Supplementary Material for "Handling Polymorphic Algebraic Effects"

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1 Definition

1.1 Surface language

1.1.1 Syntax

Effect operations
op

Effects
$\epsilon ::=$ sets of effect operations

Base types
$\iota ::= \mathsf{bool} \mid \mathsf{int} \mid \bot \mid ...$

Type variables
α, β, γ

Types
$A, B, C, D ::= \alpha \mid \iota \mid A \rightarrow \epsilon\ B$

Type schemes
$\sigma ::= A \mid \forall\, \alpha.\sigma$

Constants
$c ::= \mathsf{true} \mid \mathsf{false} \mid 0 \mid + \mid ...$

Terms
$M ::= x \mid c \mid \lambda x.M \mid M_1\, M_2 \mid \mathsf{let}\, x = M_1\, \mathsf{in}\, M_2 \mid$
$\quad \#\mathsf{op}(M) \mid \mathsf{handle}\, M\, \mathsf{with}\, H \mid \mathsf{resume}\, M$

Handlers
$H ::= \mathsf{return}\, x \rightarrow M \mid H; \mathsf{op}(x) \rightarrow M$

Typing contexts
$\Gamma ::= \emptyset \mid \Gamma, x : \sigma \mid \Gamma, \alpha$

Convention 1. *We write $\forall\, \boldsymbol{\alpha}^{i \in I}.A$ for $\forall\, \alpha_1....\forall\, \alpha_n.A$ where $I = \{1, ..., n\}$. We often omit indices (i and j) and index sets (I and J) if they are not important: for example, we often abbreviate $\forall\, \boldsymbol{\alpha}^{i \in I}.A$ to $\forall\, \boldsymbol{\alpha}^{I}.A$ or even $\forall\, \boldsymbol{\alpha}.A$. Similarly, we use a bold font for other sequences ($\boldsymbol{A}^{i \in I}$ for a sequence of types, $\boldsymbol{v}^{i \in I}$ for a sequence of values, and so on). We sometimes write $\{\boldsymbol{\alpha}\}$ to view the sequence $\boldsymbol{\alpha}$ as a set by ignoring the order. We also write $\forall\, \boldsymbol{\alpha}^{I}.\boldsymbol{\sigma}^{J}$ for a sequence $\forall\, \boldsymbol{\alpha}^{I}.\sigma_{j_1}, \ldots, \forall\, \boldsymbol{\alpha}^{I}.\sigma_{j_n}$ of type schemes (where $J = \{j_1, \ldots, j_n\}$).*

Definition 1 (Domain of typing contexts)**.** *We define $dom(\Gamma)$ as follows.*

$$
\begin{array}{lcl}
dom(\Gamma, x : \sigma) & \stackrel{\mathrm{def}}{=} & dom(\Gamma) \cup \{x\} \\
dom(\Gamma, \alpha) & \stackrel{\mathrm{def}}{=} & dom(\Gamma) \cup \{\alpha\}
\end{array}
$$

Definition 2 (Free type variables and type substitution in type schemes)**.** *Free type variables $ftv(\sigma)$ in a type scheme σ and type substitution $B[\boldsymbol{A}/\boldsymbol{\alpha}]$ of $\boldsymbol{A}$ for type variables $\boldsymbol{\alpha}$ in B are defined as usual.*

Assumption 1. *We suppose that each constant c is assigned a first-order closed type $ty(c)$ of the form $\iota_1 \rightarrow \langle\rangle \cdots \rightarrow \langle\rangle\ \iota_n$ and that each effect operation* op *is assigned a signature of the form $\forall \boldsymbol{\alpha}.A \hookrightarrow B$. We also assume that, for $ty\,(\mathsf{op}) = \forall \boldsymbol{\alpha}.A \hookrightarrow B$, $ftv(A) \subseteq \{\boldsymbol{\alpha}\}$ and $ftv(B) \subseteq \{\boldsymbol{\alpha}\}$.*

Suppose that, for any ι, there is a set K_ι of constants of ι. For any constant c, $ty(c) = \iota$ if and only if $c \in K_\iota$. The function ζ gives a denotation to pairs of constants. In particular, for any constants c_1 and c_2: (1) $\zeta(c_1, c_2)$ is defined if and only if $ty(c_1) = \iota \rightarrow \langle\rangle\ A$ and $ty(c_2) = \iota$ for some A; and (2) if $\zeta(c_1, c_2)$ is defined, $\zeta(c_1, c_2)$ is a constant and $ty(\zeta(c_1, c_2)) = A$ where $ty(c_1) = \iota \rightarrow \langle\rangle\ A$.

1.1.2 Typing

Definition 3 (Resumption type)**.** *We define resumption type R as follows.*

$$R \ ::= \ \mathsf{none} \mid (\boldsymbol{\alpha}, x : A, B \rightarrow \epsilon\ C) \quad \textit{(if } ftv(A) \cup ftv(B) \subseteq \{\boldsymbol{\alpha}\} \textit{ and } ftv(C) \cap \{\boldsymbol{\alpha}\} = \emptyset\textit{)}$$

Definition 4 (Type scheme well-formedness)**.** *We write $\Gamma \vdash \sigma$ if and only if $ftv(\sigma) \subseteq dom(\Gamma)$.*

Definition 5. *Judgments $\vdash \Gamma$ and $\Gamma; R \vdash M : A \mid \epsilon$ and $\Gamma; R \vdash H : A \mid \epsilon \Rightarrow B \mid \epsilon'$ are the least relations satisfying the rules in Figure 1.*

Term M is a well-typed program of A if and only if $\emptyset; \mathsf{none} \vdash M : A \mid \langle\rangle$.

1.2 Intermediate language

1.2.1 Syntax

Values

$$v \ ::= \ c \mid \lambda x.e$$

Polymorphic values

$$w \ ::= \ v \mid \Lambda\alpha.w$$

Terms

$$e \ ::= \ x\,\boldsymbol{A} \mid c \mid \lambda x.e \mid e_1\ e_2 \mid \mathsf{let}\ x = \Lambda\boldsymbol{\alpha}.e_1\ \mathsf{in}\ e_2 \mid \#\mathsf{op}(\boldsymbol{A}, e) \mid \#\mathsf{op}(\boldsymbol{\sigma}, w, E) \mid \mathsf{handle}\ e\ \mathsf{with}\ h \mid \mathsf{resume}\ \boldsymbol{\alpha}\ x.e$$

Handlers

$$h \ ::= \ \mathsf{return}\ x \rightarrow e \mid h; \Lambda\boldsymbol{\alpha}.\mathsf{op}(x) \rightarrow e$$

Evaluation contexts

$$E^{\boldsymbol{\alpha}^I} \ ::= \ [\,]\ (\text{if } \boldsymbol{\alpha}^I = \emptyset) \mid E^{\boldsymbol{\alpha}^I}\ e_2 \mid v_1\ E^{\boldsymbol{\alpha}^I} \mid \mathsf{let}\ x = \Lambda\boldsymbol{\beta}^{J_1}.E^{\boldsymbol{\gamma}^{J_2}}\ \mathsf{in}\ e_2\ (\text{if } \boldsymbol{\alpha}^I = \boldsymbol{\beta}^{J_1}, \boldsymbol{\gamma}^{J_2}) \mid \#\mathsf{op}(\boldsymbol{A}^J, E^{\boldsymbol{\alpha}^I}) \mid \mathsf{handle}\ E^{\boldsymbol{\alpha}^I}\ \mathsf{with}\ h$$

Top-level typing contexts

$$\Delta \ ::= \ \emptyset \mid \Delta, \alpha$$

Convention 2. *We write E for $E^{\boldsymbol{\alpha}}$ if $\boldsymbol{\alpha}$ is not important.*

Definition 6 (Free type variables)**.** *We write $ftv(e)$ and $ftv(E)$ for sets of type variables that occur free in e and E, respectively. The notion of free type variables is defined as usual.*

Definition 7 (Substitution)**.** *Substitution $e[\boldsymbol{A}/\boldsymbol{\alpha}]$ of $\boldsymbol{A}$ for $\boldsymbol{\alpha}$ in e is defined in a capture-avoiding manner as usual. Substitution $e[w/x]$ of polymorphic value w for variable x in e is also defined in a standard capture-avoiding manner: in particular,*

$$(x\,\boldsymbol{A})[\Lambda\boldsymbol{\alpha}.v/x] \ \stackrel{\text{def}}{=} \ v[\boldsymbol{A}/\boldsymbol{\alpha}]$$

Substitution $e[E^{\boldsymbol{\beta}^J}/\mathsf{resume}]^{\forall \boldsymbol{\beta}^J.\boldsymbol{A}^I}_{\Lambda\boldsymbol{\beta}^J.v}$ of continuation $E^{\boldsymbol{\beta}^J}$ for resumptions in e is defined in a capture-avoiding

Well-formed rules for typing contexts

$\boxed{\vdash \Gamma}$

$$\frac{}{\vdash \emptyset}\ \textsc{WF\_Empty} \qquad \frac{\vdash \Gamma \quad x \notin \mathit{dom}(\Gamma) \quad \Gamma \vdash \sigma}{\vdash \Gamma, x : \sigma}\ \textsc{WF\_Var} \qquad \frac{\vdash \Gamma \quad \alpha \notin \mathit{dom}(\Gamma)}{\vdash \Gamma, \alpha}\ \textsc{WF\_TyVar}$$

Typing rules

$\boxed{\Gamma; R \vdash M : A \mid \epsilon}$

$$\frac{\vdash \Gamma \quad x : \forall \boldsymbol{\alpha}.A \in \Gamma \quad \Gamma \vdash \boldsymbol{B}}{\Gamma; R \vdash x : A[\boldsymbol{B}/\boldsymbol{\alpha}] \mid \epsilon}\ \textsc{TS\_Var} \qquad \frac{\vdash \Gamma}{\Gamma; R \vdash c : \mathit{ty}(c) \mid \epsilon}\ \textsc{TS\_Const}$$

$$\frac{\Gamma, x : A; R \vdash M : B \mid \epsilon'}{\Gamma; R \vdash \lambda x.M : A \rightarrow^{\epsilon'} B \mid \epsilon}\ \textsc{TS\_Abs} \qquad \frac{\Gamma; R \vdash M_1 : A \rightarrow^{\epsilon'} B \mid \epsilon \quad \Gamma; R \vdash M_2 : A \mid \epsilon \quad \epsilon' \subseteq \epsilon}{\Gamma; R \vdash M_1\, M_2 : B \mid \epsilon}\ \textsc{TS\_App}$$

$$\frac{\mathit{ty}\,(\mathsf{op}) = \forall \boldsymbol{\alpha}.A \hookrightarrow B \quad \mathsf{op} \in \epsilon \quad \Gamma; R \vdash M : A[\boldsymbol{C}/\boldsymbol{\alpha}] \mid \epsilon \quad \Gamma \vdash \boldsymbol{C}}{\Gamma; R \vdash \#\mathsf{op}(M) : B[\boldsymbol{C}/\boldsymbol{\alpha}] \mid \epsilon}\ \textsc{TS\_Op}$$

$$\frac{\Gamma, \boldsymbol{\alpha}; R \vdash M_1 : A \mid \epsilon \quad \Gamma, x : \forall \boldsymbol{\alpha}.A; R \vdash M_2 : B \mid \epsilon}{\Gamma; R \vdash \mathsf{let}\ x = M_1\ \mathsf{in}\ M_2 : B \mid \epsilon}\ \textsc{TS\_Let}$$

$$\frac{\Gamma; R \vdash M : A \mid \epsilon' \quad \epsilon' \subseteq \epsilon}{\Gamma; R \vdash M : A \mid \epsilon}\ \textsc{TS\_Weak} \qquad \frac{\Gamma; R \vdash M : A \mid \epsilon \quad \Gamma; R \vdash H : A \mid \epsilon \Rightarrow B \mid \epsilon'}{\Gamma; R \vdash \mathsf{handle}\ M\ \mathsf{with}\ H : B \mid \epsilon'}\ \textsc{TS\_Handle}$$

$$\frac{\begin{array}{l}\vdash \Gamma_1, x : D, \Gamma_2 \quad \boldsymbol{\alpha} \in \Gamma_1 \quad \epsilon \subseteq \epsilon' \\ \Gamma_1, \Gamma_2, \boldsymbol{\beta}, x : A[\boldsymbol{\beta}/\boldsymbol{\alpha}]; (\boldsymbol{\alpha}, x : A, B \rightarrow^{\epsilon} C) \vdash M : B[\boldsymbol{\beta}/\boldsymbol{\alpha}] \mid \epsilon'\end{array}}{\Gamma_1, x : D, \Gamma_2; (\boldsymbol{\alpha}, x : A, B \rightarrow^{\epsilon} C) \vdash \mathsf{resume}\ M : C \mid \epsilon'}\ \textsc{TS\_Resume}$$

$\boxed{\Gamma; R \vdash H : A \mid \epsilon \Rightarrow B \mid \epsilon'}$

$$\frac{\Gamma, x : A; R \vdash M : B \mid \epsilon' \quad \epsilon \subseteq \epsilon'}{\Gamma; R \vdash \mathsf{return}\ x \rightarrow M : A \mid \epsilon \Rightarrow B \mid \epsilon'}\ \textsc{THS\_Return}$$

$$\frac{\begin{array}{l}\Gamma; R \vdash H : A \mid \epsilon \Rightarrow B \mid \epsilon' \\ \mathit{ty}\,(\mathsf{op}) = \forall \boldsymbol{\alpha}.C \hookrightarrow D \quad \Gamma, \boldsymbol{\alpha}, x : C; (\boldsymbol{\alpha}, x : C, D \rightarrow^{\epsilon'} B) \vdash M : B \mid \epsilon'\end{array}}{\Gamma; R \vdash H; \mathsf{op}(x) \rightarrow M : A \mid \epsilon \uplus \{\mathsf{op}\} \Rightarrow B \mid \epsilon'}\ \textsc{THS\_Op}$$

Figure 1: Typing rules in $\lambda_{\mathrm{eff}}^{\mathrm{let}}$.

manner, as follows (we describe only important cases).

$$\begin{array}{rcl}(\mathsf{resume}\,\boldsymbol{\alpha}^I\ x.e)[E^{\boldsymbol{\beta}^J}/\mathsf{resume}]^{\forall \boldsymbol{\beta}^J.\boldsymbol{A}^I}_{\Lambda\boldsymbol{\beta}^J.v} & \stackrel{\text{def}}{=} & \\ \multicolumn{3}{r}{\mathsf{let}\ y = \Lambda\boldsymbol{\beta}^J.e[E^{\boldsymbol{\beta}^J}/\mathsf{resume}]^{\forall \boldsymbol{\beta}^J.\boldsymbol{A}^I}_{\Lambda\boldsymbol{\beta}^J.v}[\boldsymbol{A}^I/\boldsymbol{\alpha}^I][v/x]\ \mathsf{in}\ E^{\boldsymbol{\beta}^J}[y\,\boldsymbol{\beta}^J]} \\ \multicolumn{3}{r}{\textit{(if } (\mathit{ftv}(e) \cup \mathit{ftv}(E^{\boldsymbol{\beta}^J})) \cap \{\boldsymbol{\beta}^J\} = \emptyset \textit{ and } y \textit{ is fresh)}} \\ (\mathsf{return}\ x \rightarrow e)[E/\mathsf{resume}]^{\boldsymbol{\sigma}}_w & \stackrel{\text{def}}{=} & \mathsf{return}\ x \rightarrow e[E/\mathsf{resume}]^{\boldsymbol{\sigma}}_w \\ (h'; \Lambda\boldsymbol{\alpha}^J.\mathsf{op}(x) \rightarrow e)[E/\mathsf{resume}]^{\boldsymbol{\sigma}^I}_w & \stackrel{\text{def}}{=} & h'[E/\mathsf{resume}]^{\boldsymbol{\sigma}^I}_w; \Lambda\boldsymbol{\alpha}^J.\mathsf{op}(x) \rightarrow e\end{array}$$

Definition 8 (Resumption type)**.** *We define resumption type r as follows.*

$$r \quad ::= \quad \mathsf{none} \mid (\boldsymbol{\alpha}, A, B \rightarrow^{\epsilon} C) \quad \textit{(if } \mathit{ftv}(A) \cup \mathit{ftv}(B) \subseteq \{\boldsymbol{\alpha}\} \textit{ and } \mathit{ftv}(C) \cap \{\boldsymbol{\alpha}\} = \emptyset\textit{)}$$

We also define a set of type variables captured by a resume type:

$$\begin{array}{lcl} \mathit{tyvars}(\mathsf{none}) & \stackrel{\text{def}}{=} & \emptyset \\ \mathit{tyvars}((\boldsymbol{\alpha}, A, B \to \epsilon\ C)) & \stackrel{\text{def}}{=} & \{\boldsymbol{\alpha}\} \end{array}$$

Reduction rules $\boxed{e_1 \rightsquigarrow e_2}$

$$c_1\; c_2 \;\rightsquigarrow\; \zeta(c_1, c_2) \quad \text{(R\_Const)} \qquad (\lambda x.e)\; v \;\rightsquigarrow\; e[v/x] \quad \text{(R\_Beta)}$$

$$\mathsf{let}\; x = \Lambda \boldsymbol{\alpha}.v \;\mathsf{in}\; e \;\rightsquigarrow\; e[\Lambda \boldsymbol{\alpha}.v/x] \quad \text{(R\_Let)} \qquad \mathsf{handle}\; v \;\mathsf{with}\; h \;\rightsquigarrow\; e[v/x] \quad \text{(R\_Return)} \quad (\text{where } h^{\mathsf{return}} = \mathsf{return}\; x \to e)$$

$$\#\mathsf{op}(\boldsymbol{A}, v) \;\rightsquigarrow\; \#\mathsf{op}(\boldsymbol{A}, v, [\,]) \quad \text{(R\_Op)}$$

$$\#\mathsf{op}(\boldsymbol{\sigma}, w, E)\; e_2 \;\rightsquigarrow\; \#\mathsf{op}(\boldsymbol{\sigma}, w, E\; e_2) \quad \text{(R\_OpApp1)}$$

$$v_1\; \#\mathsf{op}(\boldsymbol{\sigma}, w, E) \;\rightsquigarrow\; \#\mathsf{op}(\boldsymbol{\sigma}, w, v_1\; E) \quad \text{(R\_OpApp2)}$$

$$\#\mathsf{op}'(\boldsymbol{A}^I, \#\mathsf{op}(\boldsymbol{\sigma}^J, w, E)) \;\rightsquigarrow\; \#\mathsf{op}(\boldsymbol{\sigma}^J, w, \#\mathsf{op}'(\boldsymbol{A}^I, E)) \quad \text{(R\_OpOp)}$$

$$\mathsf{handle}\; \#\mathsf{op}(\boldsymbol{\sigma}, w, E) \;\mathsf{with}\; h \;\rightsquigarrow\; \#\mathsf{op}(\boldsymbol{\sigma}, w, \mathsf{handle}\; E \;\mathsf{with}\; h) \quad (\text{where } \mathsf{op} \notin \mathit{ops}(h)) \quad \text{(R\_OpHandle)}$$

$$\mathsf{let}\; x = \Lambda \boldsymbol{\alpha}^I.\#\mathsf{op}(\boldsymbol{\sigma}^J, w, E) \;\mathsf{in}\; e_2 \;\rightsquigarrow\; \#\mathsf{op}(\forall\, \boldsymbol{\alpha}^I.\boldsymbol{\sigma}^J, \Lambda \boldsymbol{\alpha}^I.w, \mathsf{let}\; x = \Lambda \boldsymbol{\alpha}^I.E \;\mathsf{in}\; e_2) \quad \text{(R\_OpLet)}$$

$$\mathsf{handle}\; \#\mathsf{op}(\forall\, \boldsymbol{\beta}^J.\boldsymbol{A}^I, \Lambda \boldsymbol{\beta}^J.v, E^{\boldsymbol{\beta}^J}) \;\mathsf{with}\; h \;\rightsquigarrow\; e[\mathsf{handle}\; E^{\boldsymbol{\beta}^J} \;\mathsf{with}\; h/\mathsf{resume}]^{\forall\,\boldsymbol{\beta}^J.\boldsymbol{A}^I}_{\Lambda\boldsymbol{\beta}^J.v}[\boldsymbol{A}^I[\bot/\boldsymbol{\beta}^J]/\boldsymbol{\alpha}^I][v[\bot/\boldsymbol{\beta}^J]/x] \quad (\text{where } h^{\mathsf{op}} = \Lambda \boldsymbol{\alpha}^I.\mathsf{op}(x) \to e) \quad \text{(R\_Handle)}$$

Evaluation rules $\boxed{e_1 \longrightarrow e_2}$

$$\frac{e_1 \rightsquigarrow e_2}{E[e_1] \longrightarrow E[e_2]} \quad \text{E\_Eval}$$

Figure 2: Semantics of $\lambda^{\Lambda}_{\mathrm{eff}}$.

1.2.2 Semantics

Definition 9. *Relations $\longrightarrow$ and $\rightsquigarrow$ are the smallest relations satisfying the rules in Figure 2.*

Definition 10 (Multi-step evaluation)**.** *Binary relation $\longrightarrow^*$ over terms is the reflexive and transitive closure of $\longrightarrow$.*

Definition 11 (Nonreducible terms)**.** *We write $e \not\longrightarrow$ if there no terms e' such that $e \longrightarrow e'$.*

Typing rules

$\boxed{\Gamma; r \vdash e : A \mid \epsilon}$

$$\frac{\vdash \Gamma \quad x : \forall \boldsymbol{\alpha}.A \in \Gamma \quad \Gamma \vdash \boldsymbol{B}}{\Gamma; r \vdash x\, \boldsymbol{B} : A[\boldsymbol{B}/\boldsymbol{\alpha}] \mid \epsilon}\ \textsc{T\_Var} \qquad \frac{\vdash \Gamma}{\Gamma; r \vdash c : ty(c) \mid \epsilon}\ \textsc{T\_Const}$$

$$\frac{\Gamma, x : A; r \vdash e : B \mid \epsilon'}{\Gamma; r \vdash \lambda x.e : A \to^{\epsilon'} B \mid \epsilon}\ \textsc{T\_Abs} \qquad \frac{\Gamma; r \vdash e_1 : A \to^{\epsilon'} B \mid \epsilon \quad \Gamma; r \vdash e_2 : A \mid \epsilon \quad \epsilon' \subseteq \epsilon}{\Gamma; r \vdash e_1\, e_2 : B \mid \epsilon}\ \textsc{T\_App}$$

$$\frac{ty\,(\mathsf{op}) = \forall \boldsymbol{\alpha}.A \hookrightarrow B \quad \mathsf{op} \in \epsilon \quad \Gamma; r \vdash e : A[\boldsymbol{C}/\boldsymbol{\alpha}] \mid \epsilon \quad \Gamma \vdash \boldsymbol{C}}{\Gamma; r \vdash \#\mathsf{op}(\boldsymbol{C}, e) : B[\boldsymbol{C}/\boldsymbol{\alpha}] \mid \epsilon}\ \textsc{T\_Op}$$

$$\frac{\begin{array}{c} ty\,(\mathsf{op}) = \forall \boldsymbol{\alpha}^I.A \hookrightarrow B \quad \mathsf{op} \in \epsilon \quad \Gamma \vdash \forall \boldsymbol{\beta}^J.\boldsymbol{C}^I \\ \Gamma, \boldsymbol{\beta}^J; r \vdash v : A[\boldsymbol{C}^I/\boldsymbol{\alpha}^I] \mid \epsilon \quad \Gamma \vdash E^{\boldsymbol{\beta}^J} : \forall \boldsymbol{\beta}^J.(B[\boldsymbol{C}^I/\boldsymbol{\alpha}^I]) \multimap D \mid \epsilon \end{array}}{\Gamma; r \vdash \#\mathsf{op}(\forall \boldsymbol{\beta}^J.\boldsymbol{C}^I, \Lambda \boldsymbol{\beta}^J.v, E^{\boldsymbol{\beta}^J}) : D \mid \epsilon}\ \textsc{T\_OpCont}$$

$$\frac{\Gamma; r \vdash e : A \mid \epsilon' \quad \epsilon' \subseteq \epsilon}{\Gamma; r \vdash e : A \mid \epsilon}\ \textsc{T\_Weak} \qquad \frac{\Gamma; r \vdash e : A \mid \epsilon \quad \Gamma; r \vdash h : A \mid \epsilon \Rightarrow B \mid \epsilon'}{\Gamma; r \vdash \mathsf{handle}\ e\ \mathsf{with}\ h : B \mid \epsilon'}\ \textsc{T\_Handle}$$

$$\frac{\Gamma, \boldsymbol{\alpha}; r \vdash e_1 : A \mid \epsilon \quad \Gamma, x : \forall \boldsymbol{\alpha}.A; r \vdash e_2 : B \mid \epsilon}{\Gamma; r \vdash \mathsf{let}\ x = \Lambda \boldsymbol{\alpha}.e_1\ \mathsf{in}\ e_2 : B \mid \epsilon}\ \textsc{T\_Let}$$

$$\frac{\boldsymbol{\alpha} \in \Gamma \quad \Gamma, \boldsymbol{\beta}, x : A[\boldsymbol{\beta}/\boldsymbol{\alpha}]; (\boldsymbol{\alpha}, A, B \to^{\epsilon} C) \vdash e : B[\boldsymbol{\beta}/\boldsymbol{\alpha}] \mid \epsilon' \quad \epsilon \subseteq \epsilon'}{\Gamma; (\boldsymbol{\alpha}, A, B \to^{\epsilon} C) \vdash \mathsf{resume}\, \boldsymbol{\beta}\, x.e : C \mid \epsilon'}\ \textsc{T\_Resume}$$

Figure 3: Typing rules for terms in $\lambda^{\Lambda}_{\mathsf{eff}}$.

1.2.3 Typing

Definition 12. *Judgments* $\Gamma; r \vdash e : A \mid \epsilon$ *and* $\Gamma; r \vdash h : A \mid \epsilon \Rightarrow B \mid \epsilon'$ *and* $\Gamma \vdash E : \sigma \multimap A \mid \epsilon$ *are the smallest relations satisfying the rules in Figures 3 and 4.*

Convention 3 (Typing judgments of values)**.** *We write* $\Gamma \vdash v : A$ *if* $\Gamma; \mathsf{none} \vdash v : A \mid \epsilon$*; effect* ϵ *given to value* v *can be any (validated by Lemma 6).*

1.2.4 Elaboration

Definition 13. *Relation* $\Gamma; R \vdash M : A \mid \epsilon \rhd^S e$ *is the smallest relation satisfying the rules in Figure 5.*

2 Proofs

2.1 Type soundness of $\lambda^{\Lambda}_{\mathsf{eff}}$

Lemma 1. *Suppose that* $\vdash \Gamma_1, \Gamma_2$ *and* $\vdash \Gamma_1, \Gamma_3$ *and* $dom(\Gamma_2) \cap dom(\Gamma_3) = \emptyset$*. Then,* $\vdash \Gamma_1, \Gamma_2, \Gamma_3$*.*

Proof. Straightforward by induction on Γ_3. □

Lemma 2 (Weakening)**.** *Suppose that* $\vdash \Gamma_1, \Gamma_2$*. Let* Γ_3 *be a typing context such that* $dom(\Gamma_2) \cap dom(\Gamma_3) = \emptyset$*.*

1. *If* $\Gamma_1, \Gamma_3; r \vdash e : A \mid \epsilon$*, then* $\Gamma_1, \Gamma_2, \Gamma_3; r \vdash e : A \mid \epsilon$*.*

2. *If* $\Gamma_1, \Gamma_3; r \vdash h : A \mid \epsilon \Rightarrow B \mid \epsilon'$*, then* $\Gamma_1, \Gamma_2, \Gamma_3; r \vdash h : A \mid \epsilon \Rightarrow B \mid \epsilon'$*.*

3. *If* $\Gamma_1, \Gamma_3 \vdash E : A \multimap B \mid \epsilon$*, then* $\Gamma_1, \Gamma_2, \Gamma_3 \vdash E : A \multimap B \mid \epsilon$*.*

$\boxed{\Gamma; r \vdash h : A \,|\, \epsilon \Rightarrow B \,|\, \epsilon'}$

$$\frac{\Gamma, x : A; r \vdash e \; : B \,|\, \epsilon' \quad \epsilon \subseteq \epsilon'}{\Gamma; r \vdash \mathsf{return}\, x \rightarrow e : A \,|\, \epsilon \Rightarrow B \,|\, \epsilon'} \;\text{TH\_RETURN}$$

$$\frac{\begin{array}{l}\Gamma; r \vdash h : A \,|\, \epsilon \Rightarrow B \,|\, \epsilon' \\ \mathit{ty}\,(\mathsf{op}) \;=\; \forall \boldsymbol{\alpha}. C \hookrightarrow D \quad \Gamma, \boldsymbol{\alpha}, x : C; (\boldsymbol{\alpha}, C, D \rightarrow^{\epsilon'} B) \vdash e \; : B \,|\, \epsilon'\end{array}}{\Gamma; r \vdash h; \Lambda\boldsymbol{\alpha}.\mathsf{op}(x) \rightarrow e : A \,|\, \epsilon \uplus \{\mathsf{op}\} \Rightarrow B \,|\, \epsilon'} \;\text{TH\_OP}$$

$\boxed{\Gamma \vdash E : \sigma \multimap A \,|\, \epsilon}$

$$\frac{}{\Gamma \vdash [\,] : A \multimap A \,|\, \epsilon} \;\text{TE\_HOLE}$$

$$\frac{\Gamma \vdash E : \sigma \multimap (A \rightarrow^{\epsilon'} B) \,|\, \epsilon \quad \Gamma; \mathsf{none} \vdash e_2 \; : A \,|\, \epsilon \quad \epsilon' \subseteq \epsilon}{\Gamma \vdash E\; e_2 : \sigma \multimap B \,|\, \epsilon} \;\text{TE\_APP1}$$

$$\frac{\Gamma; \mathsf{none} \vdash v_1 \; : (A \rightarrow^{\epsilon'} B) \,|\, \epsilon \quad \Gamma \vdash E : \sigma \multimap A \,|\, \epsilon \quad \epsilon' \subseteq \epsilon}{\Gamma \vdash v_1\, E : \sigma \multimap B \,|\, \epsilon} \;\text{TE\_APP2}$$

$$\frac{\mathit{ty}\,(\mathsf{op}) \;=\; \forall \boldsymbol{\alpha}. A \hookrightarrow B \quad \mathsf{op} \in \epsilon \quad \Gamma \vdash E : \sigma \multimap A[\boldsymbol{C}/\boldsymbol{\alpha}] \,|\, \epsilon \quad \Gamma \vdash \boldsymbol{C}}{\Gamma \vdash \#\mathsf{op}(\boldsymbol{C}, E) : \sigma \multimap B[\boldsymbol{C}/\boldsymbol{\alpha}] \,|\, \epsilon} \;\text{TE\_OP}$$

$$\frac{\Gamma \vdash E : \sigma \multimap A \,|\, \epsilon \quad \Gamma; \mathsf{none} \vdash h : A \,|\, \epsilon \Rightarrow B \,|\, \epsilon'}{\Gamma \vdash \mathsf{handle}\, E\, \mathsf{with}\, h : \sigma \multimap B \,|\, \epsilon'} \;\text{TE\_HANDLE}$$

$$\frac{\Gamma \vdash E : \sigma \multimap A \,|\, \epsilon' \quad \epsilon' \subseteq \epsilon}{\Gamma \vdash E : \sigma \multimap A \,|\, \epsilon} \;\text{TE\_WEAK}$$

$$\frac{\Gamma, \boldsymbol{\alpha} \vdash E : \sigma \multimap A \,|\, \epsilon \quad \Gamma, x : \forall\, \boldsymbol{\alpha}. A; \mathsf{none} \vdash e \; : B \,|\, \epsilon}{\Gamma \vdash \mathsf{let}\, x = \Lambda\boldsymbol{\alpha}. E\, \mathsf{in}\, e : \forall\, \boldsymbol{\alpha}. \sigma \multimap B \,|\, \epsilon} \;\text{TE\_LET}$$

Figure 4: Typing rules for handlers and continuations in $\lambda^{\Lambda}_{\text{eff}}$.

Proof. By mutual induction on the typing derivations. We mention only the interesting cases.

Case (T_VAR) and (T_CONST): By Lemma 1.

Case (T_ABS): We are given $\Gamma_1, \Gamma_3; r \vdash \lambda x.e' \; : A' \rightarrow^{\epsilon'} B' \,|\, \epsilon$ and, by inversion, $\Gamma_1, \Gamma_3, x : A'; r \vdash e' \; : B' \,|\, \epsilon'$. With loss of generality, we can suppose that $x \notin \mathit{dom}(\Gamma_1, \Gamma_2, \Gamma_3)$. By the IH, $\Gamma_1, \Gamma_2, \Gamma_3, x : A'; r \vdash e' \; : B' \,|\, \epsilon'$. Thus, by (T_ABS), $\Gamma_1, \Gamma_2, \Gamma_3; r \vdash \lambda x.e' \; : A' \rightarrow^{\epsilon'} B' \,|\, \epsilon$.

□

Lemma 3. *If* $\vdash \Gamma_1, \alpha, \Gamma_2$ *and* $\Gamma_1 \vdash A$, *then* $\vdash \Gamma_1, \Gamma_2\, [A/\alpha]$.

Proof. By induction on Γ_2. □

Lemma 4. *If* $\Gamma; r \vdash e \; : A \,|\, \epsilon$, *then* $\vdash \Gamma$.

Proof. By induction on the derivation of $\Gamma; r \vdash e \; : A \,|\, \epsilon$. □

Lemma 5 (Type substitution)**.** *Suppose that* $\Gamma_1 \vdash A$ *and* $\alpha \notin \mathit{tyvars}(r)$ *and* $\mathit{tyvars}(r) \subseteq \mathit{dom}(\Gamma_2)$.

1. *If* $\Gamma_1, \alpha, \Gamma_2; r \vdash e \; : B \,|\, \epsilon$, *then* $\Gamma_1, \Gamma_2\, [A/\alpha]; r[A/\alpha] \vdash e[A/\alpha] \; : B[A/\alpha] \,|\, \epsilon$.

Term elaboration rules $\boxed{\Gamma; R \vdash M : A \,|\, \epsilon \rhd^S e}$

$$\frac{\vdash \Gamma \quad x : \forall \boldsymbol{\alpha}.A \in \Gamma \quad \Gamma \vdash \boldsymbol{B}}{\Gamma; R \vdash x : A[\boldsymbol{B}/\boldsymbol{\alpha}] \,|\, \epsilon \rhd^S S(x)\, \boldsymbol{B}} \text{ Elab\_Var} \qquad \frac{\vdash \Gamma}{\Gamma; R \vdash c : ty(c) \,|\, \epsilon \rhd^S c} \text{ Elab\_Const}$$

$$\frac{\Gamma, x : A; R \vdash M : B \,|\, \epsilon' \rhd^{S \circ \{x \mapsto x\}} e}{\Gamma; R \vdash \lambda x.M : A \to^{\epsilon'} B \,|\, \epsilon \rhd^S \lambda x.e} \text{ Elab\_Abs}$$

$$\frac{\Gamma; R \vdash M_1 : A \to^{\epsilon'} B \,|\, \epsilon \rhd^S e_1 \quad \Gamma; R \vdash M_2 : A \,|\, \epsilon \rhd^S e_2 \quad \epsilon' \subseteq \epsilon}{\Gamma; R \vdash M_1\, M_2 : B \,|\, \epsilon \rhd^S e_1\, e_2} \text{ Elab\_App}$$

$$\frac{ty\,(\mathsf{op}) = \forall \boldsymbol{\alpha}.A \hookrightarrow B \quad \mathsf{op} \in \epsilon \quad \Gamma; R \vdash M : A[\boldsymbol{C}/\boldsymbol{\alpha}] \,|\, \epsilon \rhd^S e \quad \Gamma \vdash \boldsymbol{C}}{\Gamma; R \vdash \#\mathsf{op}(M) : B[\boldsymbol{C}/\boldsymbol{\alpha}] \,|\, \epsilon \rhd^S \#\mathsf{op}(\boldsymbol{C}, e)} \text{ Elab\_Op}$$

$$\frac{\Gamma; R \vdash M : A \,|\, \epsilon \rhd^S e \quad \Gamma; R \vdash H : A \,|\, \epsilon \Rightarrow B \,|\, \epsilon' \rhd^S h}{\Gamma; R \vdash \mathsf{handle}\, M \,\mathsf{with}\, H : B \,|\, \epsilon' \rhd^S \mathsf{handle}\, e \,\mathsf{with}\, h} \text{ Elab\_Handle}$$

$$\frac{\begin{array}{c}\Gamma, \boldsymbol{\alpha}; R \vdash M_1 : A \,|\, \epsilon \rhd^S e_1 \quad S' = S \circ \{x \mapsto x\} \\ \Gamma, x : \forall \boldsymbol{\alpha}.A; R \vdash M_2 : B \,|\, \epsilon \rhd^{S'} e_2\end{array}}{\Gamma; R \vdash \mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2 : B \,|\, \epsilon \rhd^S \mathsf{let}\, x = \Lambda \boldsymbol{\alpha}.e_1 \,\mathsf{in}\, e_2} \text{ Elab\_Let}$$

$$\frac{\begin{array}{l}R = (\boldsymbol{\alpha}, x : A, B \to^{\epsilon} C) \quad \vdash \Gamma_1, x : D, \Gamma_2 \quad \boldsymbol{\alpha} \in \Gamma_1 \quad \epsilon \subseteq \epsilon' \\ y \text{ is fresh} \quad S' = S \circ \{x \mapsto y\} \\ \Gamma_1, \Gamma_2, \boldsymbol{\beta}, x : A[\boldsymbol{\beta}/\boldsymbol{\alpha}]; R \vdash M : B[\boldsymbol{\beta}/\boldsymbol{\alpha}] \,|\, \epsilon' \rhd^{S'} e\end{array}}{\Gamma_1, x : D, \Gamma_2; R \vdash \mathsf{resume}\, M : C \,|\, \epsilon' \rhd^S \mathsf{resume}\, \boldsymbol{\beta}\, y.e} \text{ Elab\_Resume}$$

$$\frac{\Gamma; R \vdash M : A \,|\, \epsilon' \rhd^S e \quad \epsilon' \subseteq \epsilon}{\Gamma; R \vdash M : A \,|\, \epsilon \rhd^S e} \text{ Elab\_Weak}$$

Handler elaboration rules $\boxed{\Gamma; R \vdash H : A \,|\, \epsilon \Rightarrow B \,|\, \epsilon' \rhd^S h}$

$$\frac{\Gamma, x : A; R \vdash M : B \,|\, \epsilon' \rhd^{S \circ \{x \mapsto x\}} e \quad \epsilon \subseteq \epsilon'}{\Gamma; R \vdash \mathsf{return}\, x \to M : A \,|\, \epsilon \Rightarrow B \,|\, \epsilon' \rhd^S \mathsf{return}\, x \to e} \text{ ElabH\_Return}$$

$$\frac{\begin{array}{c}ty\,(\mathsf{op}) = \forall \boldsymbol{\alpha}.C \hookrightarrow D \quad \Gamma; R \vdash H : A \,|\, \epsilon \Rightarrow B \,|\, \epsilon' \rhd^S h \\ \Gamma, \boldsymbol{\alpha}, x : C; (\boldsymbol{\alpha}, x : C, D \to^{\epsilon'} B) \vdash M : B \,|\, \epsilon' \rhd^{S \circ \{x \mapsto x\}} e\end{array}}{\Gamma; R \vdash H; \mathsf{op}(x) \to M : A \,|\, \epsilon \uplus \{\mathsf{op}\} \Rightarrow B \,|\, \epsilon' \rhd^S h; \Lambda \boldsymbol{\alpha}.\mathsf{op}(x) \to e} \text{ ElabH\_Op}$$

Figure 5: Elaboration rules.

2. *If* $\Gamma_1, \alpha, \Gamma_2; r \vdash h : B \,|\, \epsilon \Rightarrow C \,|\, \epsilon'$, *then* $\Gamma_1, \Gamma_2\,[A/\alpha]; r[A/\alpha] \vdash h[A/\alpha] : B[A/\alpha] \,|\, \epsilon \Rightarrow C[A/\alpha] \,|\, \epsilon'$.

3. *If* $\Gamma_1, \alpha, \Gamma_2 \vdash E : B \multimap C \,|\, \epsilon$, *then* $\Gamma_1, \Gamma_2\,[A/\alpha] \vdash E[A/\alpha] : B[A/\alpha] \multimap C[A/\alpha] \,|\, \epsilon$.

Proof. By mutual induction on the typing derivations. We mention only the interesting cases.

Case (T_Var) and (T_Const): By Lemma 3.

Case (T_Resume): We are given $\Gamma_1, \alpha, \Gamma_2; (\boldsymbol{\beta}, A', B' \to^{\epsilon'} B) \vdash \mathsf{resume}\, \boldsymbol{\gamma}\, x.e' : B \,|\, \epsilon$ and, by inversion,

- $\boldsymbol{\beta} \in \Gamma_1, \alpha, \Gamma_2$,

- $\Gamma_1, \alpha, \Gamma_2, \boldsymbol{\gamma}, x : A' [\boldsymbol{\gamma}/\boldsymbol{\beta}]; (\boldsymbol{\beta}, A', B' \rightarrow^{\epsilon'} B) \vdash e' : B'[\boldsymbol{\gamma}/\boldsymbol{\beta}] \,|\, \epsilon'$, and
- $\epsilon' \subseteq \epsilon$.

Without loss of generality, we can suppose that $\alpha \notin \boldsymbol{\gamma}$. By the IH,

$$\Gamma_1, \Gamma_2 [A/\alpha], \boldsymbol{\gamma}, x : A' [\boldsymbol{\gamma}/\boldsymbol{\beta}]; (\boldsymbol{\beta}, A', B' \rightarrow^{\epsilon'} B[A/\alpha]) \vdash e'[A/\alpha] : B'[\boldsymbol{\gamma}/\boldsymbol{\beta}] \,|\, \epsilon'.$$

Note that (1) $A'[A/\alpha] = A'$, $B'[A/\alpha] = B'$, and $B'[\boldsymbol{\gamma}/\boldsymbol{\beta}][A/\alpha] = B'[\boldsymbol{\gamma}/\boldsymbol{\beta}]$ by the requirement to resume types and (2) $\mathit{ftv}(B[A/\alpha]) \cap \boldsymbol{\beta} = \emptyset$ since $\Gamma_1 \vdash A$ and $\boldsymbol{\beta} = \mathit{tyvars}((\boldsymbol{\beta}, A', B' \rightarrow^{\epsilon'} B)) \subseteq \mathit{dom}(\Gamma_2)$ and $\mathit{dom}(\Gamma_1) \cap \mathit{dom}(\Gamma_2) = \emptyset$ by Lemma 4. By (T_Resume), we finish.

□

Lemma 6 (Values can be given any effects)**.** *If $\Gamma; r \vdash v : A \,|\, \epsilon$, then $\Gamma; r \vdash v : A \,|\, \epsilon'$ for any ϵ'.*

Proof. Straightforward by induction on the derivation of $\Gamma; r \vdash v : A \,|\, \epsilon$. □

Lemma 7.

1. *If $\Gamma; \mathsf{none} \vdash e : A \,|\, \epsilon$, then $\Gamma; r \vdash e : A \,|\, \epsilon$ for any r.*
2. *If $\Gamma; \mathsf{none} \vdash h : A \,|\, \epsilon \Rightarrow B \,|\, \epsilon'$, then $\Gamma; r \vdash h : A \,|\, \epsilon \Rightarrow B \,|\, \epsilon'$ for any r.*

Proof. Straightforward by mutual induction on the typing derivations. □

Lemma 8. *If $\vdash \Gamma_1, x : \sigma, \Gamma_2$, then $\vdash \Gamma_1, \Gamma_2$.*

Proof. Straightforward by induction on Γ_2. □

Lemma 9 (Value substitution)**.** *Suppose that $\Gamma_1, \boldsymbol{\alpha}^I \vdash v : A$.*

1. *If $\Gamma_1, x : \forall \boldsymbol{\alpha}^I.A, \Gamma_2; r \vdash e : B \,|\, \epsilon$, then $\Gamma_1, \Gamma_2; r \vdash e[\Lambda \boldsymbol{\alpha}^I.v/x] : B \,|\, \epsilon$.*
2. *If $\Gamma_1, x : \forall \boldsymbol{\alpha}^I.A, \Gamma_2; r \vdash h : B \,|\, \epsilon \Rightarrow C \,|\, \epsilon'$, then $\Gamma_1, \Gamma_2; r \vdash h[\Lambda \boldsymbol{\alpha}^I.v/x] : B \,|\, \epsilon \Rightarrow C \,|\, \epsilon'$.*
3. *If $\Gamma_1, x : \forall \boldsymbol{\alpha}^I.A, \Gamma_2 \vdash E : B \multimap C \,|\, \epsilon$, then $\Gamma_1, \Gamma_2 \vdash E[\Lambda \boldsymbol{\alpha}^I.v/x] : B \multimap C \,|\, \epsilon$.*

Proof. By mutual induction on the typing derivations. We mention only the interesting cases.

Case (T_Var): We are given $\Gamma_1, x : \forall \boldsymbol{\alpha}^I.A, \Gamma_2; r \vdash y\, \boldsymbol{C}^J : D[\boldsymbol{C}^J/\boldsymbol{\beta}^J] \,|\, \epsilon$ and, by inversion,

- $\vdash \Gamma_1, x : \forall \boldsymbol{\alpha}^I.A, \Gamma_2$,
- $y : \forall \boldsymbol{\beta}^J.D \in \Gamma_1, x : \forall \boldsymbol{\alpha}^I.A, \Gamma_2$, and
- $\Gamma_1, x : \forall \boldsymbol{\alpha}^I.A, \Gamma_2 \vdash \boldsymbol{C}^J$.

By Lemma 8, $\vdash \Gamma_1, \Gamma_2$.

If $x \neq y$, then the conclusion is obvious by (T_Var). Otherwise, if $x = y$, then $\forall \boldsymbol{\alpha}^I.A = \forall \boldsymbol{\beta}^J.D$ and so we have to show that

$$\Gamma_1, \Gamma_2; r \vdash v[\boldsymbol{C}^I/\boldsymbol{\alpha}^I] : A[\boldsymbol{C}^I/\boldsymbol{\alpha}^I] \,|\, \epsilon.$$

Since $\Gamma_1, \boldsymbol{\alpha}^I \vdash v : A$ (i.e., $\Gamma_1, \boldsymbol{\alpha}^I; \mathsf{none} \vdash v : A \,|\, \epsilon'$ for some ϵ'), we have it by Lemmas 5, 2, 6, and 7.

Case (T_Const): By Lemma 8.

□

Lemma 10. *If $\Gamma \vdash E^{\boldsymbol{\alpha}^J} : \forall \boldsymbol{\alpha}^J.A \multimap B \,|\, \epsilon$ and $\Gamma, \boldsymbol{\alpha}^J; \mathsf{none} \vdash e : A \,|\, \langle\rangle$, then $\Gamma; \mathsf{none} \vdash E^{\boldsymbol{\alpha}^J}[e] : B \,|\, \epsilon$.*

Proof. By induction on the derivation of $\Gamma \vdash E^{\boldsymbol{\alpha}^J} : \forall \boldsymbol{\alpha}^J.A \multimap B \,|\, \epsilon$.

Case (TE_Hole): By (T_Weak).

Case (TE_App1) and (TE_App2): By the IH, (T_Weak), and (T_App).

Case (TE_Op): By the IH and (T_Op).

Case (TE_Handle): By the IH and (T_Handle).

Case (TE_Weak): By the IH and (T_Weak).

Case (TE_Let): By the IH and (T_Let).

□

Lemma 11 (Continuation substitution)**.** *Suppose that* $\Gamma \vdash E^{\boldsymbol{\beta}^J} : \forall \boldsymbol{\beta}^J.(B[\boldsymbol{C}^I/\boldsymbol{\alpha}^I]) \multimap D \,|\, \epsilon$ *and* $\Gamma, \boldsymbol{\beta}^J \vdash v : A[\boldsymbol{C}^I/\boldsymbol{\alpha}^I]$ *and* $\Gamma \vdash \forall \boldsymbol{\beta}^J.\boldsymbol{C}^I$.

1. *If* $\Gamma; (\boldsymbol{\alpha}^I, A, B \rightarrow \epsilon\ D) \vdash e : D' \,|\, \epsilon'$, *then* $\Gamma; \mathsf{none} \vdash e[E^{\boldsymbol{\beta}^J}/\mathsf{resume}]^{\forall \boldsymbol{\beta}^J.\boldsymbol{C}^I}_{\Lambda \boldsymbol{\beta}^J.v} : D' \,|\, \epsilon'$.

2. *If* $\Gamma; (\boldsymbol{\alpha}^I, A, B \rightarrow \epsilon\ D) \vdash h : D_1 \,|\, \epsilon_1 \Rightarrow D_2 \,|\, \epsilon_2$, *then* $\Gamma; \mathsf{none} \vdash h[E^{\boldsymbol{\beta}^J}/\mathsf{resume}]^{\forall \boldsymbol{\beta}^J.\boldsymbol{C}^I}_{\Lambda \boldsymbol{\beta}^J.v} : D_1 \,|\, \epsilon_1 \Rightarrow D_2 \,|\, \epsilon_2$.

Proof. By mutual induction on the typing derivations.

1. By case analysis on the typing rule applied last.

   Case (T_Var) and (T_Const): Obvious.

   Case (T_Abs), (T_App), (T_Op), (T_Weak), (T_Handle), and (T_Let): By the IH(s) with (if necessary) weakening (Lemmas 4 and 2).

   Case (T_OpCont): By the IH; note that

$$\#\mathsf{op}(\boldsymbol{\sigma}, w, E')[E^{\boldsymbol{\beta}^J}/\mathsf{resume}]^{\forall \boldsymbol{\beta}^J.\boldsymbol{C}^I}_{\Lambda \boldsymbol{\beta}^J.v} = \#\mathsf{op}(\boldsymbol{\sigma}, w[E^{\boldsymbol{\beta}^J}/\mathsf{resume}]^{\forall \boldsymbol{\beta}^J.\boldsymbol{C}^I}_{\Lambda \boldsymbol{\beta}^J.v}, E').$$

   Case (T_Resume): We are given $\Gamma; (\boldsymbol{\alpha}^I, A, B \rightarrow \epsilon\ D) \vdash \mathsf{resume}\,\boldsymbol{\gamma}^I\,x.e' : D \,|\, \epsilon'$ and, by inversion,

   - $\boldsymbol{\alpha}^I \in \Gamma$,
   - $\Gamma, \boldsymbol{\gamma}^I, x : A\,[\boldsymbol{\gamma}^I/\boldsymbol{\alpha}^I]; (\boldsymbol{\alpha}^I, A, B \rightarrow \epsilon\ D) \vdash e' : B[\boldsymbol{\gamma}^I/\boldsymbol{\alpha}^I] \,|\, \epsilon'$ and
   - $\epsilon \subseteq \epsilon'$.

   Without loss of generality, we can suppose that each type variable of $\boldsymbol{\gamma}^I$ is distinct from $\boldsymbol{\beta}^J$. Thus, by weakening (Lemmas 4 and 2) and the IH,

$$\Gamma, \boldsymbol{\gamma}^I, x : A\,[\boldsymbol{\gamma}^I/\boldsymbol{\alpha}^I]; \mathsf{none} \vdash e'[E^{\boldsymbol{\beta}^J}/\mathsf{resume}]^{\forall \boldsymbol{\beta}^J.\boldsymbol{C}^I}_{\Lambda \boldsymbol{\beta}^J.v} : B[\boldsymbol{\gamma}^I/\boldsymbol{\alpha}^I] \,|\, \epsilon'.$$

   By Lemma 2,

$$\Gamma, \boldsymbol{\beta}^J, \boldsymbol{\gamma}^I, x : A\,[\boldsymbol{\gamma}^I/\boldsymbol{\alpha}^I]; \mathsf{none} \vdash e'[E^{\boldsymbol{\beta}^J}/\mathsf{resume}]^{\forall \boldsymbol{\beta}^J.\boldsymbol{C}^I}_{\Lambda \boldsymbol{\beta}^J.v} : B[\boldsymbol{\gamma}^I/\boldsymbol{\alpha}^I] \,|\, \epsilon'.$$

   Since $\Gamma, \boldsymbol{\beta}^J \vdash \boldsymbol{C}^I$, we have

$$\Gamma, \boldsymbol{\beta}^J, x : A\,[\boldsymbol{C}^I/\boldsymbol{\alpha}^I]; \mathsf{none} \vdash e'[E^{\boldsymbol{\beta}^J}/\mathsf{resume}]^{\forall \boldsymbol{\beta}^J.\boldsymbol{C}^I}_{\Lambda \boldsymbol{\beta}^J.v}\,[\boldsymbol{C}^I/\boldsymbol{\gamma}^I] : B[\boldsymbol{C}^I/\boldsymbol{\alpha}^I] \,|\, \epsilon'$$

   by Lemma 5. Since $\Gamma, \boldsymbol{\beta}^J \vdash v : A[\boldsymbol{C}^I/\boldsymbol{\alpha}^I]$, we have

$$\Gamma, \boldsymbol{\beta}^J; \mathsf{none} \vdash e'[E^{\boldsymbol{\beta}^J}/\mathsf{resume}]^{\forall \boldsymbol{\beta}^J.\boldsymbol{C}^I}_{\Lambda \boldsymbol{\beta}^J.v}\,[\boldsymbol{C}^I/\boldsymbol{\gamma}^I][v/x] : B[\boldsymbol{C}^I/\boldsymbol{\alpha}^I] \,|\, \epsilon' \qquad (1)$$

   by Lemma 9.
   Since $\Gamma \vdash E^{\boldsymbol{\beta}^J} : \forall \boldsymbol{\beta}^J.(B[\boldsymbol{C}^I/\boldsymbol{\alpha}^I]) \multimap D \,|\, \epsilon$, we have

$$\Gamma, y : \forall \boldsymbol{\beta}^J.B\,[\boldsymbol{C}^I/\boldsymbol{\alpha}^I] \vdash E^{\boldsymbol{\beta}^J} : \forall \boldsymbol{\beta}^J.(B[\boldsymbol{C}^I/\boldsymbol{\alpha}^I]) \multimap D \,|\, \epsilon$$

   for some fresh variable y by Lemma 2. Since $\Gamma, y : \forall \boldsymbol{\beta}^J.B\,[\boldsymbol{C}^I/\boldsymbol{\alpha}^I], \boldsymbol{\beta}^J; \mathsf{none} \vdash y\,\boldsymbol{\beta}^J : B[\boldsymbol{C}^I/\boldsymbol{\alpha}^I] \,|\, \langle\rangle$ by (T_Var), we have

$$\Gamma, y : \forall \boldsymbol{\beta}^J.B\,[\boldsymbol{C}^I/\boldsymbol{\alpha}^I]; \mathsf{none} \vdash E^{\boldsymbol{\beta}^J}[y\,\boldsymbol{\beta}^J] : D \,|\, \epsilon' \qquad (2)$$

by Lemma 10 and (T_Weak).
By (1), (2), and (T_Let),

$$\Gamma; \mathsf{none} \vdash \mathsf{let}\, y = \Lambda \boldsymbol{\beta}^J . e'[E^{\boldsymbol{\beta}^J}/\mathsf{resume}]^{\forall \boldsymbol{\beta}^J . \boldsymbol{C}^I}_{\Lambda \boldsymbol{\beta}^J . v} [\boldsymbol{C}^I/\boldsymbol{\gamma}^I][v/x]\, \mathsf{in}\, E^{\boldsymbol{\beta}^J}[y\, \boldsymbol{\beta}^J] \;:\; D \mid \epsilon',$$

which is what we have to show by definition of substitution for resume.

2. By case analysis on the typing rule applied last.

   Case (TH_Return): By the IH.

   Case (TH_Op): By the IH; note that

$$\begin{aligned} & (h'; \Lambda \boldsymbol{\gamma}^{I'} . \mathsf{op}(x) \rightarrow e')[E^{\boldsymbol{\beta}^J}/\mathsf{resume}]^{\forall \boldsymbol{\beta}^J . \boldsymbol{C}^I}_{\Lambda \boldsymbol{\beta}^J . v} \\ = \;& h'[E^{\boldsymbol{\beta}^J}/\mathsf{resume}]^{\forall \boldsymbol{\beta}^J . \boldsymbol{C}^I}_{\Lambda \boldsymbol{\beta}^J . v}; \Lambda \boldsymbol{\gamma}^{I'} . \mathsf{op}(x) \rightarrow e'. \end{aligned}$$

□

Lemma 12 (Constant inversion). *If $\Gamma; r \vdash c \;:\; A \mid \epsilon$, then $ty(c) = A$.*

Proof. Straightforward by induction on the derivation of $\Gamma; r \vdash c \;:\; A \mid \epsilon$. □

Lemma 13 (Abstraction inversion). *If $\Gamma; r \vdash \lambda x.e \;:\; A \rightarrow^{\epsilon'} B \mid \epsilon$, then $\Gamma, x : A; r \vdash e \;:\; B \mid \epsilon'$.*

Proof. Straightforward by induction on the derivation of $\Gamma; r \vdash \lambda x.e \;:\; A \rightarrow^{\epsilon'} B \mid \epsilon$. □

Lemma 14 (Continuation inversion). *If $\Gamma; r \vdash \#\mathsf{op}(\boldsymbol{\sigma}^I, w, E) \;:\; D \mid \epsilon$, then*

- $\boldsymbol{\sigma}^I = \forall \boldsymbol{\beta}^J . \boldsymbol{C}^I$,
- $w = \Lambda \boldsymbol{\beta}^J . v$,
- *E captures $\boldsymbol{\beta}^J$ at the hole,*
- $\epsilon' \subseteq \epsilon$,
- $ty(\mathsf{op}) = \forall \boldsymbol{\alpha}^I . A \hookrightarrow B$,
- $\mathsf{op} \in \epsilon'$,
- $\Gamma \vdash \forall \boldsymbol{\beta}^J . \boldsymbol{C}^I$,
- $\Gamma, \boldsymbol{\beta}^J; r \vdash v \;:\; A[\boldsymbol{C}^I/\boldsymbol{\alpha}^I] \mid \epsilon'$, *and*
- $\Gamma \vdash E : \forall \boldsymbol{\beta}^J . (B[\boldsymbol{C}^I/\boldsymbol{\alpha}^I]) \multimap D \mid \epsilon'$

for some $\boldsymbol{\alpha}^I$, $\boldsymbol{\beta}^J$, $\boldsymbol{C}^I$, A, B, v, and ϵ'.

Proof. Straightforward by induction on the derivation of $\Gamma; r \vdash \#\mathsf{op}(\boldsymbol{\sigma}^I, w, E) \;:\; D \mid \epsilon$. □

Lemma 15 (Handler inversion). *Suppose that $\Gamma; r \vdash h : A \mid \epsilon \Rightarrow B \mid \epsilon'$.*

1. *If $h^{\mathsf{return}} = \mathsf{return}\, x \rightarrow e$, then $\Gamma, x : A; r \vdash e \;:\; B \mid \epsilon'$ for some x and e.*
2. *For any $\mathsf{op} \in ops(h)$,*
   - $h^{\mathsf{op}} = \Lambda \boldsymbol{\alpha}^I . \mathsf{op}(x) \rightarrow e$,
   - $ty(\mathsf{op}) = \forall \boldsymbol{\alpha}^I . C \hookrightarrow D$, *and*
   - $\Gamma, \boldsymbol{\alpha}^I, x : C; (\boldsymbol{\alpha}^I, C, D \rightarrow^{\epsilon'} B) \vdash e \;:\; B \mid \epsilon'$

   for some $\boldsymbol{\alpha}^I$, x, e, C, and D.

Proof. Straightforward by induction on the derivation of $\Gamma; r \vdash h : A \mid \epsilon \Rightarrow B \mid \epsilon'$. □

Lemma 16 (Canonical forms). *If* $\Gamma; r \vdash v : \iota \mid \epsilon$*, then* $v = c$*.*

Proof. Straightforward by induction on the derivation. □

Lemma 17. *If* $\Gamma; r \vdash h : A \mid \epsilon \Rightarrow B \mid \epsilon'$ *and* $\mathsf{op} \in \epsilon$ *and* $\mathsf{op} \notin ops(h)$*, then* $\mathsf{op} \in \epsilon'$

Proof. Straightforward by induction on the derivation of $\Gamma; r \vdash h : A \mid \epsilon \Rightarrow B \mid \epsilon'$. □

Lemma 18 (Progress). *If* $\Delta; \mathsf{none} \vdash e : A \mid \epsilon$*, then (1)* $e \longrightarrow e'$ *for some* e'*, (2)* e *is a value, or (3)* $e = \#\mathsf{op}(\boldsymbol{\sigma}, w, E)$ *for some* $\mathsf{op} \in \epsilon$*,* $\boldsymbol{\sigma}$*,* w*, and* E*.*

Proof. By induction on the derivation of $\Delta; \mathsf{none} \vdash e : A \mid \epsilon$.

Case (T_Var): Contradictory.

Case (T_Const): Obvious.

Case (T_Abs): Obvious.

Case (T_App): We are given

- $e = e_1\, e_2$,
- $\Delta; \mathsf{none} \vdash e_1\, e_2 : A \mid \epsilon$,
- $\Delta; \mathsf{none} \vdash e_1 : B \rightarrow\epsilon'\ A \mid \epsilon$,
- $\Delta; \mathsf{none} \vdash e_2 : B \mid \epsilon$, and
- $\epsilon' \subseteq \epsilon$.

By case analysis on the behavior of e_1. We have three cases to be considered by the IH.

Case $e_1 \longrightarrow e_1'$ for some e_1': We have $e \longrightarrow e_1'\, e_2$.

Case $e_1 = \#\mathsf{op}(\boldsymbol{\sigma}, w, E)$ for some $\mathsf{op} \in \epsilon$, $\boldsymbol{\sigma}$, w, and E: By (R_OpApp1) and (E_Eval).

Case $e_1 = v_1$ for some v_1: By case analysis on the behavior of e_2 with the IH.

Case $e_2 \longrightarrow e_2'$ for some e_2': We have $e \longrightarrow v_1\, e_2'$.

Case $e_2 = \#\mathsf{op}(\boldsymbol{\sigma}, w, E)$ for some $\mathsf{op} \in \epsilon$, $\boldsymbol{\sigma}$, w, and E: By (R_OpApp2) and (E_Eval).

Case $e_2 = v_2$ for some v_2: If $v_1 = c_1$ for some c_1, then $B = \iota$ and $ty(c_1) = \iota \rightarrow \langle\rangle\ A$ and $\epsilon' = \langle\rangle$ by Lemma 12. Since $\Delta; \mathsf{none} \vdash v_2 : \iota \mid \epsilon$, there exists some c_2 such that $v_2 = c_2$ and $ty(c_2) = \iota$. By the assumption about constants, $\zeta(c_1, c_2)$ is defined and $\zeta(c_1, c_2)$ is a constant and $ty(\zeta(c_1, c_2)) = A$. Thus, $e = c_1\, c_2 \longrightarrow \zeta(c_1, c_2)$ by (R_Const)/(E_Eval).
If $v_1 = \lambda x.e'$ for some x and e', then $e = (\lambda x.e')\, v_2 \longrightarrow e'[v_2/x]$ by (R_Beta)/(E_Eval). Note that substitution of v_2 for x in e' is defined since $\Delta, x\!:\!B; \mathsf{none} \vdash e' : A \mid \epsilon'$ by Lemma 13.

Case (T_Op): We are given

- $e = \#\mathsf{op}(\boldsymbol{C}^I, e')$,
- $A = B'[\boldsymbol{C}^I/\boldsymbol{\alpha}^I]$,
- $\Delta; \mathsf{none} \vdash \#\mathsf{op}(\boldsymbol{C}^I, e') : B'[\boldsymbol{C}^I/\boldsymbol{\alpha}^I] \mid \epsilon$
- $ty\,(\mathsf{op}) = \forall \boldsymbol{\alpha}^I.A' \hookrightarrow B'$,
- $\mathsf{op} \in \epsilon$,
- $\Delta; \mathsf{none} \vdash e' : A'[\boldsymbol{C}^I/\boldsymbol{\alpha}^I] \mid \epsilon$, and
- $\Delta \vdash \boldsymbol{C}^I$.

By case analysis on the behavior of e' with the IH.

Case $e' \longrightarrow e''$ for some e'': We have $e \longrightarrow \#\mathsf{op}(\boldsymbol{C}^I, e'')$.

Case $e' = \#\mathsf{op}'(\boldsymbol{\sigma}^{I'}, w, E)$ for some $\mathsf{op}' \in \epsilon$, $\boldsymbol{\sigma}^{I'}$, w, and E: By (R_OpOp) and (E_Eval).

Case $e' = v$ for some v: By (R_OP)/(E_EVAL).

Case (T_OPCONT): Obvious.

Case (T_WEAK): By the IH.

Case (T_HANDLE): We are given

- $e = \mathsf{handle}\ e'\ \mathsf{with}\ h$,
- $\Delta; \mathsf{none} \vdash \mathsf{handle}\ e'\ \mathsf{with}\ h : A \mid \epsilon$,
- $\Delta; \mathsf{none} \vdash e' : B \mid \epsilon'$, and
- $\Delta; \mathsf{none} \vdash h : B \mid \epsilon' \Rightarrow A \mid \epsilon$.

By case analysis on the behavior of e' with the IH.

Case $e' \longrightarrow e''$ for some e'': We have $e \longrightarrow \mathsf{handle}\ e''\ \mathsf{with}\ h$.

Case $e' = \#\mathsf{op}(\boldsymbol{\sigma}, w, E)$ for some $\mathsf{op} \in \epsilon'$, $\boldsymbol{\sigma}$, w, and E: If $\mathsf{op} \in \mathit{ops}(h)$, then we finish by Lemma 14 and (R_HANDLE)/(E_EV
Otherwise, if $\mathsf{op} \notin \mathit{ops}(h)$, we have $e = \mathsf{handle}\ \#\mathsf{op}(\boldsymbol{\sigma}, w, E)\ \mathsf{with}\ h \longrightarrow \#\mathsf{op}(\boldsymbol{\sigma}, w, \mathsf{handle}\ E\ \mathsf{with}\ h)$ by (R_OPHANDLE)/(E_EV
Note that $\mathsf{op} \in \epsilon$ by Lemma 17.

Case $e' = v$ for some v: By (R_RETURN)/(E_EVAL).

Case (T_RESUME): Contradictory.

Case (T_LET): We are given

- $e = \mathsf{let}\ x = \Lambda \boldsymbol{\alpha}^I.e_1\ \mathsf{in}\ e_2$,
- $\Delta, \boldsymbol{\alpha}^I; \mathsf{none} \vdash e_1 : B \mid \epsilon$, and
- $\Delta, x : \forall \boldsymbol{\alpha}^I.B; \mathsf{none} \vdash e_2 : A \mid \epsilon$.

By case analysis on the behavior of e_1.

Case $e_1 \longrightarrow e_1'$ for some e_1':

Case $e_1 = \#\mathsf{op}(\boldsymbol{\sigma}^J, w, E)$ for some $\mathsf{op} \in \epsilon$, $\boldsymbol{\sigma}^J$, w, and E: By (R_OPLET) and (E_EVAL).

Case $e_1 = v_1$ for some v_1: By (R_LET)/(E_EVAL). Note that substitution of $\Lambda \boldsymbol{\alpha}^I.v$ for x in e_2 is defined since $\Delta, x : \forall \boldsymbol{\alpha}^I.B; \mathsf{none} \vdash e_2 : A \mid \epsilon$.

□

Lemma 19.

1. *If* $\Gamma; \mathsf{none} \vdash e : A \mid \epsilon$*, then* $\Gamma \vdash A$*.*
2. *If* $\Gamma; \mathsf{none} \vdash h : A \mid \epsilon \Rightarrow B \mid \epsilon'$*, then* $\Gamma \vdash B$*.*
3. *If* $\Gamma \vdash E : \sigma \multimap A \mid \epsilon$ *and* $\Gamma \vdash \sigma$*, then* $\Gamma \vdash A$*.*

Proof. Straightforward by mutual induction on the typing derivations with Lemma 4. □

Lemma 20 (Subject reduction)**.**

1. *If* $\Delta; \mathsf{none} \vdash e_1 : A \mid \epsilon$ *and* $e_1 \rightsquigarrow e_2$*, then* $\Delta; \mathsf{none} \vdash e_2 : A \mid \epsilon$*.*
2. *If* $\Delta; \mathsf{none} \vdash e_1 : A \mid \epsilon$ *and* $e_1 \longrightarrow e_2$*, then* $\Delta; \mathsf{none} \vdash e_2 : A \mid \epsilon$*.*

Proof. 1. Suppose that $\Delta; \mathsf{none} \vdash e_1 : A \mid \epsilon$ and $e_1 \rightsquigarrow e_2$. By induction on $\Delta; \mathsf{none} \vdash e_1 : A \mid \epsilon$.

Case (T_VAR): Contradictory.

Case (T_CONST): Contradictory; no reduction rules can be applied to constants.

Case (T_ABS): Contradictory; no reduction rules can be applied to lambda abstractions.

Case (T_App): We have four reduction rules which can be applied to function applications.

Case (R_Const): We are given

- $e_1 = c_1\, c_2$,
- $e_2 = \zeta(c_1, c_2)$,
- $\Delta; \mathsf{none} \vdash c_1\, c_2 : A \,|\, \epsilon$,
- $\Delta; \mathsf{none} \vdash c_1 : B \rightarrow^{\epsilon'} A \,|\, \epsilon$,
- $\Delta; \mathsf{none} \vdash c_2 : B \,|\, \epsilon$, and
- $\epsilon' \subseteq \epsilon$.

By Lemma 12 and the assumption about constants, we have $B = ty(c_2) = \iota$ and $ty(c_1) = \iota \rightarrow^{\langle\rangle} A$ and $\epsilon' = \langle\rangle$ for some ι. By the assumption about ζ, $\zeta(c_1, c_2)$ is a constant and $ty(\zeta(c_1, c_2)) = A$. Thus, by (T_Const), $\Delta; \mathsf{none} \vdash \zeta(c_1, c_2) : A \,|\, \epsilon$; note that $\vdash \Delta$ by Lemma 4.

Case (R_Beta): We are given

- $e_1 = (\lambda x.e)\, v$,
- $e_2 = e[v/x]$,
- $\Delta; \mathsf{none} \vdash (\lambda x.e)\, v : A \,|\, \epsilon$,
- $\Delta; \mathsf{none} \vdash \lambda x.e : B \rightarrow^{\epsilon'} A \,|\, \epsilon$,
- $\Delta; \mathsf{none} \vdash v : B \,|\, \epsilon$, and
- $\epsilon' \subseteq \epsilon$.

By Lemma 13, $\Delta, x : B; \mathsf{none} \vdash e : A \,|\, \epsilon'$. By (T_Weak), $\Delta, x : B; \mathsf{none} \vdash e : A \,|\, \epsilon$. By Lemma 9 (1), $\Delta; \mathsf{none} \vdash e[v/x] : A \,|\, \epsilon$.

Case (R_OpApp1): By Lemma 14, we are given

- $e_1 = \#\mathsf{op}(\forall \boldsymbol{\beta}^J. \boldsymbol{C}^I, \Lambda\boldsymbol{\beta}^J.v, E^{\boldsymbol{\beta}^J})\, e_2'$,
- $e_2 = \#\mathsf{op}(\forall \boldsymbol{\beta}^J. \boldsymbol{C}^I, \Lambda\boldsymbol{\beta}^J.v, E^{\boldsymbol{\beta}}\, e_2')$,
- $\Delta; \mathsf{none} \vdash \#\mathsf{op}(\forall \boldsymbol{\beta}^J. \boldsymbol{C}^I, \Lambda\boldsymbol{\beta}^J.v, E^{\boldsymbol{\beta}^J})\, e_2' : A \,|\, \epsilon$,
- $\Delta; \mathsf{none} \vdash \#\mathsf{op}(\forall \boldsymbol{\beta}^J. \boldsymbol{C}^I, \Lambda\boldsymbol{\beta}^J.v, E^{\boldsymbol{\beta}^J}) : B \rightarrow^{\epsilon'} A \,|\, \epsilon$,
- $\Delta; \mathsf{none} \vdash e_2' : B \,|\, \epsilon$,
- $\epsilon' \subseteq \epsilon$,
- $ty\,(\mathsf{op}) = \forall \boldsymbol{\alpha}^I.A' \hookrightarrow B'$
- $\epsilon'' \subseteq \epsilon$,
- $\mathsf{op} \in \epsilon''$,
- $\Delta \vdash \forall \boldsymbol{\beta}^J. \boldsymbol{C}^I$,
- $\Delta, \boldsymbol{\beta}^J; \mathsf{none} \vdash v : A'[\boldsymbol{C}^I/\boldsymbol{\alpha}^I] \,|\, \epsilon''$, and
- $\Delta \vdash E^{\boldsymbol{\beta}^J} : \forall \boldsymbol{\beta}^J.(B'[\boldsymbol{C}^I/\boldsymbol{\alpha}^I]) \multimap B \rightarrow^{\epsilon'} A \,|\, \epsilon''$

for some e_2', ϵ', $\boldsymbol{\alpha}^I$, $\boldsymbol{\beta}^J$, B, $\boldsymbol{C}^I$, A', B', and ϵ''.
Since $\epsilon' \subseteq \epsilon$ and $\epsilon'' \subseteq \epsilon$, we have

$$\Delta \vdash E^{\boldsymbol{\beta}^J}\, e_2' : \forall \boldsymbol{\beta}^J.(B'[\boldsymbol{C}^I/\boldsymbol{\alpha}^I]) \multimap A \,|\, \epsilon$$

by (TE_Weak) and (TE_App1). By (T_Weak),

$$\Delta, \boldsymbol{\beta}^J; \mathsf{none} \vdash v : A'[\boldsymbol{C}^I/\boldsymbol{\alpha}^I] \,|\, \epsilon.$$

Thus, by (T_OpCont), we have the conclusion.

Case (R_OpApp2): Similar to the case of (R_OpApp1).

Case (T_Op): We have two reduction rules which can be applied to effect invocation.

Case (R_Op): We are given

- $e_1 = \#\mathsf{op}(\boldsymbol{C}^I, v)$,
- $e_2 = \#\mathsf{op}(\boldsymbol{C}^I, v, [\,])$,

- $A \ = \ B'[\boldsymbol{C}^I / \boldsymbol{\alpha}^I]$,
- $\Delta; \mathsf{none} \vdash \mathsf{\#op}(\boldsymbol{C}^I, v) \ : B'[\boldsymbol{C}^I / \boldsymbol{\alpha}^I] \,|\, \epsilon$,
- $ty\,(\mathsf{op}) \ = \ \forall \boldsymbol{\alpha}^I . A' \hookrightarrow B'$,
- $\mathsf{op} \ \in \ \epsilon$,
- $\Delta; \mathsf{none} \vdash v \ : A'[\boldsymbol{C}^I / \boldsymbol{\alpha}^I] \,|\, \epsilon$, and
- $\Delta \vdash \boldsymbol{C}^I$.

By (TE_Hole) and (T_OpCont), we have the conclusion.

Case (R_OpOp): By Lemma 14, we are given

- $e_1 \ = \ \mathsf{\#op}'(\boldsymbol{C'}^{I'}, \mathsf{\#op}(\forall \boldsymbol{\beta}^J . \boldsymbol{C}^I, \Lambda \boldsymbol{\beta}^J . v, E^{\boldsymbol{\beta}^J}))$,
- $e_2 \ = \ \mathsf{\#op}(\forall \boldsymbol{\beta}^J . \boldsymbol{C}^I, \Lambda \boldsymbol{\beta}^J . v, \mathsf{\#op}'(\boldsymbol{C'}^{I'}, E^{\boldsymbol{\beta}^J}))$,
- $A \ = \ B'[\boldsymbol{C'}^{I'} / \boldsymbol{\gamma}^{I'}]$,
- $\Delta; \mathsf{none} \vdash \mathsf{\#op}'(\boldsymbol{C'}^{I'}, \mathsf{\#op}(\forall \boldsymbol{\beta}^J . \boldsymbol{C}^I, \Lambda \boldsymbol{\beta}^J . v, E^{\boldsymbol{\beta}^J})) \ : B'[\boldsymbol{C'}^{I'} / \boldsymbol{\gamma}^{I'}] \,|\, \epsilon$
- $ty\,(\mathsf{op}') \ = \ \forall \boldsymbol{\gamma}^{I'} . A' \hookrightarrow B'$,
- $\mathsf{op}' \ \in \ \epsilon$,
- $\Delta \vdash \boldsymbol{C'}^{I'}$,
- $\Delta; \mathsf{none} \vdash \mathsf{\#op}(\forall \boldsymbol{\beta}^J . \boldsymbol{C}^I, \Lambda \boldsymbol{\beta}^J . v, E^{\boldsymbol{\beta}^J}) \ : A'[\boldsymbol{C'}^{I'} / \boldsymbol{\gamma}^{I'}] \,|\, \epsilon$,
- $ty\,(\mathsf{op}) \ = \ \forall \boldsymbol{\alpha}^I . A'' \hookrightarrow B''$,
- $\mathsf{op} \ \in \ \epsilon'$,
- $\epsilon' \ \subseteq \ \epsilon$,
- $\Delta \vdash \forall \boldsymbol{\beta}^J . \boldsymbol{C}^I$,
- $\Delta, \boldsymbol{\beta}^J; \mathsf{none} \vdash v \ : A''[\boldsymbol{C}^I / \boldsymbol{\alpha}^I] \,|\, \epsilon'$, and
- $\Delta \vdash E^{\boldsymbol{\beta}^J} : \forall \boldsymbol{\beta}^J . B''[\boldsymbol{C}^I / \boldsymbol{\alpha}^I] \multimap A'[\boldsymbol{C'}^{I'} / \boldsymbol{\gamma}^{I'}] \,|\, \epsilon'$.

By (TE_Weak) and (TE_Op),

$$\Delta \vdash \mathsf{\#op}(\boldsymbol{C'}^{I'}, E^{\boldsymbol{\beta}^J}) : \forall \boldsymbol{\beta}^J . B''[\boldsymbol{C}^I / \boldsymbol{\alpha}^I] \multimap B'[\boldsymbol{C'}^{I'} / \boldsymbol{\gamma}^{I'}] \,|\, \epsilon.$$

By (T_Weak),

$$\Delta, \boldsymbol{\beta}^J; \mathsf{none} \vdash v \ : A''[\boldsymbol{C}^I / \boldsymbol{\alpha}^I] \,|\, \epsilon.$$

By (T_OpCont), we have the conclusion.

Case (T_OpCont): Contradictory.

Case (T_Weak): By the IH and (T_Weak).

Case (T_Handle): We have three reduction rules which can be applied to handler expressions.

Case (R_Return): We are given

- $e_1 \ = \ \mathsf{handle}\, v \,\mathsf{with}\, h$,
- $h^{\mathsf{return}} \ = \ \mathsf{return}\, x \rightarrow e$,
- $e_2 \ = \ e[v/x]$,
- $\Delta; \mathsf{none} \vdash \mathsf{handle}\, v \,\mathsf{with}\, h \ : A \,|\, \epsilon$,
- $\Delta; \mathsf{none} \vdash v \ : B \,|\, \epsilon'$, and
- $\Delta; \mathsf{none} \vdash h : B \,|\, \epsilon' \Rightarrow A \,|\, \epsilon$.

By Lemma 15, $\Gamma, x : B; \mathsf{none} \vdash e \ : A \,|\, \epsilon$. By Lemma 9, we finish.

Case (R_Handle): By Lemmas 15 and 14, we are given

- $e_1 \ = \ \mathsf{handle}\, \mathsf{\#op}(\forall \boldsymbol{\beta}^J . \boldsymbol{C}^I, \Lambda \boldsymbol{\beta}^J . v, E^{\boldsymbol{\beta}^J}) \,\mathsf{with}\, h$,
- $h^{\mathsf{op}} \ = \ \Lambda \boldsymbol{\alpha}^I . \mathsf{op}(x) \rightarrow e$,
- $e_2 \ = \ e[\mathsf{handle}\, E^{\boldsymbol{\beta}^J} \,\mathsf{with}\, h / \mathsf{resume}]^{\forall \boldsymbol{\beta}^J . \boldsymbol{C}^I}_{\Lambda \boldsymbol{\beta}^J . v}[\boldsymbol{C}[\bot^J / \boldsymbol{\beta}^J]^I / \boldsymbol{\alpha}^I][v[\bot^J / \boldsymbol{\beta}^J] / x]$,
- $\Delta; \mathsf{none} \vdash \mathsf{handle}\, \mathsf{\#op}(\forall \boldsymbol{\beta}^J . \boldsymbol{C}^I, \Lambda \boldsymbol{\beta}^J . v, E^{\boldsymbol{\beta}^J}) \,\mathsf{with}\, h \ : A \,|\, \epsilon$,

- $\Delta; \mathsf{none} \vdash h : B \mid \epsilon' \Rightarrow A \mid \epsilon$,
- $\Delta; \mathsf{none} \vdash \#\mathsf{op}(\forall \boldsymbol{\beta}^J . \boldsymbol{C}^I, \Lambda \boldsymbol{\beta}^J . v, E^{\boldsymbol{\beta}^J}) : B \mid \epsilon'$,
- $ty(\mathsf{op}) = \forall \boldsymbol{\alpha}^I . A' \hookrightarrow B'$,
- $\Delta, \boldsymbol{\alpha}^I, x : A'; (\boldsymbol{\alpha}^I, A', B' \to \epsilon\ A) \vdash e : A \mid \epsilon$,
- $\mathsf{op} \in \epsilon''$,
- $\epsilon'' \subseteq \epsilon'$,
- $\Delta \vdash \forall \boldsymbol{\beta}^J . \boldsymbol{C}^I$,
- $\Delta, \boldsymbol{\beta}^J; \mathsf{none} \vdash v : A'[\boldsymbol{C}^I / \boldsymbol{\alpha}^I] \mid \epsilon''$, and
- $\Delta \vdash E^{\boldsymbol{\beta}^J} : \forall \boldsymbol{\beta}^J . B'[\boldsymbol{C}^I / \boldsymbol{\alpha}^I] \multimap B \mid \epsilon''$.

Since $\Delta \vdash E^{\boldsymbol{\beta}^J} : \forall \boldsymbol{\beta}^J . B'[\boldsymbol{C}^I / \boldsymbol{\alpha}^I] \multimap B \mid \epsilon''$ and $\Delta; \mathsf{none} \vdash h : B \mid \epsilon' \Rightarrow A \mid \epsilon$, we have

$$\Delta, \boldsymbol{\alpha}^I, x : A' \vdash \mathsf{handle}\ E^{\boldsymbol{\beta}^J}\ \mathsf{with}\ h : \forall \boldsymbol{\beta}^J . B'[\boldsymbol{C}^I / \boldsymbol{\alpha}^I] \multimap A \mid \epsilon \tag{3}$$

by (TE_Weak), (TE_Handle), and weakening (Lemmas 4 and 2). Since $\Delta, \boldsymbol{\beta}^J; \mathsf{none} \vdash v : A'[\boldsymbol{C}^I / \boldsymbol{\alpha}^I] \mid \epsilon''$, we have

$$\Delta, \boldsymbol{\alpha}^I, x : A', \boldsymbol{\beta}^J \vdash v : A'[\boldsymbol{C}^I / \boldsymbol{\alpha}^I] \tag{4}$$

by weakening. Since $\Delta \vdash \forall \boldsymbol{\beta}^J . \boldsymbol{C}^I$, we have

$$\Delta, \boldsymbol{\alpha}^I, x : A' \vdash \forall \boldsymbol{\beta}^J . \boldsymbol{C}^I . \tag{5}$$

Since $\Delta, \boldsymbol{\alpha}^I, x : A'; (\boldsymbol{\alpha}^I, A', B' \to \epsilon\ A) \vdash e : A \mid \epsilon$, we have

$$\Delta, \boldsymbol{\alpha}^I, x : A'; \mathsf{none} \vdash e[\mathsf{handle}\ E^{\boldsymbol{\beta}^J}\ \mathsf{with}\ h / \mathsf{resume}]^{\forall \boldsymbol{\beta}^J . \boldsymbol{C}^I}_{\Lambda \boldsymbol{\beta}^J . v} : A \mid \epsilon$$

by Lemma 11 with (3), (4), and (5). Since $\Delta \vdash \forall \boldsymbol{\beta}^J . \boldsymbol{C}^I$, we have $\Delta \vdash \boldsymbol{C}[\bot^J / \boldsymbol{\beta}^J]^I$. Thus,

$$\begin{array}{l} \Delta, x : A'\,[\boldsymbol{C}[\bot^J / \boldsymbol{\beta}^J]^I / \boldsymbol{\alpha}^I]; \mathsf{none} \\ \quad \vdash e[\mathsf{handle}\ E^{\boldsymbol{\beta}^J}\ \mathsf{with}\ h / \mathsf{resume}]^{\forall \boldsymbol{\beta}^J . \boldsymbol{C}^I}_{\Lambda \boldsymbol{\beta}^J . v}\,[\boldsymbol{C}[\bot^J / \boldsymbol{\beta}^J]^I / \boldsymbol{\alpha}^I] : A \mid \epsilon \end{array}$$

by Lemma 5, where note that $A[\boldsymbol{C}[\bot^J / \boldsymbol{\beta}^J]^I / \boldsymbol{\alpha}^I] = A$ because $\Delta \vdash A$ by Lemma 19. Since $\Delta, \boldsymbol{\beta}^J; \mathsf{none} \vdash v : A'[\boldsymbol{C}^I / \boldsymbol{\alpha}^I] \mid \epsilon''$, we have

$$\Delta \vdash v[\bot^J / \boldsymbol{\beta}^J] : A'[\boldsymbol{C}[\bot^J / \boldsymbol{\beta}^J]^I / \boldsymbol{\alpha}^I]$$

by Lemma 5, where note that $\boldsymbol{\beta}^J$ do not occur free in A' since A' is the argument type of op. By Lemma 9,

$$\begin{array}{l} \Delta; \mathsf{none} \\ \quad \vdash e[\mathsf{handle}\ E^{\boldsymbol{\beta}^J}\ \mathsf{with}\ h / \mathsf{resume}]^{\forall \boldsymbol{\beta}^J . \boldsymbol{C}^I}_{\Lambda \boldsymbol{\beta}^J . v}\,[\boldsymbol{C}[\bot^J / \boldsymbol{\beta}^J]^I / \boldsymbol{\alpha}^I]\, v[\bot^J / \boldsymbol{\beta}^J] : A \mid \epsilon, \end{array}$$

which is what we have to show.

Case (R_OpHandle): By Lemma 14, we are given

- $e_1 = \mathsf{handle}\ \#\mathsf{op}(\forall \boldsymbol{\beta}^J . \boldsymbol{C}^I, \Lambda \boldsymbol{\beta}^J . v, E^{\boldsymbol{\beta}^J})\ \mathsf{with}\ h$,
- $e_2 = \#\mathsf{op}(\forall \boldsymbol{\beta}^J . \boldsymbol{C}^I, \Lambda \boldsymbol{\beta}^J . v, \mathsf{handle}\ E^{\boldsymbol{\beta}^J}\ \mathsf{with}\ h)$,
- $\mathsf{op} \notin ops(h)$,
- $\Delta; \mathsf{none} \vdash \mathsf{handle}\ \#\mathsf{op}(\forall \boldsymbol{\beta}^J . \boldsymbol{C}^I, \Lambda \boldsymbol{\beta}^J . v, E^{\boldsymbol{\beta}^J})\ \mathsf{with}\ h : A \mid \epsilon$,
- $\Delta; \mathsf{none} \vdash \#\mathsf{op}(\forall \boldsymbol{\beta}^J . \boldsymbol{C}^I, \Lambda \boldsymbol{\beta}^J . v, E^{\boldsymbol{\beta}^J}) : B \mid \epsilon'$,
- $\Delta; \mathsf{none} \vdash h : B \mid \epsilon' \Rightarrow A \mid \epsilon$,
- $\epsilon'' \subseteq \epsilon'$,
- $ty(\mathsf{op}) = \forall \boldsymbol{\alpha}^I . A' \hookrightarrow B'$,
- $\mathsf{op} \in \epsilon''$,
- $\Delta \vdash \forall \boldsymbol{\beta}^J . \boldsymbol{C}^I$,

- $\Delta, \boldsymbol{\beta}^J; \mathsf{none} \vdash v : A'[\boldsymbol{C}^I/\boldsymbol{\alpha}^I] \mid \epsilon''$, and
- $\Delta \vdash E^{\boldsymbol{\beta}^J} : \forall \boldsymbol{\beta}^J.(B'[\boldsymbol{C}^I/\boldsymbol{\alpha}^I]) \multimap B \mid \epsilon''$.

By (TE_Weak) and (TE_Handle),

$$\Delta \vdash \mathsf{handle}\, E^{\boldsymbol{\beta}^J}\, \mathsf{with}\, h : \forall \boldsymbol{\beta}^J.(B'[\boldsymbol{C}^I/\boldsymbol{\alpha}^I]) \multimap A \mid \epsilon.$$

Since $\Delta, \boldsymbol{\beta}^J; \mathsf{none} \vdash v : A'[\boldsymbol{C}^I/\boldsymbol{\alpha}^I] \mid \epsilon''$, we have

$$\Delta, \boldsymbol{\beta}^J; \mathsf{none} \vdash v : A'[\boldsymbol{C}^I/\boldsymbol{\alpha}^I] \mid \epsilon$$

by Lemma 6. Since $\mathsf{op} \in \epsilon'' \subseteq \epsilon'$ and $\Delta; \mathsf{none} \vdash h : B \mid \epsilon' \Rightarrow A \mid \epsilon$ and $\mathsf{op} \notin ops(h)$, we have $\mathsf{op} \in \epsilon$ by Lemma 17. Thus, we finish by (T_OpCont).

Case (T_Resume): Contradictory.

Case (T_Let): We have two reduction rules which can be applied to let expressions.

Case (R_Let): We are given

- $e_1 = \mathsf{let}\, x = \Lambda \boldsymbol{\alpha}.v\, \mathsf{in}\, e$,
- $e_2 = e[\Lambda \boldsymbol{\alpha}.v/x]$,
- $\Delta; \mathsf{none} \vdash \mathsf{let}\, x = \Lambda \boldsymbol{\alpha}.v\, \mathsf{in}\, e : A \mid \epsilon$,
- $\Delta, \boldsymbol{\alpha}; \mathsf{none} \vdash v : B \mid \epsilon$, and
- $\Delta, x : \forall \boldsymbol{\alpha}.B; \mathsf{none} \vdash e : A \mid \epsilon$.

We have the conclusion by Lemma 9.

Case (R_OpLet): By Lemma 14, we are given

- $e_1 = \mathsf{let}\, x = \Lambda \boldsymbol{\alpha}^I.\#\mathsf{op}(\forall \boldsymbol{\beta}^J.\boldsymbol{C}^{I'}, \Lambda \boldsymbol{\beta}^J.v, E^{\boldsymbol{\beta}^J})\, \mathsf{in}\, e$,
- $e_2 = \#\mathsf{op}(\forall \boldsymbol{\alpha}^I.\forall \boldsymbol{\beta}^J.\boldsymbol{C}^{I'}, \Lambda \boldsymbol{\alpha}^I.\Lambda \boldsymbol{\beta}^J.v, \mathsf{let}\, x = \Lambda \boldsymbol{\alpha}^I.E^{\boldsymbol{\beta}^J}\, \mathsf{in}\, e)$,
- $\Delta; \mathsf{none} \vdash \mathsf{let}\, x = \Lambda \boldsymbol{\alpha}^I.\#\mathsf{op}(\forall \boldsymbol{\beta}^J.\boldsymbol{C}^{I'}, \Lambda \boldsymbol{\beta}^J.v, E^{\boldsymbol{\beta}^J})\, \mathsf{in}\, e : A \mid \epsilon$,
- $\Delta, \boldsymbol{\alpha}^I; \mathsf{none} \vdash \#\mathsf{op}(\forall \boldsymbol{\beta}^J.\boldsymbol{C}^{I'}, \Lambda \boldsymbol{\beta}^J.v, E^{\boldsymbol{\beta}^J}) : B \mid \epsilon$,
- $\Delta, x : \forall \boldsymbol{\alpha}^I.B; \mathsf{none} \vdash e : A \mid \epsilon$,
- $\epsilon' \subseteq \epsilon$,
- $ty(\mathsf{op}) = \forall \boldsymbol{\gamma}^{I'}.A' \hookrightarrow B'$,
- $\mathsf{op} \in \epsilon'$,
- $\Delta, \boldsymbol{\alpha}^I \vdash \forall \boldsymbol{\beta}^J.\boldsymbol{C}^{I'}$,
- $\Delta, \boldsymbol{\alpha}^I, \boldsymbol{\beta}^J; \mathsf{none} \vdash v : A'[\boldsymbol{C}^{I'}/\boldsymbol{\gamma}^{I'}] \mid \epsilon'$, and
- $\Delta, \boldsymbol{\alpha}^I \vdash E^{\boldsymbol{\beta}^J} : \forall \boldsymbol{\beta}^J.(B'[\boldsymbol{C}^{I'}/\boldsymbol{\gamma}^{I'}]) \multimap B \mid \epsilon'$.

By (TE_Weak) and (TE_Let),

$$\Delta \vdash \mathsf{let}\, x = \Lambda \boldsymbol{\alpha}^I.E^{\boldsymbol{\beta}^J}\, \mathsf{in}\, e : \forall \boldsymbol{\alpha}^I.\forall \boldsymbol{\beta}^J.(B'[\boldsymbol{C}^{I'}/\boldsymbol{\gamma}^{I'}]) \multimap A \mid \epsilon.$$

Since $\Delta, \boldsymbol{\alpha}^I \vdash \forall \boldsymbol{\beta}^J.\boldsymbol{C}^{I'}$,

$$\Delta \vdash \forall \boldsymbol{\alpha}^I.\forall \boldsymbol{\beta}^J.\boldsymbol{C}^{I'}.$$

Thus, by (T_OpCont), we have the conclusion.

2. Suppose that $\Delta; \mathsf{none} \vdash e_1 : A \mid \epsilon$ and $e_1 \longrightarrow e_2$. By definition, there exists some E, e_1', and e_2' such that $e_1 = E[e_1']$, $e_2 = E[e_2']$, and $e_1' \rightsquigarrow e_2'$. By induction on the derivation of $\Delta; \mathsf{none} \vdash E[e_1'] : A \mid \epsilon$. If $E = []$, then we have the conclusion by the first case. In what follows, we suppose that $E \neq []$. By case analysis on the typing rule applied last to derive $\Delta; \mathsf{none} \vdash E[e_1'] : A \mid \epsilon$.

Case (T_Var), (T_Const), (T_Abs), (T_OpCont), and (T_Resume): It is contradictory because $E = []$.

Case (T_App): By case analysis on E.

Case $E = E'\, e$: We are given

- $\Delta; \mathsf{none} \vdash E'[e'_1] : B \rightarrow \epsilon' A \,|\, \epsilon$,
- $\Delta; \mathsf{none} \vdash e : B \,|\, \epsilon$, and
- $\epsilon' \subseteq \epsilon$

for some B and ϵ'. By the IH, $\Delta; \mathsf{none} \vdash E'[e'_2] : B \rightarrow \epsilon' A \,|\, \epsilon$. By (T_App), we finish.

Case $E = v\, E'$: By the IH.

Case (T_Op): By the IH.

Case (T_Weak): By the IH.

Case (T_Handle): By the IH.

Case (T_Let): By the IH.

□

Theorem 1 (Type Soundness of $\lambda^{\Lambda}_{\mathrm{eff}}$). *If $\Delta; \mathsf{none} \vdash e : A \,|\, \epsilon$ and $e \longrightarrow^* e'$ and $e' \not\longrightarrow$, then (1) e' is a value or (2) $e' = \#\mathsf{op}(\boldsymbol{\sigma}, w, E)$ for some $\mathsf{op} \in \epsilon$, $\boldsymbol{\sigma}$, w, and E.*

Proof. By Lemma 20, $\Delta; \mathsf{none} \vdash e' : A \,|\, \epsilon$. We have the conclusion by Lemma 18. □

2.2 Elaboration is type-preserving

Definition 14. *Elaboration $\Gamma \rhd^S \Gamma'$ of Γ to Γ' with S is the least relation that satisfies the following rules.*

$$\frac{}{\emptyset \rhd^S \emptyset}\ \textsc{ElabG\_Empty} \qquad \frac{\Gamma \rhd^S \Gamma'}{\Gamma, x : \sigma \rhd^S \Gamma', S(x) : \sigma}\ \textsc{ElabG\_Var}$$

$$\frac{\Gamma \rhd^S \Gamma'}{\Gamma, \alpha \rhd^S \Gamma', \alpha}\ \textsc{ElabG\_TyVar}$$

Definition 15. *Elaboration $R \rhd r$ of R to r is defined as follows.*

$$\mathsf{none} \rhd \mathsf{none} \qquad (\boldsymbol{\alpha}, x : A, B \rightarrow \epsilon\ C) \rhd (\boldsymbol{\alpha}, A, B \rightarrow \epsilon\ C)$$

Lemma 21. *If $\Gamma \rhd^S \Gamma'$, then, for any $x : \sigma \in \Gamma$, $S(x) : \sigma \in \Gamma'$.*

Proof. By induction on the derivation of $\Gamma \rhd^S \Gamma'$. □

Lemma 22. *If $\Gamma \rhd^S \Gamma'$, then, for any α, $\alpha \in \Gamma$ if and only if $\alpha \in \Gamma'$.*

Proof. By induction on the derivation of $\Gamma \rhd^S \Gamma'$. □

Lemma 23.

1. *If $\Gamma; R \vdash M : A \,|\, \epsilon$, then $\vdash \Gamma$.*
2. *If $\Gamma; R \vdash H : A \,|\, \epsilon \Rightarrow B \,|\, \epsilon'$, then $\vdash \Gamma$.*

Proof. Straightforward by induction on the typing derivations. □

Lemma 24. *If $\Gamma_1, x : \sigma, \Gamma_2 \rhd^S \Gamma'$, then $\Gamma' = \Gamma'_1, S(x) : \sigma, \Gamma'_2$ and $\Gamma_1, \Gamma_2 \rhd^S \Gamma'_1, \Gamma'_2$ for some Γ'_1 and Γ'_2.*

Proof. Straightforward by induction on Γ_2. □

Lemma 25. *Suppose that $R \rhd r$ and $\Gamma \rhd^S \Gamma'$ and $\vdash \Gamma'$.*

1. *If $\Gamma; R \vdash M : A \,|\, \epsilon$, then there exists some e such that $\Gamma; R \vdash M : A \,|\, \epsilon \rhd^S e$ and $\Gamma'; r \vdash e : A \,|\, \epsilon$.*
2. *If $\Gamma; R \vdash H : A \,|\, \epsilon \Rightarrow B \,|\, \epsilon'$, then there exists some h such that $\Gamma; R \vdash H : A \,|\, \epsilon \Rightarrow B \,|\, \epsilon' \rhd^S h$ and $\Gamma'; r \vdash h : A \,|\, \epsilon \Rightarrow B \,|\, \epsilon'$.*

Proof. By mutual induction on the typing derivations.

1. By case analysis on the typing rule applied last.

Case (TS_VAR): We are given $\Gamma; R \vdash x : B[\boldsymbol{C}/\boldsymbol{\alpha}] \mid \epsilon$ and, by inversion, $\vdash \Gamma$ and $x : \forall \boldsymbol{\alpha}.B \in \Gamma$ and $\Gamma \vdash \boldsymbol{C}$. By Lemma 21, $S(x) : \forall \boldsymbol{\alpha}.B \in \Gamma'$. Thus, $S(x)$ is defined, so

$$\Gamma; R \vdash x : B[\boldsymbol{C}/\boldsymbol{\alpha}] \mid \epsilon \rhd^S S(x)\, \boldsymbol{C}$$

by (ELAB_VAR). By Lemma 22, $\Gamma' \vdash \boldsymbol{C}$. By (T_VAR), we finish.

Case (TS_CONST): By (ELAB_CONST) and (T_CONST).

Case (TS_ABS): We are given $\Gamma; R \vdash \lambda x.M' : B \to \epsilon'\ C \mid \epsilon$ and, by inversion, $\Gamma, x : B; R \vdash M' : C \mid \epsilon'$. Without loss of generality, we can suppose that x does not occur in S and Γ'. Since $\Gamma \rhd^S \Gamma'$, we have $\Gamma, x : B \rhd^{S \circ \{x \mapsto x\}} \Gamma, x : B$ by (ELABG_VAR). By Lemma 23, $\vdash \Gamma, x : B$. Thus, $\Gamma \vdash B$. By Lemma 22, $\Gamma' \vdash B$. Thus, by (WF_VAR), $\vdash \Gamma', x : B$. By the IH, $\Gamma, x : B; R \vdash M' : C \mid \epsilon' \rhd^{S \circ \{x \mapsto x\}} e'$ for some e' such that $\Gamma', x : B; r \vdash e' : C \mid \epsilon'$. By (ELAB_ABS) and (T_ABS), we finish.

Case (TS_APP): By the IHs, (ELAB_APP), and (T_APP).

Case (TS_OP): By the IH, (ELAB_OP), and (T_OP) with Lemma 22.

Case (TS_LET): Similar to (TS_ABS).

Case (TS_WEAK): By the IH, (ELAB_WEAK), and (T_WEAK).

Case (TS_HANDLE): By the IH, (ELAB_HANDLE), and (T_HANDLE).

Case (TS_RESUME): We are given $\Gamma_1, x : D, \Gamma_2; (\boldsymbol{\alpha}, x : B, C \to \epsilon'\ A) \vdash \mathsf{resume}\, M' : A \mid \epsilon$ and, by inversion,

- $\vdash \Gamma_1, x : D, \Gamma_2$,
- $\boldsymbol{\alpha} \in \Gamma_1$,
- $\epsilon' \subseteq \epsilon$, and
- $\Gamma_1, \Gamma_2, \boldsymbol{\beta}, x : B\, [\boldsymbol{\beta}/\boldsymbol{\alpha}]; (\boldsymbol{\alpha}, x : B, C \to \epsilon'\ A) \vdash M' : C[\boldsymbol{\beta}/\boldsymbol{\alpha}] \mid \epsilon$.

Let y be a fresh variable. Since $\Gamma_1, x : D, \Gamma_2 \rhd^S \Gamma'$, there exist some Γ'_1 and Γ'_2 such that $\Gamma' = \Gamma'_1, S(x) : D, \Gamma'_2$ and $\Gamma_1, \Gamma_2 \rhd^S \Gamma'_1, \Gamma'_2$ by Lemma 24. Since $\vdash \Gamma_1, x : D, \Gamma_2$, $x \notin \mathit{dom}(\Gamma_1, \Gamma_2)$. Thus, $\Gamma_1, \Gamma_2 \rhd^{S \circ \{x \mapsto y\}} \Gamma'_1, \Gamma'_2$. By (ELABG_VAR) and (ELABG_TYVAR), $\Gamma_1, \Gamma_2, \boldsymbol{\beta}, x : B\, [\boldsymbol{\beta}/\boldsymbol{\alpha}] \rhd^{S \circ \{x \mapsto y\}} \Gamma'_1, \Gamma'_2, \boldsymbol{\beta}, y : B\, [\boldsymbol{\beta}/\boldsymbol{\alpha}]$. Since $\vdash \Gamma'_1, S(x) : D, \Gamma'_2$, we have $\vdash \Gamma'_1, \Gamma'_2, \boldsymbol{\beta}, y : B\, [\boldsymbol{\beta}/\boldsymbol{\alpha}]$. by Lemmas 8, 23, 22, and 1. Thus, by the IH,

$$\Gamma_1, \Gamma_2, \boldsymbol{\beta}, x : B\, [\boldsymbol{\beta}/\boldsymbol{\alpha}]; (\boldsymbol{\alpha}, x : B, C \to \epsilon'\ A) \vdash M' : C[\boldsymbol{\beta}/\boldsymbol{\alpha}] \mid \epsilon \rhd^{S \circ \{x \mapsto y\}} e'$$

for some e' such that $\Gamma'_1, \Gamma'_2, \boldsymbol{\beta}, y : B\, [\boldsymbol{\beta}/\boldsymbol{\alpha}]; r \vdash e' : C[\boldsymbol{\beta}/\boldsymbol{\alpha}] \mid \epsilon$. By applying (ELAB_RESUME),

$$\Gamma_1, x : D, \Gamma_2; (\boldsymbol{\alpha}, x : B, C \to \epsilon'\ A) \vdash \mathsf{resume}\, M' : A \mid \epsilon \rhd^S \mathsf{resume}\, \boldsymbol{\beta}\, y.e'.$$

Since $\Gamma'_1, S(x) : D, \Gamma'_2, \boldsymbol{\beta}, y : B\, [\boldsymbol{\beta}/\boldsymbol{\alpha}]; r \vdash e' : C[\boldsymbol{\beta}/\boldsymbol{\alpha}] \mid \epsilon$ by Lemma 2 and $\boldsymbol{\alpha} \in \Gamma'_1, S(x) : D, \Gamma'_2$ by Lemma 22, we have

$$\Gamma'_1, S(x) : D, \Gamma'_2; r \vdash \mathsf{resume}\, \boldsymbol{\beta}\, y.e' : A \mid \epsilon$$

by (T_RESUME).

2. By case analysis on the typing rule applied last.

Case (THS_RETURN): Similar to (TS_ABS).

Case (THS_OP): Similar to (TS_ABS).

□

Theorem 2 (Elaboration is type-preserving). *If M is a well-typed program of A, then $\emptyset; \mathsf{none} \vdash M : A \mid \langle\rangle \rhd^{\emptyset} e$ and $\emptyset; \mathsf{none} \vdash e : A \mid \langle\rangle$ for some e.*

Proof. By Lemma 25. □