\alpha.A]\!]\ W\ .$$

Let $W_1 \sqsupseteq W$ such that $0 < W_1.n$. It suffices to show that

$$(\varsigma_{\mathrm{fst}}(\Lambda\alpha.M_1), \varsigma_{\mathrm{snd}}(\Lambda\alpha.M_2))_{W_1} \in \mathcal{R}[\![\forall\alpha.A]\!]\ W_1\ .$$

Suppose that

- $W_2 \sqsupseteq W_1$,
- $\omega\, W_2 \vdash (B_1, B_2, r)$, and
- $\{\alpha\}\#\omega\, W_2$,

for some W_2, B_1, B_2, and r, and then it suffices to show that

$$(\varsigma_{\mathrm{fst}}(\Lambda\alpha.M_1)\, B_1, \varsigma_{\mathrm{snd}}(\Lambda\alpha.M_2)\, B_2)_{\omega\, W_2} \in \mathcal{E}[\![A]\!]\, \{\alpha \mapsto (B_1, B_2, r)\} \uplus \omega\, W_2\ .$$

Suppose that

- $W_3 \sqsupseteq \{\alpha \mapsto (B_1, B_2, r)\} \uplus \omega\, W_2$,
- $0 < n < W_3.n$, and
- $W_3.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(\Lambda\alpha.M_1)\, B_1) \longrightarrow^{n} R_1$

for some W_3, n, and R_1, and then it suffices to show that there exists some R_2 such that

- $W_3.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(\Lambda\alpha.M_2)\, B_2) \longrightarrow^{*} R_2$ and
- $(R_1, R_2) \in \mathcal{R}[\![A]\!]\, (W_3 - n)$.

Since $W_3.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(\Lambda\alpha.M_1)\, B_1) \longrightarrow^{n} R_1$, we can find

$$W_3.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(\Lambda\alpha.M_1)\, B_1) \longrightarrow W_3.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(M_1[B_1/\alpha])) \longrightarrow^{n-1} R_1\ .$$

Then, it suffices to show that

$$(\varsigma_{\mathrm{fst}}(M_1[B_1/\alpha]), \varsigma_{\mathrm{snd}}(M_2[B_2/\alpha]))_{W_3} \in \mathcal{E}[\![A]\!]\, (W_3 - 1)\ .$$

Since $W_3 \sqsupseteq \{\alpha \mapsto (B_1, B_2, r)\} \uplus \omega\, W_2$, there exists some W_3' such that

- $W'_3 \sqsupseteq \omega\, W_2$
- $W_3 \;=\; W'_3.\rho(\{\alpha \;\Mapsto\; (B_1, B_2, r)\}) \uplus W'_3$ and

by Lemma 138. Since $(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$ and $W_2 \sqsupseteq W_1 \sqsupseteq W$, we have $(W_2, W_2.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma]\!]$ by Lemmas 110 and 119. Noting that $\omega\, W_2.\rho \;=\; W_2.\rho$, by Lemma 126, $(\omega\, W_2, \omega\, W_2.\rho(\varsigma)) \in \mathcal{G}[\![\omega\Gamma]\!]$. Since $W'_3 - 1 \sqsupseteq W'_3 \sqsupseteq \omega\, W_2$, we have $(W'_3, W'_3.\rho(\varsigma)) \in \mathcal{G}[\![\omega\Gamma]\!]$ by Lemmas 119 and 114. By Lemmas 126 and 5, $(\omega\, W'_3, W'_3.\rho(\varsigma)) \in \mathcal{G}[\![\omega\Gamma]\!]$. Since $W'_3 \sqsupseteq \omega\, W_2$ and $\omega\, W_2 \vdash (B_1, B_2, r)$, we have $W'_3 \vdash W'_3.\rho(\{\alpha \Mapsto (B_1, B_2, r)\})(\alpha)$, i.e., $\omega\, W'_3 \vdash \omega\, W'_3.\rho(\{\alpha \Mapsto (B_1, B_2, r)\})(\alpha)$, by Lemma 136. Thus, by Lemma 144,

$$(\omega\, W'_3.\rho(\{\alpha \Mapsto (B_1, B_2, r)\}) \uplus \omega\, W'_3,\; W'_3.\rho(\varsigma)) \in \mathcal{G}[\![\omega\Gamma, \alpha^{\mathbf{0}}]\!] \;.$$

Since $\omega\Gamma, \alpha^{\mathbf{0}} \vdash M_1 \preceq M_2 : A$, we have

$$(\varsigma_{\mathrm{fst}}(M_1[B_1/\alpha]), \varsigma_{\mathrm{snd}}(M_2[B_2/\alpha]))_{\omega\, W'_3} \in \mathcal{E}[\![A]\!]\,(\omega\, W'_3.\rho(\{\alpha \Mapsto (B_1, B_2, r)\}) \uplus \omega\, W'_3) \;.$$

Since α can be supposed not to occur free in B_1, B_2, nor ς, and $\omega\, W_3 \;=\; \omega(W'_3.\rho(\{\alpha \Mapsto (B_1, B_2, r)\}) \uplus W'_3) = \omega\, W'_3.\rho(\{\alpha \Mapsto (B_1, B_2, r)\}) \uplus \omega\, W'_3$, we have

$$(\varsigma_{\mathrm{fst}}(M_1[B_1/\alpha]), \varsigma_{\mathrm{snd}}(M_2[B_2/\alpha]))_{W_3} \in \mathcal{E}[\![A]\!]\,\omega\, W_3 \;.$$

Since $W_3 - 1 \sqsupseteq \omega\, W_3 - 1 \sqsupseteq \omega\, W_3$ by Lemma 118, we have the conclusion by Lemmas 110, 116, and 113. ☐

Lemma 146. *If $\{\overline{\alpha_1}, \overline{\alpha_2}\} \# W$, then:*

- *$(R_1, R_2) \in \mathcal{R}[\![A]\!]\; W$ implies $(\nu\overline{\alpha_1}.\, R_1, \nu\overline{\alpha_2}.\, R_2) \in \mathcal{R}[\![A]\!]\; W$; and*
- *$(M_1, M_2) \in \mathcal{E}[\![A]\!]\; W$ implies $(\nu\overline{\alpha_1}.\, M_1, \nu\overline{\alpha_2}.\, M_2) \in \mathcal{E}[\![A]\!]\; W$.*

Proof. By induction on A. We first consider the first property on $\mathcal{R}$ and then the second one on $\mathcal{E}$ with the first property.

- Let $(R_1, R_2) \in \mathcal{R}[\![A]\!]\; W$. We show $(\nu\overline{\alpha_1}.\, R_1, \nu\overline{\alpha_2}.\, R_2) \in \mathcal{R}[\![A]\!]\; W$ by case analysis on A.

  Case $A = \iota$: By definition.

  Case $A = \beta$: Let $(B_1, B_2, r) = W.\rho(\beta)$. Since $(R_1, R_2) \in \mathcal{R}[\![\beta]\!]\; W$, we have $(R_1, R_2) \in r(\blacktriangleright W)$ by definition. It suffices to show that $(\nu\overline{\alpha_1}.\, R_1, \nu\overline{\alpha_2}.\, R_2) \in r(\blacktriangleright W)$. Since $\vdash W$, we have $r \in \mathrm{Rel}_{W.n}[B_1, B_2]$. Thus, the conclusion is implied by the fourth condition on $\mathrm{Rel}_{W.n}[B_1, B_2]$ since $\{\overline{\alpha_1}, \overline{\alpha_2}\} \# W$.

  Case $A = B \multimap C$: Suppose that

  - $W' \sqsupseteq W$,
  - $(W_1, W_2) \Supset W'$,
  - $W_1 \sqsupseteq W$, and
  - $(R'_1, R'_2) \in \mathcal{R}[\![B]\!]\; W_2$

  for some W', W_1, W_2, R'_1, and R'_2, and then it suffices to show that

  $$((\nu\overline{\alpha_1}.\, R_1)\, R'_1, (\nu\overline{\alpha_2}.\, R_2)\, R'_2)_{W'} \in \mathcal{E}[\![C]\!]\; W' \;.$$

  We can find that $W'.\rho_{\mathrm{fst}}((\nu\overline{\alpha_1}.\, R_1)\, R'_1) \longrightarrow^n W'.\rho_{\mathrm{fst}}(\nu\overline{\alpha_1}.\,(R_1\, R'_1))$ for some n, and $W'.\rho_{\mathrm{snd}}((\nu\overline{\alpha_2}.\, R_2)\, R'_2) \longrightarrow^* W'.\rho_{\mathrm{snd}}(\nu\overline{\alpha_2}.\,(R_2\, R'_2))$. Then, it suffices to show that

  $$(\nu\overline{\alpha_1}.\,(R_1\, R'_1), \nu\overline{\alpha_2}.\,(R_2\, R'_2))_{W'} \in \mathcal{E}[\![C]\!]\; W'$$

  by Lemmas 116 and 113. Without loss of generality, we can suppose that $\{\overline{\alpha_1}, \overline{\alpha_2}\} \# W'$. Since $(R_1, R_2) \in \mathcal{R}[\![B \multimap C]\!]\; W$, we have

  $$(R_1\, R'_1, R_2\, R'_2)_{W'} \in \mathcal{E}[\![C]\!]\; W' \;.$$

  By the IH, we have the conclusion.

  Case $A = \forall\beta.B$: Suppose that

- $W' \sqsupseteq W$,
- $\omega\, W' \vdash (C_1, C_2, r)$, and
- $\{\beta\} \# \omega\, W'$

for some W', C_1, C_2, and r, and then it suffices to show that

$$((\nu\overline{\alpha_1}.\, R_1)\, C_1, (\nu\overline{\alpha_2}.\, R_2)\, C_2)_{\omega\, W'} \in \mathcal{E}[\![B]\!]\, \{\beta \Mapsto (C_1, C_2, r)\} \uplus \omega\, W' \ .$$

Since $W'.\rho_{\mathrm{fst}}((\nu\overline{\alpha_1}.\, R_1)\, C_1) \longrightarrow^n W'.\rho_{\mathrm{fst}}(\nu\overline{\alpha_1}.\,(R_1\, C_1))$ for some n, and $W'.\rho_{\mathrm{snd}}((\nu\overline{\alpha_2}.\, R_2)\, C_2) \longrightarrow^* W'.\rho_{\mathrm{snd}}(\nu\overline{\alpha_2}.\,(R_2\, C_2))$, it suffices to show that

$$(\nu\overline{\alpha_1}.\,(R_1\, C_1), \nu\overline{\alpha_2}.\,(R_2\, C_2))_{\omega\, W'} \in \mathcal{E}[\![B]\!]\, \{\beta \Mapsto (C_1, C_2, r)\} \uplus \omega\, W'$$

by Lemmas 116 and 113. Without loss of generality, we can suppose that $\{\overline{\alpha_1}, \overline{\alpha_2}\} \# W'$. Since $(R_1, R_2) \in \mathcal{R}[\![\forall\beta.B]\!]\, W$, we have

$$(R_1\, C_1, R_2\, C_2)_{\omega\, W'} \in \mathcal{E}[\![B]\!]\, \{\beta \Mapsto (C_1, C_2, r)\} \uplus \omega\, W' \ .$$

Since we can suppose that $\beta \notin \{\overline{\alpha_1}, \overline{\alpha_2}\}$ without loss of generality, we have $\{\overline{\alpha_1}, \overline{\alpha_2}\} \# (\{\beta \Mapsto (C_1, C_2, r)\} \uplus \omega\, W')$. Thus, by the IH, we have the conclusion.

Case $A = !B$: It suffices to show that $(\nu\overline{\alpha_1}.\, R_1, \nu\overline{\alpha_2}.\, R_2) \in \mathcal{R}[\![!B]\!]\, W$, that is,

$$(\mathsf{let}\, !x = \nu\overline{\alpha_1}.\, R_1\, \mathsf{in}\, x, \mathsf{let}\, !x = \nu\overline{\alpha_2}.\, R_2\, \mathsf{in}\, x) \in \mathcal{E}[\![B]\!]\, \omega\, W \ .$$

By Lemma 35, there exist some $\overline{\beta_1}$, $\overline{\beta_2}$, R_1', and R_2' such that

- $R_1 = \nu\overline{\beta_1}.\, !R_1'$ and
- $R_2 = \nu\overline{\beta_2}.\, !R_2'$.

Let $W' \sqsupseteq \omega\, W$ such that $1 < W'.n$. We have

- $W'.\rho_{\mathrm{fst}}(\mathsf{let}\, !x = \nu\overline{\alpha_1}.\, \nu\overline{\beta_1}.\, !R_1'\, \mathsf{in}\, x) \longrightarrow W'.\rho_{\mathrm{fst}}(\nu\overline{\alpha_1}.\, \nu\overline{\beta_1}.\, R_1')$ and
- $W'.\rho_{\mathrm{snd}}(\mathsf{let}\, !x = \nu\overline{\alpha_2}.\, \nu\overline{\beta_2}.\, !R_2'\, \mathsf{in}\, x) \longrightarrow W'.\rho_{\mathrm{snd}}(\nu\overline{\alpha_2}.\, \nu\overline{\beta_2}.\, R_2')$.

Thus, it suffices to show that

$$(\nu\overline{\alpha_1}.\, \nu\overline{\beta_1}.\, R_1', \nu\overline{\alpha_2}.\, \nu\overline{\beta_2}.\, R_2')_{W'} \in \mathcal{R}[\![B]\!]\, (W' - 1) \ .$$

Since $(R_1, R_2) \in \mathcal{R}[\![!B]\!]\, W$, we have $(\nu\overline{\beta_1}.\, R_1', \nu\overline{\beta_2}.\, R_2')_{W'} \in \mathcal{R}[\![B]\!]\, (W' - 1)$. By the IH, we have the conclusion.

- Let $(M_1, M_2) \in \mathcal{E}[\![A]\!]\, W$. We show that $(\nu\overline{\alpha_1}.\, M_1, \nu\overline{\alpha_2}.\, M_2) \in \mathcal{E}[\![A]\!]\, W$ with the first property. Suppose that
  - $W' \sqsupseteq W$,
  - $n < W'.n$, and
  - $W'.\rho_{\mathrm{fst}}(\nu\overline{\alpha_1}.\, M_1) \longrightarrow^n R_1$

  for some W', n, and R_1, and the it suffices to show that there exists some R_2 such that
  - $W'.\rho_{\mathrm{snd}}(\nu\overline{\alpha_1}.\, M_2) \longrightarrow^* R_2$ and
  - $(R_1, R_2) \in \mathcal{R}[\![A]\!]\, (W' - n)$.

  By the semantics, $R_1 = \nu\overline{\alpha_1}.\, R_1'$ for some R_1' such that $W'.\rho_{\mathrm{fst}}(M_1) \longrightarrow^n R_1'$. Since $(M_1, M_2) \in \mathcal{E}[\![A]\!]\, W$ and $W' \sqsupseteq W$, there exists some R_2' such that
  - $W'.\rho_{\mathrm{snd}}(M_2) \longrightarrow^* R_2'$ and
  - $(R_1', R_2') \in \mathcal{R}[\![A]\!]\, W' - n$.

  By the first property on $\mathcal{R}$, we have the conclusion $(\nu\overline{\alpha_1}.\, R_1', \nu\overline{\alpha_2}.\, R_2') \in \mathcal{R}[\![A]\!]\, (W' - n)$ where let $R_2 = \nu\overline{\alpha_2}.\, R_2'$.

□

Lemma 147. *If* $\Gamma \vdash M_1 \preceq M_2 : \forall\alpha.B$ *and* $\Gamma \vdash A$*, then* $\Gamma \vdash M_1\, A \preceq M_2\, A : B[A/\alpha]$.

Proof. Let $(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$. It suffices to show that

$$(\varsigma_{\mathrm{fst}}(M_1\, A), \varsigma_{\mathrm{snd}}(M_2\, A))_W \in \mathcal{E}[\![B[A/\alpha]]\!]\, W\ .$$

Suppose that

- $W' \sqsupseteq W$,
- $n < W'.n$, and
- $W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(M_1\, A)) \longrightarrow^n R_1$

for some W', n, and R_1, and then it suffices to show that there exists some R_2 such that

- $W'.\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(M_2\, A)) \longrightarrow^* R_2$ and
- $(R_1, R_2) \in \mathcal{R}[\![B[A/\alpha]]\!]\,(W' - n)$.

Since $W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(M_1\, A)) \longrightarrow^n R_1$, we can find that there exist some $\overline{\beta_1}$, M_1', and n_1 such that

- $W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(M_1)) \longrightarrow^{n_1} \nu\overline{\beta_1}.\,\Lambda\alpha.M_1'$ and
- $W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(M_1\, A)) \longrightarrow^{n_1} (\nu\overline{\beta_1}.\,\Lambda\alpha.M_1')\; W'.\rho_{\mathrm{fst}}(A) \longrightarrow^{n_2} \nu\overline{\beta_1}.\, M_1'[W'.\rho_{\mathrm{fst}}(A)/\alpha] \longrightarrow^{n-n_1-n_2} R_1$ for some n_2 (note that we can suppose that type variables $\overline{\beta_1}$ do not occur in $W'.\rho_{\mathrm{fst}}(A)$ without loss of generality).

Since $(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$ and $W' \sqsupseteq W$, we have $(W', W'.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma]\!]$ by Lemma 119. Since $\Gamma \vdash M_1 \preceq M_2 : \forall\alpha.B$, we have

$$(\varsigma_{\mathrm{fst}}(M_1), \varsigma_{\mathrm{snd}}(M_2))_{W'} \in \mathcal{E}[\![\forall\alpha.B]\!]\, W'\ .$$

Since $W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{fst}}(M_1)) \longrightarrow^{n_1} \nu\overline{\beta_1}.\,\Lambda\alpha.M_1'$ and $n_1 < n < W'.n$, there exist some $\overline{\beta_2}$ and M_2' such that

- $\rho_{\mathrm{snd}}(\varsigma_{\mathrm{snd}}(M_2)) \longrightarrow^* \nu\overline{\beta_2}.\,\Lambda\alpha.M_2'$ and
- $(\nu\overline{\beta_1}.\,\Lambda\alpha.M_1', \nu\overline{\beta_2}.\,\Lambda\alpha.M_2') \in \mathcal{R}[\![\forall\alpha.B]\!]\,(W' - n_1)$.

Let $A_1 = W'.\rho_{\mathrm{fst}}(A)$, and $A_2 = W'.\rho_{\mathrm{snd}}(A)$. Since $\Gamma \vdash A$ and $(W', W'.\rho(\varsigma)) \in \mathcal{G}[\![\Gamma]\!]$, we have $W'.\Delta \vdash A_1$ and $W'.\Delta \vdash A_2$. Let r be a function given in Lemma 143 for α and A, that is, given a world W_0, r returns $\mathcal{R}[\![A]\!]\,(W_0.n+1,\ W_0.\Delta,\ W_0.\rho)$. By

- Lemma 116 (for monotonicity),
- Lemma 140 (for the irrelevance condition on Rel_n),
- Lemma 131 (for the third condition on Rel_n), and
- Lemma 146 (for the fourth condition on Rel_n),

we have $\omega(W' - n_1) \vdash (A_1, A_2, r)$. Since $(\nu\overline{\beta_1}.\,\Lambda\alpha.M_1', \nu\overline{\beta_2}.\,\Lambda\alpha.M_2') \in \mathcal{R}[\![\forall\alpha.B]\!]\, W' - n_1$ and we can suppose that $\{\alpha\}\#\omega(W' - n_1)$ without loss of generality, we have

$$((\nu\overline{\beta_1}.\,\Lambda\alpha.M_1')\, A_1, (\nu\overline{\beta_2}.\,\Lambda\alpha.M_2')\, A_2) \in \mathcal{E}[\![B]\!]\,\{\alpha \Mapsto (A_1, A_2, r)\} \uplus \omega(W' - n_1)$$

with Lemma 113. Further, we have $\vdash \omega(W' - n_1)$ by Lemma 105 with $\vdash W'$ implied by $W' \sqsupseteq W$. Since $\omega(W' - n_1).\rho(\{\alpha \Mapsto (A, A, r)\}) = \{\alpha \Mapsto (A_1, A_2, r)\}$, we have

$$((\nu\overline{\beta_1}.\,\Lambda\alpha.M_1')\, A_1, (\nu\overline{\beta_2}.\,\Lambda\alpha.M_2')\, A_2) \in \mathcal{E}[\![B[A/\alpha]]\!]\,\omega(W' - n_1)$$

by Lemma 143. Since $(\nu\overline{\beta_1}.\,\Lambda\alpha.M_1')\, A_1 \longrightarrow^{n_2} \nu\overline{\beta_1}.\, M_1'[A_1/\alpha] \longrightarrow^{n-n_1-n_2} R_1$, we can find that there exists some R_2 such that

- $W'.\rho_{\mathrm{fst}}(\varsigma_{\mathrm{snd}}(M_2\, A)) \longrightarrow^* (\nu\overline{\beta_2}.\,\Lambda\alpha.M_2')\, A_2 \longrightarrow^* R_2$ and
- $(R_1, R_2) \in \mathcal{E}[\![B[A/\alpha]]\!]\, \omega\, W' - n$.

Since $W' - n \sqsupseteq \omega\, W' - n$ by Lemma 118, we have the conclusion by Lemmas 116 and 113. ☐

Theorem 5 (Parametricity / Fundamental Property)**.** *If $\Gamma \vdash M : A$, then $\Gamma \vdash M \approx M : A$*

Proof. It suffices to show that $\Gamma \vdash M \preceq M : A$, which is shown by induction on the typing derivation of $\Gamma \vdash M : A$ withe the compatibility lemmas (Lemmas 120, 121, 123, 125, 127, 128, 133, 141, 145, and 147). ☐

Theorem 6 (Soundness with respect to Contextual Equivalence)**.** *If $\Gamma \vdash M_1 \approx M_2 : A$, then $\Gamma \vdash M \approx_{ctx} M : A$*

Proof. Let ι be a base type, c be a constant of ι, and $\mathbb{C}$ be a context such that $\mathbb{C} : (\Gamma \vdash A) \rightsquigarrow (\emptyset \vdash \iota)$. Now, we suppose that $\mathbb{C}[M_1] \longrightarrow^n \nu\overline{\alpha_1}.\, c$ for some n and $\overline{\alpha_1}$, and then show that $\mathbb{C}[M_2] \longrightarrow^* \nu\overline{\alpha_2}.\, c$ for some $\overline{\alpha_2}$; the reverse direction can be proven in a similar way. By induction on the typing derivation of $\mathbb{C}$ with the compatibility lemmas (Lemmas 120, 121, 123, 125, 127, 128, 133, 141, 145, 147), we have $\emptyset \vdash \mathbb{C}[M_1] \preceq \mathbb{C}[M_2] : \iota$; note that, for any Γ, M, A, $\Gamma \vdash M : A$ implies $\Gamma \vdash M \preceq M : A$ (which is shown in a way similar to parametricity). Let $W = (n+1, \emptyset, \emptyset)$. Since $(W, \emptyset) \in \mathcal{G}[\![\emptyset]\!]$, we have $(\mathbb{C}[M_1], \mathbb{C}[M_2]) \in \mathcal{E}[\![\iota]\!]\, W$. Since $\mathbb{C}[M_1] \longrightarrow^n \nu\overline{\alpha_1}.\, c$, there exists some R_2 such that

- $\mathbb{C}[M_2] \longrightarrow^* R_2$ and
- $(\nu\overline{\alpha_1}.\, c, R_2) \in \mathcal{R}[\![\iota]\!]\,(W - n)$.

By the definition of $\mathcal{R}$, we have $R_2 = \nu\overline{\alpha_2}.\, c$ for some $\overline{\alpha_2}$. ☐

3.6 Examples of Free Theorems

Example 1 (Free Theorem for the Empty Type)**.** *If $\Delta \vdash M : \forall\alpha.\alpha$ and $\Delta \vdash A$, then there exists no result R such that $M\, A \longrightarrow^* R$.*

Proof. Assume that $M\, A \longrightarrow^n R$ for some n and R. Since $\Delta \vdash M : \forall\alpha.\alpha$, we have $\Delta \vdash M \preceq M : \forall\alpha.\alpha$ by Theorem 5. Let $W = (n+1, \Delta, \emptyset)$. We have $(W, \emptyset) \in \mathcal{G}[\![\Delta]\!]$ by definition. Thus, $(M, M) \in \mathcal{E}[\![\forall\alpha.\alpha]\!]\, W$. Since $M\, A \longrightarrow^n R$, there exist some R', n_1, and n_2 such that

- $M \longrightarrow^{n_1} R'$,
- $R'\, A \longrightarrow^{n_2} R$, and
- $n = n_1 + n_2$.

Since $n_1 \leq n < n+1 = W.n$ and $W \sqsupseteq W$ by Lemma 111, we have

$$(R', R') \in \mathcal{R}[\![\forall\alpha.\alpha]\!]\; W - n_1 \;.$$

Let r be a relational interpretation that maps any world to the empty set. Without loss of generality, we can suppose that $\{\alpha\}\#\omega(W - n_1)$. Since $\omega(W - n_1).\Delta = \omega\Delta$ and $\omega\Delta \vdash A$ from $\Delta \vdash A$, we have $\omega(W - n_1) \vdash (A, A, r)$. Thus,

$$(R'\, A, R'\, A) \in \mathcal{E}[\![\alpha]\!]\; W'$$

where $W' = \{\alpha \Mapsto (A, A, r)\} \uplus \omega(W - n_1)$. Since $R'\, A \longrightarrow^{n_2} R$ and $n_2 = n - n_1 < W'.n$ and $W' \sqsupseteq W'$ by Lemma 111, we have

$$(R, R) \in \mathcal{R}[\![\alpha]\!]\; W' - n_2 \;.$$

However, the relational interpretation r returns the empty set for any world, so there is a contradiction. ☐

Lemma 148. *If $\Gamma_1, \Gamma_2 \vdash R : \forall\alpha.A$, then $\omega\Gamma_1, \Gamma_2 \vdash R : \forall\alpha.A$.*

Proof. Straightforward by induction on the derivation of $\Gamma_1, \Gamma_2 \vdash R : \forall\alpha.A$. ☐

Lemma 149. *Suppose that $\Delta \vdash M : A$ and $\Delta \succ \rho$ and $\forall\, \alpha \in dom(\rho) \cap dom(\Delta).\ \alpha^{\mathbf{0}} \in \Delta$.*

1. *$M \rightsquigarrow M'$ implies $\rho_{fst}(M) \rightsquigarrow \rho_{fst}(M')$ and $\rho_{snd}(M) \rightsquigarrow \rho_{snd}(M')$.*

2. $M \longrightarrow M'$ *implies* $\rho_{fst}(M) \longrightarrow \rho_{fst}(M')$ *and* $\rho_{snd}(M) \longrightarrow \rho_{snd}(M')$*.*

Proof.

1. By case analysis on the reduction rule applied to derive $M \rightsquigarrow M'$. It is easy to prove the conclusion if $M \rightsquigarrow M'$ is derived by the rules other than (R_CLOSING).

   Consider the case that $M \rightsquigarrow M'$ is derived by (R_CLOSING). Then, $M = \Lambda^{\circ}\langle \beta, !R \rangle$ and $M' = !\Lambda\beta.R$ for some β and R (i.e., $\Lambda^{\circ}\langle \beta, !R \rangle \rightsquigarrow !\Lambda\beta.R$ is derived). Since $\Delta \vdash \Lambda^{\circ}\langle \beta, !R \rangle : A$, we have

   - $\Delta = \Delta_1, \beta^{\mathbf{1}}, \Delta_2$,
   - $A = !\forall\beta.B$, and
   - $\Delta_1, \beta^{\mathbf{0}}, \Delta_2 \vdash !R : !B$

   for some Δ_1 and Δ_2. $\beta^{\mathbf{1}} \in \Delta$ and the assumption $\Delta \succ \rho$ implies $\beta \notin \mathit{ftv}(\rho)$. $\beta^{\mathbf{1}} \in \Delta$ and the assumption $\forall\, \alpha \in \mathit{dom}(\rho) \cap \mathit{dom}(\Delta).\ \alpha^{\mathbf{0}} \in \Delta$ implies $\beta \notin \mathit{dom}(\rho)$. Thus, we have

   - $\rho_{\mathrm{fst}}(M) = \Lambda^{\circ}\langle \beta, !\rho_{\mathrm{fst}}(R) \rangle \rightsquigarrow !\Lambda\beta.\rho_{\mathrm{fst}}(R) = \rho_{\mathrm{fst}}(!\Lambda\beta.R)$ and
   - $\rho_{\mathrm{snd}}(M) = \Lambda^{\circ}\langle \beta, !\rho_{\mathrm{snd}}(R) \rangle \rightsquigarrow !\Lambda\beta.\rho_{\mathrm{snd}}(R) = \rho_{\mathrm{snd}}(!\Lambda\beta.R)$.

2. Straightforward by induction on the derivation of $M \longrightarrow M'$.

□

Definition 32 (Normalizing Terms)**.** *A term M is normalizing if and only if the evaluation of M and that of any term derived from M by applying operations allowed on its type (e.g., type substitution, result substitution, type application, and term application) terminate.*

Example 2 (Free Theorem for the Polymorphic Identity Type)**.** *Suppose that terms M_1 and M_2 are normalizing. If $\Gamma \vdash M_1 : \forall\alpha.\alpha \multimap \alpha$ and $\Gamma \vdash M_2 : \forall\alpha.\alpha \multimap \alpha$, then $\Gamma \vdash M_1 \preceq M_2 : \forall\alpha.\alpha \multimap \alpha$. Therefore, for any normalizing term M and typing context Γ, $\Gamma \vdash M : \forall\alpha.\alpha \multimap \alpha$ implies $\Gamma \vdash \Lambda\alpha.\lambda x.x \approx M : \forall\alpha.\alpha \multimap \alpha$.*

Proof. Let $(W, \varsigma) \in \mathcal{G}[\![\Gamma]\!]$. It suffices to show that

$$(M_1, M_2)_W \in \mathcal{E}[\![\forall\alpha.\alpha \multimap \alpha]\!]\ W\ .$$

Since M_1 and M_2 are normalizing, $W.\rho_{\mathrm{fst}}(M_1) \longrightarrow^{n_{11}} R_{11}$ and $W.\rho_{\mathrm{snd}}(M_2) \longrightarrow^{n_{21}} R_{21}$ for some n_{11}, n_{21}, R_{11}, and R_{21}. Let $W_1 \sqsupseteq W$ such that $n_{11} < W_1.n$. By the definition of $\mathcal{E}$, it suffices to show that

$$(R_{11}, R_{21})_{W_1} \in \mathcal{R}[\![\forall\alpha.\alpha \multimap \alpha]\!]\ W_1 - n_{11}\ .$$

Suppose that

- $W_2 \sqsupseteq W_1 - n_{11}$,
- $\omega\, W_2 \vdash (A_1, A_2, r)$, and
- $\{\alpha\} \# \omega\, W_2$,

for some W_2, A_1, A_2, and r. Then, it suffices to show that

$$(R_{11}\ A_1, R_{21}\ A_2)_{\omega\, W_2} \in \mathcal{E}[\![\alpha \multimap \alpha]\!]\ \{\alpha \Mapsto (A_1, A_2, r)\} \uplus \omega\, W_2\ .$$

Since M_1 and M_2 are normalizing, we have $W_2.\rho_{\mathrm{fst}}(R_{11})\ A_1 \longrightarrow^{n_{12}} R_{12}$ and $W_2.\rho_{\mathrm{snd}}(R_{21})\ A_2 \longrightarrow^{n_{22}} R_{22}$ for some n_{12}, n_{22}, R_{12}, and R_{22}. Let $W_3 \sqsupseteq \{\alpha \Mapsto (A_1, A_2, r)\} \uplus \omega\, W_2$ such that $n_{12} < W_3.n$. Then, it suffices to show that

$$(R_{12}, R_{22})_{W_3} \in \mathcal{R}[\![\alpha \multimap \alpha]\!]\ W_3 - n_{12}\ .$$

Suppose that

- $W_4 \sqsupseteq W_3 - n_{12}$,

- $(W_{41}, W_{42}) \Supset W_4$,
- $W_{41} \sqsupseteq W_4$, and
- $(R'_1, R'_2) \in \mathcal{R}[\![\alpha]\!]\, W_{42}$

for some W_4, W_{41}, W_{42}, R'_1, and R'_2. Then, it suffices to show that

$$(R_{12}\, R'_1, R_{22}\, R'_2)_{W_4} \in \mathcal{E}[\![\alpha]\!]\, W_4 \ .$$

Since $(R'_1, R'_2) \in \mathcal{R}[\![\alpha]\!]\, W_{42}$, we have $(R'_1, R'_2) \in W_{42}.\rho[\alpha](\blacktriangleright W_{42})$. Since $W_{42}.\rho = W_4.\rho$ and $W_4 \sqsupseteq W_3 - n_{12} \sqsupseteq W_3 \sqsupseteq \{\alpha \Mapsto (A_1, A_2, r)\} \uplus \omega\, W_2$, we have $(R'_1, R'_2) \in r(\blacktriangleright W_{42})$ by Lemmas 110 and 138.

Since M_1 and M_2 are normalizing, we have $W_4.\rho_{\mathrm{fst}}(R_{12})\, R'_1 \longrightarrow^{n_{13}} R_{13}$ and $W_4.\rho_{\mathrm{snd}}(R_{22})\, R'_2 \longrightarrow^{n_{23}} R_{23}$ for some n_{13}, n_{23}, R_{13}, and R_{23}. Let $W_5 \sqsupseteq W_4$ such that $n_{13} < W_5.n$. Then, it suffices to show that

$$(R_{13}, R_{23})_{W_5} \in \mathcal{R}[\![\alpha]\!]\, W_5 - n_{13} \ .$$

Since $W_5 - n_{13} \sqsupseteq W_5 \sqsupseteq W_4 \sqsupseteq \{\alpha \Mapsto (A_1, A_2, r)\} \uplus \omega\, W_2$, it suffices to show that

$$(W_5.\rho_{\mathrm{fst}}(R_{13}), W_5.\rho_{\mathrm{snd}}(R_{23})) \in r(\blacktriangleright W_5)$$

by Lemmas 110 and 138. Since $W_5 \sqsupseteq W_4 \sqsupseteq W_{42}$ by Lemmas 118 and 110, monotonicity of the relational interpretation r implies that it suffices to show that

$$(R_{13}, R_{23}) \in r(\blacktriangleright W_{42}) \ .$$

In what follows, let $i \in \{1,2\}$. Let $R'_{11} = W_2.\rho_{\mathrm{fst}}(R_{11})$ and $R'_{21} = W_2.\rho_{\mathrm{snd}}(R_{21})$ and n_0 be the maximum number between $n_{11} + n_{12} + n_{13} + 1$ and $n_{21} + n_{22} + n_{23} + 1$. Let $W_{02} = (n_0, W_2.\Delta, W_2.\rho)$. We have $\omega\, W_2.\Delta \vdash {R'}_{i1} : \forall \alpha.\alpha \multimap \alpha$ by Lemmas 148 and 34. Thus, $\omega\, W_2.\Delta \vdash {R'}_{i1} \preceq {R'}_{i1} : \forall \alpha.\alpha \multimap \alpha$ by Theorem 5. Since $(\omega\, W_{02}, \emptyset) \in \mathcal{G}[\![\omega\, W_2.\Delta]\!]$, we have

$$({R'}_{i1}, {R'}_{i1}) \in \mathcal{R}[\![\forall \alpha.\alpha \multimap \alpha]\!]\, \omega\, W_{02}$$

by the definition and Lemma 113. Since $W_4 \sqsupseteq \{\alpha \Mapsto (A_1, A_2, r)\} \uplus \omega\, W_2$, there exists some W'_4 such that

- $W'_4 \sqsupseteq \omega\, W_2$ and
- $W_4 = W'_4.\rho(\{\alpha \Mapsto (A_1, A_2, r)\}) \uplus W'_4$

by Lemma 138. Since $(W_{41}, W_{42}) \Supset W_4$, there exist some W'_{41} and W'_{42} such that

- $W_{41} = W'_4.\rho(\{\alpha \Mapsto (A_1, A_2, r)\}) \uplus W'_{41}$,
- $W_{42} = W'_4.\rho(\{\alpha \Mapsto (B_1, B_2, r)\}) \uplus W'_{42}$, and
- $(W'_{41}, W'_{42}) \Supset W'_4$.

Let

- ${W'}_{i4} = (n_0 - n_{i2},\ W'_4.\Delta,\ W'_4.\rho)$,
- ${W'}_{i41} = (n_0 - n_{i2},\ W'_{41}.\Delta,\ W'_4.\rho)$,
- ${W'}_{i42} = (n_0 - n_{i2},\ W'_{42}.\Delta,\ W'_4.\rho)$,
- $A'_1 = W_{42}.\rho_{\mathrm{fst}}(A_1)$, and
- $A'_2 = W_{42}.\rho_{\mathrm{snd}}(A_2)$.

Further, Let r_i be a function that maps a well-formed world W to a set

$$\{\ (\nu\overline{\alpha_1}.\,(\rho \uplus W).\rho_{\mathrm{fst}}(R'_i), \nu\overline{\alpha_2}.\,(\rho \uplus W).\rho_{\mathrm{snd}}(R'_i)) \quad | \quad \exists\, \overline{\beta}, \overline{\gamma_1}, \overline{\gamma_2}.\ \{\overline{\alpha_1}\} = \{\overline{\beta}, \overline{\gamma_1}\} \wedge \{\overline{\alpha_2}\} = \{\overline{\beta}, \overline{\gamma_2}\} \wedge \{\overline{\gamma_1}, \overline{\gamma_2}\} \# (\rho \uplus W @ \overline{\beta}) \wedge \rho \uplus W @ \overline{\beta} \sqsupseteq {W'}_{i42} \wedge (\{\overline{\beta}\} \cup \mathit{dom}(\rho)) \# \mathit{ftv}(A'_i)\ \}\ .$$

We show that $r_i \in \mathrm{Rel}_{\omega\, {W'}_{i4}.n}[A'_i, A'_i]$. In the proof of it, let

- ρ and $\overline{\beta}$ such that $(\{\overline{\beta}\} \cup dom(\rho)) \# ftv(A'_i)$,
- W be a world such that $\vdash W$ and $\rho \uplus W@\overline{\beta} \sqsupseteq W'_{i42}$,
- $\overline{\gamma_1}$ and $\overline{\gamma_2}$ such that $\{\overline{\gamma_1}, \overline{\gamma_2}\}\#(\rho \uplus W@\overline{\beta})$,
- $\overline{\alpha_1}$ and $\overline{\alpha_2}$ such that $\{\overline{\alpha_1}\} = \{\overline{\beta}, \overline{\gamma_1}\}$ and $\{\overline{\alpha_2}\} = \{\overline{\beta}, \overline{\gamma_2}\}$, and
- $(\nu\overline{\alpha_1}.\,(\rho \uplus W).\rho_{\mathrm{fst}}(R'_i), \nu\overline{\alpha_2}.\,(\rho \uplus W).\rho_{\mathrm{snd}}(R'_i)) \in r_i(W)$.

Let's go to the proof.

- We show that $r_i(W) \in \mathcal{P}(\mathrm{Atom}^{\mathrm{res}}\,[W.\Delta, W.\rho_{\mathrm{fst}}(A'_i), W.\rho_{\mathrm{snd}}(A'_i)])$.

  It suffices to show that

  - $W.\Delta \vdash \nu\overline{\alpha_1}.\,(\rho \uplus W).\rho_{\mathrm{fst}}(R'_i) : W.\rho_{\mathrm{fst}}(A'_i)$ and
  - $W.\Delta \vdash \nu\overline{\alpha_2}.\,(\rho \uplus W).\rho_{\mathrm{snd}}(R'_i) : W.\rho_{\mathrm{snd}}(A'_i)$.

  Since $(R'_1, R'_2) \in r(\blacktriangleright W_{42})$ and $W_{42}.\Delta = W'_{42}.\Delta = W'_{i42}.\Delta$, we have

  $$W'_{i42}.\Delta \vdash R'_i : A'_i \ .$$

  Since $\rho \uplus W@\overline{\beta} \sqsupseteq W'_{i42}$, there exists some ρ_0 such that

  - $\vdash \rho \uplus W@\overline{\beta}$,
  - $(\rho \uplus W@\overline{\beta}).\Delta, \dagger(\rho_0) \gg W'_{i42}.\Delta$,
  - $(\rho \uplus W@\overline{\beta}).\rho = \rho_0 \circ W'_{i42}.\rho$, and
  - $W'_{i42}.\Delta \succ \rho_0$.

  Since $(\rho \uplus W@\overline{\beta}).\Delta, \dagger(\rho) \gg W'_{i42}.\Delta$, there exists some Δ_1 and Δ_2 such that

  $$(\rho \uplus W@\overline{\beta}).\Delta, \dagger(\rho) = (W'_{i42}.\Delta + \Delta_1), \Delta_2 \ .$$

  Thus, there exist some Δ_{421}, Δ_{422}, Δ_{11}, Δ_{21}, and Δ_{22} such that

  - $W'_{i42}.\Delta = \Delta_{421}, \omega\Delta_{422}$,
  - $\Delta_1 = \Delta_{11}, \omega\Delta_{422}$,
  - $\Delta_2 = \Delta_{21}, \omega\Delta_{22}$,
  - $(\rho \uplus W@\overline{\beta}).\Delta = (\Delta_{421} + \Delta_{11}), \Delta_{21}$, and
  - $dom(\rho_0) = dom(\omega\Delta_{422}, \omega\Delta_{22})$.

  Thus, we have

  $$\Delta_{421}, \omega\Delta_{422} \vdash R'_i : A'_i \ .$$

  By Lemma 20,

  $$\Delta_{421}, \omega\Delta_{422}, \omega\Delta_{21} \vdash R'_i : A'_i \ .$$

  Since $\vdash \rho \uplus W@\overline{\beta}$, we have

  $$\forall \alpha' \in dom(\rho_0).\ (\rho \uplus W@\overline{\beta}).\Delta \vdash \rho_{0\mathrm{fst}}(\alpha') \wedge (\rho \uplus W@\overline{\beta}).\Delta \vdash \rho_{0\mathrm{snd}}(\alpha') \ .$$

  Thus, with $W'_{i42}.\Delta \succ \rho_0$, we have

  - $\Delta_{421}, \omega\Delta_{21} \vdash \rho_{0\mathrm{fst}}(R'_i) : \rho_{0\mathrm{fst}}(A'_i)$ and
  - $\Delta_{421}, \omega\Delta_{21} \vdash \rho_{0\mathrm{snd}}(R'_i) : \rho_{0\mathrm{snd}}(A'_i)$.

  Since $\alpha' \in dom(W'_{i42}.\rho)$ does not occur in R'_1, R'_2, A'_1, nor A'_2 by Lemma 114, noting that $\rho_0 \circ W'_{i42}.\rho = (\rho \uplus W@\overline{\beta}).\rho$, we have

– $\Delta_{421}, \omega\Delta_{21} \vdash (\rho \uplus W@\overline{\beta}).\rho_{\mathrm{fst}}(R'_i) : (\rho \uplus W@\overline{\beta}).\rho_{\mathrm{fst}}(A'_i)$ and
– $\Delta_{421}, \omega\Delta_{21} \vdash (\rho \uplus W@\overline{\beta}).\rho_{\mathrm{snd}}(R'_i) : (\rho \uplus W@\overline{\beta}).\rho_{\mathrm{snd}}(A'_i)$.

Since $dom(\rho) \# ftv(A'_i)$, we have

– $\Delta_{421}, \omega\Delta_{21} \vdash (\rho \uplus W@\overline{\beta}).\rho_{\mathrm{fst}}(R'_i) : W.\rho_{\mathrm{fst}}(A'_i)$ and
– $\Delta_{421}, \omega\Delta_{21} \vdash (\rho \uplus W@\overline{\beta}).\rho_{\mathrm{snd}}(R'_i) : W.\rho_{\mathrm{snd}}(A'_i)$.

By Lemmas 20 and 25,

– $(W@(\overline{\beta}, \overline{\gamma_1})).\Delta \vdash (\rho \uplus W@\overline{\beta}).\rho_{\mathrm{fst}}(R'_i) : W.\rho_{\mathrm{fst}}(A'_i)$ and
– $(W@(\overline{\beta}, \overline{\gamma_2})).\Delta \vdash (\rho \uplus W@\overline{\beta}).\rho_{\mathrm{snd}}(R'_i) : W.\rho_{\mathrm{snd}}(A'_i)$.

Since $\{\overline{\beta}\} \# ftv(A'_i)$ and $\{\overline{\beta}\} \# W$ and $\vdash W$, we have $\forall \beta' \in \{\overline{\beta}\}.\ \beta \notin ftv(W.\rho_{\mathrm{fst}}(A'_i)) \cup ftv(W.\rho_{\mathrm{snd}}(A'_i))$. Further, $\{\overline{\gamma_1}, \overline{\gamma_2}\} \# dom(\Delta_{421}, \omega\Delta_{21})$ implies $\{\overline{\gamma_1}, \overline{\gamma_2}\} \# (ftv(W.\rho_{\mathrm{fst}}(A'_i)) \cup ftv(W.\rho_{\mathrm{snd}}(A'_i)))$ by Lemma 18. Thus, we have the conclusion

– $W.\Delta \vdash \nu\overline{\alpha_1}.\,(\rho \uplus W@\overline{\beta}).\rho_{\mathrm{fst}}(R'_i) : W.\rho_{\mathrm{fst}}(A'_i)$ and
– $W.\Delta \vdash \nu\overline{\alpha_2}.\,(\rho \uplus W@\overline{\beta}).\rho_{\mathrm{snd}}(R'_i) : W.\rho_{\mathrm{snd}}(A'_i)$

by (T_Nu).

- Monotinicity. Let $W' \sqsupseteq W$. We show that

$$(\nu\overline{\alpha_1}.\,(\rho \uplus W).\rho_{\mathrm{fst}}(R'_i), \nu\overline{\alpha_2}.\,(\rho \uplus W).\rho_{\mathrm{snd}}(R'_i))_{W'} \in r_i(W')\ .$$

  Without loss of generality, we can suppose that $dom(\rho) \# W'$ and $\{\overline{\beta}\} \# W'$ and $\{\overline{\gamma_1}, \overline{\gamma_2}\} \# W'$. Then, by Lemmas 130, 137, and 110, we have $\rho \uplus W'@\overline{\beta} \sqsupseteq \rho \uplus W@\overline{\beta} \sqsupseteq W'_{i42}$. Thus, we have

$$(\nu\overline{\alpha_1}.\,(\rho \uplus W').\rho_{\mathrm{fst}}(R'_i), \nu\overline{\alpha_2}.\,(\rho \uplus W').\rho_{\mathrm{snd}}(R'_i)) \in r_i(W')\ .$$

  By Lemma 114, we have the conclusion.

- Irrelevance. Let W' and ρ' such that $W = \rho' \uplus W'$ and $dom(\rho') \# ftv(A'_i)$. We have

$$(\nu\overline{\alpha_1}.\,(\rho \uplus (\rho' \uplus W')).\rho_{\mathrm{fst}}(R'_i), \nu\overline{\alpha_2}.\,(\rho \uplus (\rho' \uplus W')).\rho_{\mathrm{snd}}(R'_i)) \in r_i(\rho' \uplus W')\ .$$

  Since $dom(\rho') \# ftv(A'_i)$, we have

$$(\nu\overline{\alpha_1}.\,((\rho \uplus \rho') \uplus W').\rho_{\mathrm{fst}}(R'_i), \nu\overline{\alpha_2}.\,((\rho \uplus \rho') \uplus W').\rho_{\mathrm{snd}}(R'_i)) \in r_i(W')\ .$$

  Thus, we have the conclusion.

- Let W' and α' such that $W = W'@\alpha'$ and $\{\alpha'\} \# ftv(A'_i)$ and $\vdash W'$. We have

$$(\nu\overline{\alpha_1}.\,(\rho \uplus (W'@\alpha')).\rho_{\mathrm{fst}}(R'_i), \nu\overline{\alpha_2}.\,(\rho \uplus (W'@\alpha')).\rho_{\mathrm{snd}}(R'_i)) \in r_i(W'@\alpha')\ .$$

  Since $\{\alpha'\} \# ftv(A'_i)$, we have

$$(\nu\alpha'.\,\nu\overline{\alpha_1}.\,(\rho \uplus W').\rho_{\mathrm{fst}}(R'_i), \nu\alpha'.\,\nu\overline{\alpha_2}.\,(\rho \uplus W').\rho_{\mathrm{snd}}(R'_i)) \in r_i(W')\ .$$

- Let α' such that $\{\alpha'\} \# W$. Without loss of generality, we can suppose that $\{\alpha'\} \# (dom(\rho) \cup \{\overline{\beta}\})$. Then, we have

  – $(\nu\alpha'.\,\nu\overline{\alpha_1}.\,(\rho \uplus W).\rho_{\mathrm{fst}}(R'_i), \nu\overline{\alpha_2}.\,(\rho \uplus W).\rho_{\mathrm{snd}}(R'_i)) \in r_i(W)$ and
  – $(\nu\overline{\alpha_1}.\,(\rho \uplus W).\rho_{\mathrm{fst}}(R'_i), \nu\alpha'.\,\nu\overline{\alpha_2}.\,(\rho \uplus W).\rho_{\mathrm{snd}}(R'_i)) \in r_i(W)$.

Thus, we have $\omega W'_{i4} \vdash (A'_i, A'_i, r_i)$. Now, we have the following.

- $(R'_{i1}, R'_{i1}) \in \mathcal{R}[\![\forall\alpha.\alpha \multimap \alpha]\!]\,\omega W_{02}$.

- $\omega\, W'_{i4} \sqsupseteq \omega\, W_{02}$ because $\omega\, W'_4 \sqsupseteq \omega\, W_2$ by Lemmas 112 and 5 with $W'_4 \sqsupseteq \omega\, W_2$.
- $\omega\, W'_{i4} \vdash (A'_i, A'_i, r_i)$, and
- $\{\alpha\}\#\omega\, W'_{i4}$.

Thus, we have

$$(R'_{i1}\, A_i, R'_{i1}\, A_i)_{\omega\, W'_{i4}} \in \mathcal{E}[\![\alpha \multimap \alpha]\!]\, \{\alpha \Mapsto (A_i, A_i, r_i)\} \uplus \omega\, W'_{i4} \; .$$

Since we have found $R'_{i1}\, A_i \longrightarrow^{n_{i2}} R_{i2}$, we have

- $\omega\, W'_{i4}.\rho_{\mathrm{fst}}(R'_{i1})\, A_i \longrightarrow^{n_{i2}} \omega\, W'_{i4}.\rho_{\mathrm{fst}}(R_{i2})$
- $\omega\, W'_{i4}.\rho_{\mathrm{snd}}(R'_{i1})\, A_i \longrightarrow^{n_{i2}} \omega\, W'_{i4}.\rho_{\mathrm{snd}}(R_{i2})$

by Lemma 149 with $\omega\, W'_{i4} \sqsupseteq \omega\, W_{02}$. Thus,

$$(R_{i2}, R_{i2})_{\omega\, W'_{i4}} \in \mathcal{R}[\![\alpha \multimap \alpha]\!]\, ((\{\alpha \Mapsto (A_i, A_i, r_i)\} \uplus W'_{i4}) - n_{i2})$$

by the definition of $\mathcal{E}$ with $W'_{i4} \sqsupseteq \omega\, W'_{i4}$ by Lemma 118. We have the following.

- $\{\alpha \Mapsto (A_i, A_i, r_i)\} \uplus W'_{i4} \sqsupseteq \{\alpha \Mapsto (A_i, A_i, r_i)\} \uplus W'_{i4}$ by Lemma 111.
- $(\{\alpha \Mapsto (A_i, A_i, r_i)\} \uplus W'_{i41}, \{\alpha \Mapsto (A_i, A_i, r_i)\} \uplus W'_{i42}) \Supset \{\alpha \Mapsto (A_i, A_i, r_i)\} \uplus W'_{i4}$ from $(W'_{41}, W'_{42}) \Supset W'_4$.
- $\{\alpha \Mapsto (A_i, A_i, r_i)\} \uplus W'_{i41} \sqsupseteq \{\alpha \Mapsto (A_i, A_i, r_i)\} \uplus W'_{i4}$ by Lemma 137 with $W'_{i41} \sqsupseteq W'_{i4}$, which is implied by $W'_{41} \sqsupseteq W'_4$; $W'_{41} \sqsupseteq W'_4$ is implied by Lemma 138 with $W'_4.\rho(\{\alpha \Mapsto (A_1, A_2, r)\}) \uplus W'_{41} = W_{41} \sqsupseteq W_4 = W'_4.\rho(\{\alpha \Mapsto (A_1, A_2, r)\}) \uplus W'_4$.
- $(R'_i, R'_i) \in \mathcal{R}[\![\alpha]\!]\, \{\alpha \Mapsto (A_i, A_i, r_i)\} \uplus W'_{i42}$ because we can find that $(\{\alpha \Mapsto (A_i, A_i, r_i)\} \uplus W'_{i42}).\rho_{\mathrm{fst}}(R'_i) = (\{\alpha \Mapsto (A_i, A_i, r_i)\} \uplus W'_{i42}).\rho_{\mathrm{snd}}(R'_i) = R'_i$ by $\{\alpha\} \# \mathit{ftv}(R'_i)$ and Lemma 113.

Thus, we have

$$(R_{i2}\, R'_i, R_{i2}\, R'_i)_{W'_{i4}} \in \mathcal{E}[\![\alpha]\!]\, \{\alpha \Mapsto (A_i, A_i, r_i)\} \uplus W'_{i4} \; .$$

Because type substitution does not change the number of evaluation steps, we can find that $W'_{i4}.\rho_{\mathrm{fst}}(R_{i2})\, R'_i$ terminates by n_{i3} steps. We have had $W'_{i4}.\rho_{\mathrm{fst}}(R_{12})\, R'_1 \longrightarrow^{n_{13}} R_{13}$ and $W'_{i4}.\rho_{\mathrm{snd}}(R_{22})\, R'_2 \longrightarrow^{n_{23}} R_{23}$. Since $\{\alpha \Mapsto (A_i, A_i, r)\} \uplus W'_{i4} \sqsupseteq W'_{i4} \sqsupseteq W'_{i42}$ by Lemmas 118, 139, and 110, the definition of $\mathcal{R}$ at α implies

$$R_{i3} = \nu\overline{\alpha_i}.\, (\rho_i \uplus \{\alpha \Mapsto (A_i, A_i, r_i)\} \uplus W'_{i4}).\rho_{\mathrm{fst}}(R'_i)$$

for some $\overline{\alpha_i}$ and ρ_i such that there exist some $\overline{\beta_i}$ and $\overline{\gamma_i}$ such that

- $\{\overline{\alpha_i}\} = \{\overline{\beta_i}, \overline{\gamma_i}\}$,
- $\{\overline{\gamma_i}\}\#(\rho_i \uplus \{\alpha \Mapsto (A_i, A_i, r_i)\} \uplus W'_{i4}@\overline{\beta_i})$,
- $\rho_i \uplus \{\alpha \Mapsto (A_i, A_i, r_i)\} \uplus W'_{i4}@\overline{\beta_i} \sqsupseteq W'_{i42}$, and
- $(\{\overline{\beta_i}\} \cup \mathit{dom}(\rho_i)) \# \mathit{ftv}(A'_i)$.

Since $W'_{i42}.\Delta \vdash R'_i : A'_i$, we have $\mathit{ftv}(R'_i) \subseteq \mathit{dom}(W'_{i42}.\Delta) = \mathit{dom}(W'_{i4}.\Delta)$. Thus,

$$R_{i3} = \nu\overline{\alpha_i}.\, W'_{i4}.\rho_{\mathrm{fst}}(R'_i) \; .$$

Further, by Lemma 113,

$$R_{i3} = \nu\overline{\alpha_i}.\, R'_i \; .$$

Since $\{\overline{\beta_i}\}\#(\{\alpha \Mapsto (A_i, A_i, r_i)\} \uplus W'_{i4})$ and $\{\overline{\gamma_i}\}\#(\{\alpha \Mapsto (A_i, A_i, r_i)\} \uplus W'_{i4})$, we have $\{\overline{\alpha_i}\} = \{\overline{\beta_i}, \overline{\gamma_i}\}\# W_{42}$. Since $(R'_1, R'_2) \in r(\blacktriangleright W_{42})$, we have the conclusion

$$(\nu\overline{\alpha_1}.\, R'_1, \nu\overline{\alpha_2}.\, R'_2) \in r(\blacktriangleright W_{42})$$

by the fourth condition of Rel_n on r.

□

References

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