Supplementary Material for "On Higher-Order Model Checking of Effectful Answer-Type-Polymorphic Programs"

Taro Sekiyama, Ugo Dal Lago, and Hiroshi Unno

August 22, 2025

Contents

List of Theorems

1 Outline

This is the supplementary material of the paper titled "On Higher-Order Model Checking of Effectful Answer-Type-Polymorphic Programs" published at OOPSLA'25, including all the definitions, lemmas, theorems, and proofs mentioned in the paper.

This supplementary material formalizes the subtyping extension in the paper, which subsumes $\mathsf{HEPCF}^{\mathsf{ATM}}_{\square}$ presented in the paper. The definition and examples of HOMC and alternating parity tree automata (APTAs) are found in Section 2.2.5.

2 Definition

2.1 Trees

Definition 1 (Tree Constructor Signatures)**.** A tree constructor signature S is a map from tree constructors, ranged over by s, to natural numbers that represent the arities of the constructors. We write $S(s)$ for the arity of s assigned by S.

Definition 2 (Finitely Branching Infinite Trees)**.** The set $\mathbf{Tree}_S$ of finitely branching (possibly) infinite trees generated by a tree constructor signature S is defined coinductively by the following grammar (where s is in the domain of S):

$$t \quad ::= \quad \bot \mid s(t_1, \cdots, t_{S(s)}) \; .$$

2.2 $\mathsf{HEPCF}^{\mathsf{ATM}}_{\square}$: PCF unifying Answer-Type Modification and GEPCF restrictions

2.2.1 Syntax

Variables x, y, z, f, h, k **Algebraic operations** σ, ς

Base types	B	$::=$	$\mathsf{bool} \mid \mathsf{unit} \mid \cdots$
Enum types	E	$::=$	$1 \mid 2 \mid \cdots$
Value types	T, U	$::=$	$B \mid E \mid T \to C$
Computation types	C, D	$::=$	$\Sigma \triangleright T \,/\, A$
Operation signatures	Σ	$::=$	$\{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, A_i\}^{1 \le i \le n}$
Answer type modifications	A	$::=$	$\square \mid C_1 \Rightarrow C_2$
Base constants	c	$::=$	$\mathsf{true} \mid \mathsf{false} \mid () \mid \cdots$
Enum constants	ε	$::=$	$\underline{1} \mid \underline{2} \mid \cdots$
Values	V, W	$::=$	$x \mid c \mid \varepsilon \mid \lambda x.M \mid \mathsf{fix}\, x.\, V$
Terms	L, M, N	$::=$	$\mathsf{return}\ V \mid \mathsf{let}\, x = M \,\mathsf{in}\, N \mid V\ W \mid \mathsf{case}(V; M_1, \cdots, M_n) \mid$ $\sigma(V; x.\, M) \mid \mathsf{with}\ H\ \mathsf{handle}\ M$
Handlers	H	$::=$	$\{\mathsf{return}\, x \mapsto L\} \uplus \{\sigma_i(x_i; k_i) \mapsto M_i\}^{1 \le i \le m} \uplus \{\varsigma_i(y_i) \mapsto N_i\}^{1 \le i \le n}$
Typing contexts	Γ	$::=$	$\emptyset \mid \Gamma, x : T$

Convention 1. We write Γ_1, Γ_2 for the concatenation of Γ_1 and Γ_2. Given a computation type $C = \Sigma \triangleright T \,/\, A$, we write $C.\Sigma$, $C.T$, and $C.A$ for the operation signature Σ, value type T, and answer type modification (ATM) A, respectively.

Definition 3 (Free variables and substitution)**.** The set $\mathit{fv}(M)$ of free variables in a term M is defined in a standard manner. Value substitution $M[V/x]$ and $W[V/x]$ of V for x in M and W, respectively, are defined in a capture-avoiding manner as usual.

Assumption 1. We assume a function ty that assigns a base type $\mathit{ty}(c)$ to every constant c.

Definition 4 (Top-Level Operation Signatures)**.** An operation signature Σ is *top-level* if, for any $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, A \in \Sigma$, $T^{\mathrm{par}} = B$ for some B, $T^{\mathrm{ari}} = E$ for some E, and $A = \square$.

Definition 5 (Ground Types)**.** A type T is ground if and only if $T = B$ for some B or $T = E$ for some E.

2.2.2 Semantics

Definition 6 (Semantics)**.** The evaluation relation $M_1 \longrightarrow M_2$ is the smallest relation satisfying the rules in Figure 1.

Definition 7 (Multi-step evaluation)**.** We write $M \longrightarrow^n N$ if and only if there exist some terms $L_0, \cdots, L_n$ such that: $M = L_0$; $\forall\, i < n.\ L_i \longrightarrow L_{i+1}$; and $L_n = N$. We write $M \longrightarrow^* N$ if and only if $M \longrightarrow^n N$ for some $n \in \mathbb{N}$.

Definition 8 (Infinite Evaluation)**.** We write $M \longrightarrow^\omega$ if and only if $\forall n \in \mathbb{N}. \exists M'. M \longrightarrow^n M'$.

Definition 9 (Nonreducible terms)**.** We write $M \not\longrightarrow$ if and only if there is no M' such that $M \longrightarrow M'$.

2.2.3 Type System

Definition 10 (Domains of Typing Contexts)**.** Given a typing context Γ, its domain $\mathit{dom}(\Gamma)$ is defined by induction on Γ as follows.

$$\begin{aligned} \mathit{dom}(\emptyset) &\stackrel{\mathrm{def}}{=} \emptyset \\ \mathit{dom}(\Gamma, x : T) &\stackrel{\mathrm{def}}{=} \{x\} \cup \mathit{dom}(\Gamma) \end{aligned}$$

Evaluation rules $\boxed{M_1 \longrightarrow M_2}$

$$
\begin{array}{rcllr}
(\lambda x.M)\ V & \longrightarrow & M[V/x] & & \text{HE\_BETA} \\
(\mathsf{fix}\,x.\,V)\ W & \longrightarrow & V[\mathsf{fix}\,x.\,V/x]\ W & & \text{HE\_FIX} \\
\mathsf{case}(\underline{i}; M_1, \cdots, M_n) & \longrightarrow & M_i & (\text{if } 0 < i \leq n) & \text{HE\_CASE} \\
\mathsf{let}\,x = \mathsf{return}\ V_1\ \mathsf{in}\ M_2 & \longrightarrow & M_2[V_1/x] & & \text{HE\_LETV} \\
\mathsf{let}\,x = \sigma(V_1; y.\,M_1)\ \mathsf{in}\ M_2 & \longrightarrow & \sigma(V_1; y.\,\mathsf{let}\,x = M_1\ \mathsf{in}\ M_2) & (\text{if } y \notin fv(M_2)) & \text{HE\_LETOP} \\
\mathsf{with}\ H\ \mathsf{handle}\ \mathsf{return}\ V & \longrightarrow & M[V/x] & (\text{if } \mathsf{return}\,x \mapsto M \in H) & \text{HE\_HANDLEV} \\
\mathsf{with}\ H\ \mathsf{handle}\ \sigma(V; y.\,M) & \longrightarrow & N[V/x][\lambda y.\mathsf{with}\ H\ \mathsf{handle}\ M/k] & (\text{if } \sigma(x;k) \mapsto N \in H) & \text{HE\_HANDLEOP} \\
\mathsf{with}\ H\ \mathsf{handle}\ \sigma(V; y.\,M) & \longrightarrow & \mathsf{let}\,y = N[V/x]\ \mathsf{in}\ \mathsf{with}\ H\ \mathsf{handle}\ M & (\text{if } \sigma(x) \mapsto N \in H) & \text{HE\_HANDLEOPTAIL}
\end{array}
$$

$$
\frac{M \longrightarrow N}{\mathsf{let}\,x = M\ \mathsf{in}\ L \longrightarrow \mathsf{let}\,x = N\ \mathsf{in}\ L}\ \text{HE\_LETE} \qquad \frac{M \longrightarrow N}{\mathsf{with}\ H\ \mathsf{handle}\ M \longrightarrow \mathsf{with}\ H\ \mathsf{handle}\ N}\ \text{HE\_HANDLEE}
$$

Figure 1: Semantics.

Definition 11 (Typing Contexts as Functions)**.** We view Γ as a function that maps a variable to a type. $\Gamma(x) = T$ if and only if $x : T \in \Gamma$.

Definition 12 (Pure and Impure Operation Signatures)**.** Given an operation signature $\Sigma = \{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, C_i^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{fin}}\}^{1 \leq i \leq m} \uplus \{\varsigma_i : U_i^{\mathrm{ari}} \rightsquigarrow U_i^{\mathrm{ari}} \,/\, \square\}^{1 \leq i \leq n}$, $\not\square(\Sigma)$ and $\square(\Sigma)$ denote $\{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, C_i^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{fin}}\}^{1 \leq i \leq m}$ and $\{\varsigma_i : U_i^{\mathrm{ari}} \rightsquigarrow U_i^{\mathrm{ari}} \,/\, \square\}^{1 \leq i \leq n}$, respectively.

Definition 13 (Subtyping)**.** The subtyping (with judgments of the form $T_1 <: T_2$, $C_1 <: C_2$, $\Sigma_1 <: \Sigma_2$, and $A_1 <: A_2$) is the smallest relation satisfying the rules in Figure 2.

Definition 14 (Typing)**.** The typing of values (with judgments of the form $\Gamma \vdash V : T$) and terms (with judgments of the form $\Gamma \vdash M : C$) is the smallest relation satisfying the rules in Figure 3.

2.2.4 Effect Trees

Definition 15 (Effect Trees for $\mathsf{HEPCF}^{\mathsf{ATM}}_{\square}$ Computations)**.** Given an operation signature Σ and a type T, the tree constructor signature S_T^{Σ} is defined as follows:

$$
\begin{array}{rcll}
S_T^{\Sigma}(\sigma) & \stackrel{\text{def}}{=} & n+1 & \forall \sigma : B \rightsquigarrow \mathsf{n} \,/\, A \in \Sigma \\
S_T^{\Sigma}(\mathsf{return}\ V) & \stackrel{\text{def}}{=} & 0 & \forall V \text{ such that } \emptyset \vdash V : T \\
S_T^{\Sigma}(c) & \stackrel{\text{def}}{=} & 0 & \forall c
\end{array}
$$

Given a term M such that $\emptyset \vdash M : \Sigma \rhd T \,/\, A$, the effect tree of M, denoted by $\mathbf{ET}(M)$, is a tree in $\mathbf{Tree}_{S_T^{\Sigma}}$ defined by the following (possibly infinite) process:

- if $M \longrightarrow^{\omega}$, then $\mathbf{ET}(M) = \bot$;
- if $M \longrightarrow^{*} \mathsf{return}\ V$, then $\mathbf{ET}(M) = \mathsf{return}\ V$; and
- if $M \longrightarrow^{*} \sigma(c; x.\,N)$ and $\sigma : B \rightsquigarrow \mathsf{n} \,/\, A \in \Sigma$, then $\mathbf{ET}(M) = \sigma(c, \mathbf{ET}(N[\underline{1}/x]), \cdots, \mathbf{ET}(N[\underline{\mathsf{n}}/x]))$.

2.2.5 Higher-Order Model Checking

Definition. To define higher-order model checking for $\mathsf{HEPCF}^{\mathsf{ATM}}_{\square}$, we first introduce *alternating parity tree automata* (APTAs) along with auxiliary notions.

Definition 16 (Positive Boolean Formulas)**.** The set $\mathbf{B}^{+}(X)$ of *positive Boolean formulas* over a finite set X is defined as follows:

$$\mathbf{B}^{+}(X) \ni \theta ::= \mathsf{tt} \mid \mathsf{ff} \mid \mathbf{x} \mid \theta_1 \vee \theta_2 \mid \theta_1 \wedge \theta 2$$

where $\mathbf{x} \in X$. A subset Y of X satisfies $\theta \in \mathbf{B}^{+}(X)$ if θ holds under the interpretation that assigns true to the elements in Y and false to the elements in $Y \setminus X$.

Subtyping rules for value types $\boxed{T_1 <: T_2}$

$$\frac{}{B <: B}\ \text{HS\_Base} \qquad \frac{}{E <: E}\ \text{HS\_Const} \qquad \frac{T_2 <: T_1 \quad C_1 <: C_2}{T_1 \to C_1 <: T_2 \to C_2}\ \text{HS\_Fun}$$

Subtyping rules for computation types $\boxed{C_1 <: C_2}$

$$\frac{\Sigma_2 <: \Sigma_1 \quad T_1 <: T_2 \quad A_1 <: A_2 \quad A_1 \neq \square \Longrightarrow \square(\Sigma_1) <: \square(\Sigma_2)}{\Sigma_1 \rhd T_1 \,/\, A_1 <: \Sigma_2 \rhd T_2 \,/\, A_2}\ \text{HS\_Comp}$$

Subtyping rules for answer type modifications $\boxed{A_1 <: A_2}$

$$\frac{}{\square <: \square}\ \text{HS\_AnsBox} \qquad \frac{C_1 <: C_2}{\square <: C_1 \Rightarrow C_2}\ \text{HS\_AnsEmb} \qquad \frac{C_2^{\text{ini}} <: C_1^{\text{ini}} \quad C_1^{\text{fin}} <: C_2^{\text{fin}}}{C_1^{\text{ini}} \Rightarrow C_1^{\text{fin}} <: C_2^{\text{ini}} \Rightarrow C_2^{\text{fin}}}\ \text{HS\_AnsMod}$$

Subtyping rules for operation signatures $\boxed{\Sigma_1 <: \Sigma_2}$

$$\frac{\forall\, i \in [1,n].\ T_{2i}^{\text{par}} <: T_{1i}^{\text{par}} \ \wedge\ T_{1i}^{\text{ari}} <: T_{2i}^{\text{ari}} \ \wedge\ A_{1i} <: A_{2i}}{\{\sigma_i : T_{1i}^{\text{par}} \rightsquigarrow T_{1i}^{\text{ari}} \,/\, A_{1i}\}^{1\le i\le n} \uplus \Sigma <: \{\sigma_i : T_{2i}^{\text{par}} \rightsquigarrow T_{2i}^{\text{ari}} \,/\, A_{2i}\}^{1\le i\le n}}\ \text{HS\_Sig}$$

Figure 2: Subtyping.

Definition 17 (Positions of Trees)**.** The set $dom(t)$ of the *positions* of a tree t generated by a tree constructor signature S with maximal arity n is a set of finite sequences over alphabet $\{1, \cdots, n\}$ defined as $dom(\bot) = \{\epsilon\}$ and $dom(s(t_1, \cdots, t_{S(s)})) = \{\epsilon\} \cup \bigcup_{i \in \{1,\cdots,S(s)\}} \{i \cdot p \mid p \in dom(t_i)\}$, where ϵ is the empty sequence and $\cdot$ is the concatenation of finite sequences. The node $t(p)$ of a tree t at a position $p \in dom(t)$ is defined by $\bot(\epsilon) = \bot$, $s(t_1, \cdots, t_{S(s)})(\epsilon) = s$, and $s(t_1, \cdots, t_{S(s)})(i \cdot p) = t_i(p)$.

Definition 18 (Alternating Parity Tree Automata)**.** An *alternating parity tree automaton* (APTA) over a tree constructor signature S is a tuple $\mathcal{A} = (S, Q, \delta, q_I, \Omega)$ satisfying the following:

- Q is a finite set of states with $q_I \in Q$ as the initial state.
- δ is a transition function, mapping $(q, s) \in Q \times dom(S)$ to a formula in $\mathbf{B}^+(\{1, \cdots, S(s)\} \times Q)$.
- Ω is a priority function, mapping states in Q to natural numbers.

A *run-tree* of an APTA $\mathcal{A} = (S, Q, \delta, q_I, \Omega)$ over a tree $t \in \mathbf{Tree}_S$ is a tree satisfying the following:

- Every node is labeled with some $(p, q) \in dom(t) \times Q$.
- The root node is (ϵ, q_I).
- For each node (p, q), there is a set $X \subseteq \{1, \cdots, S(t(p))\} \times Q$ satisfying the positive Boolean formula $\delta(q, t(p))$ and, for each $(i, q') \in X$, the node $(p \cdot i, q')$ is a child of the node (p, q).

A tree $t \in \mathbf{Tree}_S$ is *accepted* by an APTA $\mathcal{A}$ if there exists a run-tree of $\mathcal{A}$ over t such that every infinite path $(\epsilon, q_I), (p_1, q_1), (p_2, q_2) \cdots$ of the run-tree meets the parity condition, that is, the largest priority infinitely occurring in $\Omega(q_I), \Omega(q_1), \Omega(q_2), \cdots$ is even.

Definition 19 (Higher-Order Model Checking Problem for EPCF)**.** Given an APTA and a term M such that $\emptyset \vdash M : \Sigma \rhd T \,/\, A$ for some top-level operation signature Σ and ground type T, is $\mathbf{ET}(M)$ accepted by the APTA?

Typing rules for values $\boxed{\Gamma \vdash V : T}$

$$\frac{}{\Gamma \vdash x : \Gamma(x)}\ \textsc{HT\_Var} \qquad \frac{}{\Gamma \vdash c : ty(c)}\ \textsc{HT\_Const} \qquad \frac{0 < i \leq n}{\Gamma \vdash \underline{\mathsf{i}} : \mathsf{n}}\ \textsc{HT\_EConst}$$

$$\frac{\Gamma, x : T \vdash M : C}{\Gamma \vdash \lambda x.M : T \to C}\ \textsc{HT\_Abs} \qquad \frac{\Gamma, x : T \to C \vdash V : T \to C}{\Gamma \vdash \mathsf{fix}\, x.V : T \to C}\ \textsc{HT\_Fix} \qquad \frac{\Gamma \vdash V : T \quad T <: U}{\Gamma \vdash V : U}\ \textsc{HT\_SubV}$$

Typing rules for terms $\boxed{\Gamma \vdash M : C}$

$$\frac{\Gamma \vdash V : T}{\Gamma \vdash \mathsf{return}\ V : \Sigma \triangleright T \,/\, \square}\ \textsc{HT\_Return} \qquad \frac{\Gamma \vdash M : C \quad C <: D}{\Gamma \vdash M : D}\ \textsc{HT\_SubC}$$

$$\frac{\Gamma \vdash M_1 : \Sigma \triangleright T_1 \,/\, \square \quad \Gamma, x : T_1 \vdash M_2 : \Sigma \triangleright T_2 \,/\, A}{\Gamma \vdash \mathsf{let}\, x = M_1\ \mathsf{in}\ M_2 : \Sigma \triangleright T_2 \,/\, A}\ \textsc{HT\_Let}$$

$$\frac{\Gamma \vdash M_1 : \Sigma \triangleright T_1 \,/\, C \Rightarrow C^{\mathrm{fin}} \quad \Gamma, x : T_1 \vdash M_2 : \Sigma \triangleright T_2 \,/\, C^{\mathrm{ini}} \Rightarrow C}{\Gamma \vdash \mathsf{let}\, x = M_1\ \mathsf{in}\ M_2 : \Sigma \triangleright T_2 \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}}\ \textsc{HT\_LetATM}$$

$$\frac{\Gamma \vdash V_1 : T \to C \quad \Gamma \vdash V_2 : T}{\Gamma \vdash V_1\ V_2 : C}\ \textsc{HT\_App} \qquad \frac{\Gamma \vdash V : \mathsf{n} \quad \forall i \in [1,n].\, \Gamma \vdash M_i : C}{\Gamma \vdash \mathsf{case}(V; M_1, \cdots, M_n) : C}\ \textsc{HT\_Case}$$

$$\frac{\Sigma \ni \sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, \square \quad \Gamma \vdash V : T^{\mathrm{par}} \quad \Gamma, x : T^{\mathrm{ari}} \vdash M : \Sigma \triangleright T \,/\, A}{\Gamma \vdash \sigma(V; x.\, M) : \Sigma \triangleright T \,/\, A}\ \textsc{HT\_Op}$$

$$\frac{\Sigma \ni \sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}} \quad \Gamma \vdash V : T^{\mathrm{par}} \quad \Gamma, x : T^{\mathrm{ari}} \vdash M : \Sigma \triangleright T \,/\, C \Rightarrow C^{\mathrm{ini}}}{\Gamma \vdash \sigma(V; x.\, M) : \Sigma \triangleright T \,/\, C \Rightarrow C^{\mathrm{fin}}}\ \textsc{HT\_OpATM}$$

$$\frac{\begin{array}{c} H = \{\mathsf{return}\, x \mapsto L\} \uplus \{\sigma_i(y_i; k_i) \mapsto M_i\}^{1 \leq i \leq m} \uplus \{\varsigma_i(z_i) \mapsto N_i\}^{1 \leq i \leq n} \\ \Sigma = \{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, C_i^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{fin}}\}^{1 \leq i \leq m} \uplus \{\varsigma_i : U_i^{\mathrm{par}} \rightsquigarrow U_i^{\mathrm{ari}} \,/\, \square\}^{1 \leq i \leq n} \\ \Gamma \vdash M : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}} \quad \Gamma, x : T \vdash L : C^{\mathrm{ini}} \\ \forall i \in [1,m].\, \Gamma, y_i : T_i^{\mathrm{par}}, k_i : T_i^{\mathrm{ari}} \to C_i^{\mathrm{ini}} \vdash M_i : C_i^{\mathrm{fin}} \\ \forall i \in [1,n].\, \Gamma, z_i : U_i^{\mathrm{par}} \vdash N_i : \Sigma_i \triangleright U_i^{\mathrm{ari}} \,/\, \square \\ \forall C \in \{\overline{C_i^{\mathrm{ini}}}^{1 \leq i \leq m}, C^{\mathrm{fin}}\}.\ \forall i \in [1,n].\ C.\Sigma <: \Sigma_i \end{array}}{\Gamma \vdash \mathsf{with}\ H\ \mathsf{handle}\ M : C^{\mathrm{fin}}}\ \textsc{HT\_Handle}$$

Figure 3: Type System.

Examples.

Example 1. Let Σ be an operation signature for **Set**, **Get**, and **Raise**, that is,

$$\Sigma \stackrel{\mathrm{def}}{=} \{\mathbf{Set} : \mathsf{bool} \rightsquigarrow \mathsf{1} \,/\, \square, \mathbf{Get} : \mathsf{unit} \rightsquigarrow \mathsf{2} \,/\, \square, \mathbf{Raise} : \mathsf{unit} \rightsquigarrow \mathsf{0} \,/\, \square\}\ .$$

Let $\mathcal{A}_{\mathbf{SGR}} \stackrel{\mathrm{def}}{=} (S^{\Sigma}_{\mathsf{unit}}, Q, \delta, q_1, \{q \mapsto 0 \mid q \in Q\})$ where $Q = \{q_1, q_2, q_{\mathsf{true}}, q_{\mathsf{false}}\}$ and δ is defined as

- $\delta(q_i, \mathbf{Set}) = ((1, q_{\mathsf{true}}) \wedge (2, q_1)) \vee ((1, q_{\mathsf{false}}) \wedge (2, q_2))$,
- $\delta(q_i, \mathbf{Get}) = (i + 1, q_i)$,
- $\delta(q_i, \mathbf{Raise}) = \mathsf{ff}$,
- $\delta(q_i, \mathsf{return}\ v) = \delta(q_b, b) = \mathsf{tt}$ and $\delta(q_b, b') = \mathsf{ff}$

for each $i \in \{1, 2\}$, $b \in \{\mathsf{true}, \mathsf{false}\}$, and $b' \in \{\mathsf{true}, \mathsf{false}\} \setminus \{b\}$ (in the other cases, δ returns ff).

The states q_1 and q_2 express the program states where the global reference manipulated by **Set** and **Get** refers to true and false, respectively. Based on this idea, the transition rules for **Set** and **Get** encode the semantics of mutable state in the APTA: if **Set** is called with the parameter true (resp. false), the continuation is executed under the state q_1 (resp. q_2); if **Get** is called in the state q_1 (resp. q_2), the continuation supposing the return value of **Get** to be $\underline{1}$ (resp. $\underline{2}$) is chosen. The conjunct $(1, q_{\mathsf{true}})$ (resp. $(1, q_{\mathsf{false}})$) in the transition of **Set** requires that the parameter of **Set** be true (resp. false) to set the state of the continuation to q_1 (resp. q_2).

The transition rule for **Raise** expresses the specification that **Raise** must not be called. This is indicated by the fact that we cannot make a run-tree of $\mathcal{A}_{\mathbf{SGR}}$ over a tree t that involves a path where a **Raise** node is reachable and **Set** and **Get** interact in accordance with the semantics of mutable state. If there exists such a run-tree, it would contain a node (p, q_i) for some $p \in \mathit{dom}(t)$ and $i \in \{1, 2\}$ such that $t(p) = \mathbf{Raise}$. By the definition of run-trees, there should be some X that satisfies the positive Boolean formula $\delta(q_i, \mathbf{Raise})$, but there is no such X because $\delta(q_i, \mathbf{Raise}) = \mathsf{ff}$. Thus, there is no run-tree of $\mathcal{A}_{\mathbf{SGR}}$ over the tree t, which means that the tree t is not accepted by $\mathcal{A}_{\mathbf{SGR}}$.

Example 2. Consider verifying the use of global file manipulation operations **Open** that opens the file, **Read** that reads the contents of the opened file, **EOF** that checks whether there remains readable data, and **Close** that closes the opened file. The use of these operations is valid if their call sequences conform to the regular expression $(\mathbf{Open}\,(\mathbf{EOF}^+\,\mathbf{Read})^*\,\mathbf{Close})^*$. Let Σ be an operation signature for the file operations defined as:

$$\Sigma \stackrel{\text{def}}{=} \{\mathbf{Open} : \mathsf{unit} \rightsquigarrow 1 \,/\, \square, \mathbf{Read} : \mathsf{unit} \rightsquigarrow 1 \,/\, \square, \mathbf{EOF} : \mathsf{unit} \rightsquigarrow 2 \,/\, \square, \mathbf{Close} : \mathsf{unit} \rightsquigarrow 1 \,/\, \square\} \ .$$

Here, we assume simpler file operations than the operations in practice (as in POSIX): **Open** does not take a file path nor return a file descriptor to identify the opened file object (thus, the file to be manipulated is predetermined) and **Read** only returns the enum constant $\underline{1}$. Nevertheless, it is still nontrivial to verify the valid use of the file operations even for this simplified version. We can treat more practical file operations, such as ones that can manipulate multiple files, by adapting the techniques in the previous work [2].

An APTA $\mathcal{A}_{\mathbf{File}}$ that only accepts effect trees where file operations are used in a valid manner is given by $(S_T^\Sigma, \{q_1, q_2, q_3\}, \delta, q_1, \{q_1 \mapsto 2, q_2 \mapsto 1, q_3 \mapsto 1\})$ where the transition function δ is defined by

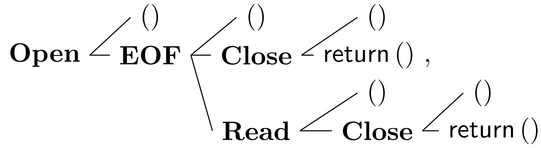
- $\delta(q_1, \mathbf{Open}) = (2, q_2)$ and $\delta(q_2, \mathbf{Open}) = \delta(q_3, \mathbf{Open}) = \mathsf{ff}$,
- $\delta(q_3, \mathbf{Read}) = (2, q_2)$ and $\delta(q_1, \mathbf{Read}) = \delta(q_2, \mathbf{Read}) = \mathsf{ff}$,
- $\delta(q_2, \mathbf{EOF}) = \delta(q_3, \mathbf{EOF}) = (2, q_2) \wedge (3, q_3)$ and $\delta(q_1, \mathbf{EOF}) = \mathsf{ff}$,
- $\delta(q_2, \mathbf{Close}) = \delta(q_3, \mathbf{Close}) = (2, q_1)$ and $\delta(q_1, \mathbf{Close}) = \mathsf{ff}$,
- $\delta(q_1, \mathsf{return}\ v) = \mathsf{tt}$ and $\delta(q_2, \mathsf{return}\ v) = \delta(q_3, \mathsf{return}\ v) = \mathsf{ff}$, and
- $\delta(q_1, c) = \delta(q_2, c) = \delta(q_3, c) = \mathsf{tt}$

(the type T is of $\mathsf{HEPCF}^{\mathsf{ATM}}_{\square}$ terms to be verified). The state q_1 and q_2 represent that the file is closed and opened, respectively, and q_3 represents the state where it is ensured that some data is readable. Thus, a call to **Open** in the state q_1, a call to **Read** in q_3, and a call to **EOF** or **Close** in q_2 or q_3 are valid, whereas a call to **Open** in q_2 or q_3, a call to **Read** in q_1 or q_2, and a call to **EOF** or **Close** in q_1 are invalid. The transition of **EOF** at the state q_2 or q_3 expresses that **EOF** returns $\underline{1}$ if there remains no readable data and, otherwise, $\underline{2}$. The transition of $\mathsf{return}\ v$ at q_2 or q_3 means that a term to be verified must not terminate in the state that the file are left open.

For example, a term

$$\mathbf{Open}((); \_.\, \mathbf{EOF}((); x.\, \mathsf{case}(x; \mathbf{Close}((); \_.\, \mathsf{return}\ ()), \mathbf{Read}((); \_.\, \mathbf{Close}((); \_.\, \mathsf{return}\ ())))))$$

generates the effect tree t as



and a run-tree over t can be given as:

$$
(\epsilon, q_1) — (2, q_2) \begin{array}{l} — (2\cdot 2, q_2) — (2\cdot 2\cdot 2, q_1) \\ \diagdown\, (2\cdot 3, q_3) — (2\cdot 3\cdot 2, q_2) — (2\cdot 3\cdot 2\cdot 2, q_1) \end{array}
$$

For a term $\mathbf{Open}((); \_. \mathbf{EOF}((); \_. \mathbf{Read}((); \_. \mathbf{Close}((); \_. \mathsf{return}\,())))))$, the effect tree t' like

$$
\mathbf{Open} \begin{array}{l} \diagup\, () \\ — \mathbf{EOF} \begin{array}{l} \diagup\, () \\ — \mathbf{Read} \begin{array}{l} \diagup\, () \\ — \mathbf{Close} \begin{array}{l} \diagup\, () \\ — \mathsf{return}\,() \end{array} \end{array} \\ \diagdown\, \mathbf{Read} \begin{array}{l} \diagup\, () \\ — \mathbf{Close} \begin{array}{l} \diagup\, () \\ — \mathsf{return}\,() \end{array} \end{array} \end{array} \end{array} ,
$$

is generated, but we cannot make a run-tree over t'; the process to make a run-tree will stop at

$$
(\epsilon, q_1) — (2, q_2) \begin{array}{l} — (2\cdot 2, q_2) \\ \diagdown\, (2\cdot 3, q_3) — (2\cdot 3\cdot 2, q_2) \cdot (2\cdot 3\cdot 2\cdot 2, q_1) \end{array}
$$

but there is no finite set X satisfying $\delta(q_2, t'(2\cdot 2)) = \delta(q_2, \mathbf{Read}) = \mathsf{ff}$.

The priority function ensures that the opened file will be closed eventually for divergent programs. To see it, consider the effect tree t generated by a term $\mathbf{Open}((); x. (\mathsf{fix}\, f.\lambda y.\mathbf{EOF}((); z.\, f\, y))\, ())$, which infinitely calls $\mathbf{EOF}$ after performing $\mathbf{Open}$. Given a run-tree over t, it should include an infinite path where q_2 or q_3 occurs infinitely and q_1 occurs only at the root node. The largest priority infinitely occurring in such a path is 1, which results in breaking the parity condition. Thus, the effect tree t is not accepted by $\mathcal{A}_{\mathbf{File}}$. Note that, if the opened file is eventually closed as in a term $(\mathsf{fix}\, f.\lambda x'.\mathbf{Open}((); x.\, \mathbf{EOF}((); y.\, \mathbf{Close}((); z.\, f\, x'))))\, ()$, we can make a run-tree where the state q_1 appears infinitely many times in every infinite path, so the largest priority infinitely occurring in an infinite path is even and the parity condition holds.

Evaluation rules $\boxed{e_1 \longrightarrow e_2}$

$$
\begin{array}{rcllr}
(\lambda x. e_1)\, v_2 & \longrightarrow & e_1[v_2/x] & & \text{E\_BETA} \\
(\mathsf{fix}\, x. v_1)\, v_2 & \longrightarrow & v_1[\mathsf{fix}\, x. v_1/x]\, v_2 & & \text{E\_FIX} \\
\mathsf{case}(\underline{i}; e_1, \cdots, e_n) & \longrightarrow & e_i & (\text{if } 0 < i \leq n) & \text{E\_CASE} \\
\mathsf{let}\, x = \mathsf{return}\, v_1\, \mathsf{in}\, e_2 & \longrightarrow & e_2[v_1/x] & & \text{E\_LETV} \\
\mathsf{let}\, x = \sigma(v_1; y.\, e_1)\, \mathsf{in}\, e_2 & \longrightarrow & \sigma(v_1; y.\, \mathsf{let}\, x = e_1\, \mathsf{in}\, e_2) & (\text{if } y \notin fv(e_2)) & \text{E\_LETOP}
\end{array}
$$

$$
\frac{e_1 \longrightarrow e_1'}{\mathsf{let}\, x = e_1\, \mathsf{in}\, e_2 \longrightarrow \mathsf{let}\, x = e_1'\, \mathsf{in}\, e_2}\ (\text{E\_LETE})
$$

Figure 4: Semantics.

2.3 EPCF: PCF with Algebraic Effects

2.3.1 Syntax

Variables x, y, z, f, h, k **Algebraic operations** σ, ς

$$
\begin{array}{rrcl}
\textbf{Base types} & B & ::= & \mathsf{bool} \mid \mathsf{unit} \mid \cdots \\
\textbf{Enum types} & E & ::= & 1 \mid 2 \mid \cdots \\
\textbf{Types} & \tau & ::= & B \mid E \mid \tau_1 \to \tau_2 \\
\textbf{Operation signatures} & \Xi & ::= & \{\sigma_i : B_i \rightsquigarrow E_i\}^{1 \leq i \leq n} \\
\textbf{Base constants} & c & ::= & \mathsf{true} \mid \mathsf{false} \mid () \mid \cdots \\
\textbf{Enum constants} & \varepsilon & ::= & \underline{1} \mid \underline{2} \mid \cdots \\
\textbf{Values} & v, w & ::= & x \mid c \mid \varepsilon \mid \lambda x. e \mid \mathsf{fix}\, x. v \\
\textbf{Terms} & e & ::= & \mathsf{return}\, v \mid \mathsf{let}\, x = e_1\, \mathsf{in}\, e_2 \mid v_1\, v_2 \mid \mathsf{case}(v; e_1, \cdots, e_n) \mid \sigma(v; x.\, e) \\
\textbf{Typing contexts} & \Delta & ::= & \emptyset \mid \Delta, x : \tau
\end{array}
$$

For the syntactic operations common in $\mathsf{HEPCF}^{\mathsf{ATM}}_{\square}$ and EPCF, we use the same notation (e.g., $fv(e)$ is the set of free variables in e and $e[v/x]$ is the term obtained by substituting v for x in e).

Definition 20 (Ground Types)**.** A type τ is ground if and only if $\tau = B$ for some B or $\tau = E$ for some E.

2.3.2 Semantics

Definition 21 (Semantics)**.** The evaluation relation $e_1 \longrightarrow e_2$ is the smallest relation satisfying the rules in Figure 4.

Definition 22 (Multi-step evaluation)**.** We write $e \longrightarrow^n e'$ if and only if there exist some terms $e_0, \cdots, e_n$ such that: $e = e_0$; $\forall i < n.\ e_i \longrightarrow e_{i+1}$; and $e_n = e'$. We write $e \longrightarrow^* e'$ if and only if $e \longrightarrow^n e'$ for some $n \in \mathbb{N}$, and $e \longrightarrow^+ e'$ if and only if $e \longrightarrow^n e'$ for some $n > 0$.

Definition 23 (Infinite Evaluation)**.** We write $e \longrightarrow^\omega$ if and only if $\forall n \in \mathbb{N}. \exists e'. e \longrightarrow^n e'$.

Definition 24 (Nonreducible terms)**.** We write $e \not\longrightarrow$ if and only if there is no e' such that $e \longrightarrow e'$.

2.3.3 Type System

Definition 25 (Typing)**.** Fix an operation signature Ξ. Then, the typing of values (with judgments of the form $\Xi \parallel \Delta \vdash v : \tau$) and terms (with judgments of the form $\Xi \parallel \Delta \vdash e : \tau$) is the smallest relation satisfying the rules in Figure 5.

Typing rules $\boxed{\Xi \parallel \Delta \vdash v : \tau}$ $\boxed{\Xi \parallel \Delta \vdash e : \tau}$

$$\frac{}{\Xi \parallel \Delta \vdash x : \Delta(x)}\ \text{T\_VAR} \qquad \frac{}{\Xi \parallel \Delta \vdash c : ty(c)}\ \text{T\_CONST} \qquad \frac{0 < i \leq n}{\Xi \parallel \Delta \vdash \underline{\mathsf{i}} : \mathsf{n}}\ \text{T\_ECONST}$$

$$\frac{\Xi \parallel \Delta, x : \tau_1 \vdash e : \tau_2}{\Xi \parallel \Delta \vdash \lambda x.e : \tau_1 \to \tau_2}\ \text{T\_ABS} \qquad \frac{\Xi \parallel \Delta, x : \tau_1 \to \tau_2 \vdash v : \tau_1 \to \tau_2}{\Xi \parallel \Delta \vdash \mathsf{fix}\, x.v : \tau_1 \to \tau_2}\ \text{T\_FIX}$$

$$\frac{\Xi \parallel \Delta \vdash v : \tau}{\Xi \parallel \Delta \vdash \mathsf{return}\, v : \tau}\ \text{T\_RETURN} \qquad \frac{\Xi \parallel \Delta \vdash e_1 : \tau_1 \quad \Xi \parallel \Delta, x : \tau_1 \vdash e_2 : \tau_2}{\Xi \parallel \Delta \vdash \mathsf{let}\, x = e_1 \,\mathsf{in}\, e_2 : \tau_2}\ \text{T\_LET}$$

$$\frac{\Xi \parallel \Delta \vdash v_1 : \tau_1 \to \tau_2 \quad \Xi \parallel \Delta \vdash v_2 : \tau_1}{\Xi \parallel \Delta \vdash v_1\, v_2 : \tau_2}\ \text{T\_APP} \qquad \frac{\Xi \parallel \Delta \vdash v : \mathsf{n} \quad \forall i \in [1, n].\ \Xi \parallel \Delta \vdash e_i : \tau}{\Xi \parallel \Delta \vdash \mathsf{case}(v; e_1, \cdots, e_n) : \tau}\ \text{T\_CASE}$$

$$\frac{\Xi \ni \sigma : B \rightsquigarrow E \quad \Xi \parallel \Delta \vdash v : B \quad \Xi \parallel \Delta, x : E \vdash e : \tau}{\Xi \parallel \Delta \vdash \sigma(v; x.\, e) : \tau}\ \text{T\_OP}$$

Figure 5: Type system.

2.3.4 Effect Trees

Definition 26 (Effect Trees for EPCF Computations)**.** Given an operation signature Ξ and a type τ, the tree constructor signature S^{Ξ}_{τ} is defined as follows:

$$\begin{array}{rcll} S^{\Xi}_{\tau}(\sigma) & \overset{\text{def}}{=} & n+1 & \forall \sigma : B \rightsquigarrow \mathsf{n} \in \Xi \\ S^{\Xi}_{\tau}(\mathsf{return}\, v) & \overset{\text{def}}{=} & 0 & \forall v \text{ such that } \Xi \parallel \emptyset \vdash v : \tau \\ S^{\Xi}_{\tau}(c) & \overset{\text{def}}{=} & 0 & \forall c \end{array}$$

Given a term e such that $\Xi \parallel \emptyset \vdash e : \tau$, the effect tree of e, denoted by $\mathbf{ET}(e)$, is a tree in $\mathbf{Tree}_{S^{\Xi}_{\tau}}$ defined by the following (possibly infinite) process:

- if $e \longrightarrow^{\omega}$, then $\mathbf{ET}(e) = \bot$;
- if $e \longrightarrow^{*} \mathsf{return}\, v$, then $\mathbf{ET}(e) = \mathsf{return}\, v$; and
- if $e \longrightarrow^{*} \sigma(c; x.\, e')$ and $\sigma : B \rightsquigarrow \mathsf{n} \in \Xi$, then $\mathbf{ET}(e) = \sigma(c, \mathbf{ET}(e'[\underline{1}/x]), \cdots, \mathbf{ET}(e'[\underline{\mathsf{n}}/x]))$.

2.3.5 Contextual Relations

Definition 27 (Program Contexts)**.** Term contexts, ranged over by P, and value contexts, ranged over by Q, are defined as follows:

$$\begin{array}{rcl} P & \overset{\text{def}}{=} & [\,] \mid \mathsf{return}\, Q \mid \mathsf{let}\, x = P \,\mathsf{in}\, e_2 \mid \mathsf{let}\, x = e_1 \,\mathsf{in}\, P \mid Q\, v_2 \mid v_1\, Q \mid \\ & & \mathsf{case}(Q; M_1, \cdots, M_n) \mid \mathsf{case}(V; M_1, \cdots, M_i, P, M_{i+1}, M_n) \mid \sigma(Q; x.\, M) \mid \sigma(V; x.\, P) \\ Q & \overset{\text{def}}{=} & \lambda x.P \mid \mathsf{fix}\, x.Q \end{array}$$

We write $P[e]$ for the term obtained by filling the hole $[\,]$ in P with the term e.

Definition 28 (Contextual Equivalence)**.** Terms e_1 and e_2 are contextually equivalent at type τ under typing context Δ and operation signature Ξ, written as $\Xi \parallel \Delta \vdash e_1 \overset{\mathsf{ctx}}{=} e_2 : \tau$, if:

- $\Xi \parallel \Delta \vdash e_1 : \tau$;
- $\Xi \parallel \Delta \vdash e_2 : \tau$; and
- for any term context P and ground type τ' such that $\Xi \parallel \emptyset \vdash P[e_1] : \tau'$ and $\Xi \parallel \emptyset \vdash P[e_2] : \tau'$, $\mathbf{ET}(P[e_1]) = \mathbf{ET}(P[e_2])$.

Definition 29 (Contextual Improvement)**.** Assume that $\Xi \parallel \Delta \vdash e_1 : \tau$ and $\Xi \parallel \Delta \vdash e_2 : \tau$. The contextual improvement $\Xi \parallel \Delta \vdash e_1 \overset{\mathsf{ctx}}{\leq} e_2 : \tau$ is the largest relation such that, for any term context P and ground type τ', if $\Xi \parallel \emptyset \vdash P[e_1] : \tau'$ and $\Xi \parallel \emptyset \vdash P[e_2] : \tau'$, then $\mathbf{ET}(P[e_1]) = \mathbf{ET}(P[e_2])$ and $\forall\, n, e_1'.\ P[e_1] \longrightarrow^n e_1' \implies \exists\, e_2'.\ P[e_2] \longrightarrow^n e_2'$.

2.4 Selective CPS Transformation from $\mathsf{HEPCF}^{\mathsf{ATM}}_{\square}$ to EPCF

Our CPS transformation is defined using the following shorthand:

- A sequence of entities $a_1, \cdots, a_n$ is abbreviated to $\overline{a}$, and its length is denoted by $|\overline{a}|$. Given $\overline{a}$, we write a_i to designate the i-th element of the sequence $\overline{a}$.
- Given a variable sequence $\overline{x} = x_1, \cdots, x_n$, we write $\lambda\overline{x}.e$ for the EPCF term $\lambda x_1.\mathsf{return}\,\lambda x_2.(\cdots(\mathsf{return}\,\lambda x_n.e)\cdots)$.
- Let X denote an EPCF term or value. Given an EPCF value v and $\overline{X}^{1\le i\le n}$ $(n > 0)$, we write $v\,\overline{X}^{1\le i\le n}$ for the EPCF term defined as follows:

$$\begin{array}{rcl@{\qquad}rcll} v\,w & \stackrel{\text{def}}{=} & v\,w & v\,(w,\overline{X}) & \stackrel{\text{def}}{=} & \mathsf{let}\,x = v\,w\,\mathsf{in}\,x\,\overline{X} & (\text{if } |\overline{X}| > 0) \\ v\,e & \stackrel{\text{def}}{=} & \mathsf{let}\,x = e\,\mathsf{in}\,v\,x & v\,(e,\overline{X}) & \stackrel{\text{def}}{=} & \mathsf{let}\,x = e\,\mathsf{in}\,v\,x\,\overline{X} & (\text{if } |\overline{X}| > 0) \end{array}$$

  where the variable x is assumed to be fresh. Similarly, given a term e and $\overline{X}^{1\le i\le n}$ $(n > 0)$, $e\,\overline{X}^{1\le i\le n}$ means the EPCF term

$$\mathsf{let}\,x = e\,\mathsf{in}\,x\,\overline{X}^{1\le i\le n}$$

  for some fresh variable x.
- Given a type τ and a type sequence $\overline{\tau_i}^{1\le i\le n}$, we write $\overline{\tau_i \to}^{1\le i\le n}\,\tau$ for the type $\tau_1 \to \tau_2 \to \cdots \to \tau_n \to \tau$ (when $n = 0$, it denotes τ).

Definition 30 (Static Functions and Applications)**.** The static lambda calculus is defined by the following syntax:

$$\mathbf{t} \quad \stackrel{\text{def}}{=} \quad \mathbf{x} \mid \boldsymbol{\lambda}(\mathbf{x}_1, \cdots, \mathbf{x}_n).\,\mathbf{t} \mid \mathbf{t}@(\mathbf{t}_1, \cdots, \mathbf{t}_n) \mid e \mid v$$

where e and v are EPCF terms and values, respectively, that may refer to static variables $\mathbf{x}$ bounded in the enclosing context. A static application $(\boldsymbol{\lambda}(\mathbf{x}_1, \cdots, \mathbf{x}_n).\,\mathbf{t})@(\mathbf{t}_1, \cdots, \mathbf{t}_n)$ is identified with the β-reduction result $\mathbf{t}[\mathbf{x}_1 := \mathbf{t}_1, \cdots, \mathbf{x}_n := \mathbf{t}_n]$.

Definition 31 (Domains of Operation Signatures)**.** Given an operation signature Σ, its domain $dom(\Sigma)$ is the set of operations defined by induction on Σ as follows.

$$\begin{array}{lcl} dom(\emptyset) & \stackrel{\text{def}}{=} & \emptyset \\ dom(\Sigma \uplus \{\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, A\}) & \stackrel{\text{def}}{=} & dom(\Sigma) \uplus \{\sigma\} \end{array}$$

Assumption 2 (Order of Operations)**.** We assume that the set of all the operations is totally ordered by a binary relation $\preceq_{\mathsf{op}}$. We write $\sigma \preceq_{\mathsf{op}} \Sigma$ if $\forall\varsigma \in dom(\Sigma).\ \sigma \preceq_{\mathsf{op}} \varsigma$, and $\Sigma \preceq_{\mathsf{op}} \sigma$ if $\forall\varsigma \in dom(\Sigma).\ \varsigma \preceq_{\mathsf{op}} \sigma$.

Remark 1. Assumption 2 is imposed for making the CPS-transformation defined in Definition 32 deterministic.

Definition 32 (CPS Transformation of Types, Values, and Terms)**.** CPS transformation $[\![-]\!]$ from $\mathsf{HEPCF}^{\mathsf{ATM}}_{\square}$ to EPCF is defined in Figures 6, 7, 8, and 9, mapping

- value types T to EPCF types $[\![T]\!]$ (Figure 6),
- computation types C to EPCF types $[\![C]\!]$ (Figure 6),
- operation signatures Σ to functions that, given a EPCF type τ, return the EPCF type $[\![\Sigma]\!][\,\tau\,]$ (Figure 6),
- subtyping $T_1 <: T_2$, $C_1 <: C_2$, and $\Sigma_1 <: \Sigma_2$ to EPCF functions $[\![C_1 <: C_2]\!]$ or static functions $[\![T_1 <: T_2]\!]$, $[\![C_1 <: C_2]\!]^{\mathsf{e}}$, and $[\![\Sigma_1 <: \Sigma_2]\!]$ (Figure 7),
- typing derivations of values $\Gamma \vdash V : T$ to EPCF values $[\![V]\!]$ (Figure 8), and
- typing derivations of $\Gamma \vdash M : C$ to EPCF values $[\![M]\!]$ (Figure 8) or static functions $[\![M]\!]^{\mathsf{e}}$ (Figure 9).

The CPS transformation for values and terms is defined on their typing derivations, which are ranged over by $\mathcal{D}$. In referring to the typing derivations, we attach the derivations to $[\![-]\!]$, as $[\![V]\!]_{\mathcal{D}}$ and $[\![M]\!]^{\mathsf{e}}_{\mathcal{D}}$. If the typing derivation $\mathcal{D}$ is constructed by a typing rule HT_*, we may use HT_* instead of $\mathcal{D}$ (for example, $[\![M]\!]^{\mathsf{e}}_{\mathrm{HT\_VAR}}$ means $[\![M]\!]^{\mathsf{e}}_{\mathcal{D}}$ for some $\mathcal{D}$ that is constructed by the typing rule (HT_VAR)).

We write $[\![\Gamma]\!]$ for the EPCF typing context obtained by CPS-transforming the types of all the bindings of typing context Γ.

$\boxed{[\![T]\!]}$ **for value types**

$$
\begin{array}{rcl}
[\![B]\!] & \stackrel{\text{def}}{=} & B \\
[\![E]\!] & \stackrel{\text{def}}{=} & E \\
[\![T \to C]\!] & \stackrel{\text{def}}{=} & [\![T]\!] \to [\![C]\!]
\end{array}
$$

$\boxed{[\![C]\!]}$ **for computation types**

$$
\begin{array}{rcl}
[\![\Sigma \triangleright T \,/\, \square]\!] & \stackrel{\text{def}}{=} & [\![\Sigma]\!][\, 1 \to [\![T]\!]\,] \\
[\![\Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}]\!] & \stackrel{\text{def}}{=} & [\![\Sigma]\!][\, ([\![T]\!] \to [\![\square(\Sigma)]\!][\, [\![C^{\mathrm{ini}}]\!]\,]) \to [\![C^{\mathrm{fin}}]\!]\,]
\end{array}
$$

$\boxed{[\![\Sigma]\!][\,\tau\,]}$ **for operation signatures**

$$
[\![\{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, C_i^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{fin}}\}^{1 \le i \le m} \uplus \{\varsigma_i : U_i^{\mathrm{par}} \rightsquigarrow U_i^{\mathrm{ari}} \,/\, \square\}^{1 \le i \le n}]\!][\,\tau\,] \stackrel{\text{def}}{=}
$$

$$
\overline{([\![T_i^{\mathrm{par}}]\!] \to ([\![T_i^{\mathrm{ari}}]\!] \to \overline{([\![U_j^{\mathrm{par}}]\!] \to [\![U_j^{\mathrm{ari}}]\!]) \to}^{1 \le j \le n} [\![C_i^{\mathrm{ini}}]\!]) \to [\![C_i^{\mathrm{fin}}]\!]) \to}^{1 \le i \le m} \overline{([\![U_i^{\mathrm{par}}]\!] \to [\![U_i^{\mathrm{ari}}]\!]) \to}^{1 \le i \le n} \tau
$$

Figure 6: CPS transformation for types.

2.4.1 Auxiliary Definitions for Proofs

Definition 33 (Evaluation Contexts)**.** We define EPCF evaluation contexts, ranged over by E, as follows:

$$
E \stackrel{\text{def}}{=} [\,] \mid \mathsf{let}\, x = E \,\mathsf{in}\, e_2
$$

Given $n \ge 1$ and a computation type sequence $\overline{C} = C_1, \cdots, C_n$ such that $\forall\, i \in [1, n-1].\ C_i <: C_{i+1}$, we define the evaluation context $E^{\overline{C}}$ as follows:

$$
\begin{array}{rcl}
E^{C} & \stackrel{\text{def}}{=} & [\,] \\
E^{\overline{C}, C_1, C_2} & \stackrel{\text{def}}{=} & \mathsf{let}\, x_1 = E^{\overline{C}, C_1} \,\mathsf{in\ return}\, [\![C_1 <: C_2]\!] @ (x_1)\ .
\end{array}
$$

We write $E[e]$ for the term obtained by filling the hole of the evaluation context E with the term e. We also write $E[E']$ for the evaluation context obtained by filling the hole of the evaluation context E with the one E'.

Definition 34 (Full $\beta\eta$ Monadic Reduction)**.** We define a binary relation $\hookrightarrow$ over EPCF terms and over EPCF values, called *full $\beta\eta$ monadic reduction*, to be the reflexive, transitive, compatible closure satisfying the following axioms:

$$
\begin{array}{l}
\forall\, x, v, e.\ (\lambda x.e)\, v \hookrightarrow e[v/x] \\
\forall\, x, v.\ x \notin \mathit{fv}(v) \Longrightarrow \lambda x.v\, x \hookrightarrow v \\
\forall\, x, v, e.\ \mathsf{let}\, x = \mathsf{return}\, v \,\mathsf{in}\, e \hookrightarrow e[v/x] \\
\forall\, x, e.\ \mathsf{let}\, x = e \,\mathsf{in\ return}\, x \hookrightarrow e \\
\forall\, x, y, e_1, e_2, e_3.\ y \notin \mathit{fv}(e_3) \Longrightarrow \mathsf{let}\, x = (\mathsf{let}\, y = e_1 \,\mathsf{in}\, e_2) \,\mathsf{in}\, e_3 \hookrightarrow \mathsf{let}\, y = e_1 \,\mathsf{in\ let}\, x = e_2 \,\mathsf{in}\, e_3
\end{array}
$$

For value sequences $\overline{v_{1i}}^{1 \le i \le n}$ and $\overline{v_{2i}}^{1 \le i \le n}$, we write $\overline{v_{1i}}^{1 \le i \le n} \hookrightarrow \overline{v_{2i}}^{1 \le i \le n}$ if, for any $i \in [1, n]$, $v_{1i} \hookrightarrow v_{2i}$.

Definition 35 (Value Sequence Split)**.** Given $\overline{v}$ and Σ, $\mathit{split}(\overline{v}, \Sigma)$ returns a pair of value sequences $\overline{v^{\not\square}}$ and $\overline{v^{\square}}$ such that: $\overline{v} = \overline{v^{\not\square}}, \overline{v^{\square}}$; $|\overline{v^{\not\square}}| = |\not\square(\Sigma)|$; and $|\overline{v^{\square}}| = |\square(\Sigma)|$.

$\boxed{[\![T_1 <: T_2]\!]}$ **for value subtyping**

$$
\begin{array}{rcl}
[\![B <: B]\!] & \stackrel{\text{def}}{=} & \boldsymbol{\lambda}\mathbf{x}_1.\,\mathbf{x}_1 \\
[\![E <: E]\!] & \stackrel{\text{def}}{=} & \boldsymbol{\lambda}\mathbf{x}_1.\,\mathbf{x}_1 \\
[\![T_1 \to C_1 <: T_2 \to C_2]\!] & \stackrel{\text{def}}{=} & \boldsymbol{\lambda}\mathbf{x}_1.\,\lambda y_2.\mathsf{let}\, z_1 = \mathbf{x}_1\,([\![T_2 <: T_1]\!]@(y_2))\,\mathsf{in\,return}\,[\![C_1 <: C_2]\!]@(z_1)
\end{array}
$$

$\boxed{[\![C_1 <: C_2]\!]}$ **for computation subtyping**

$$
\begin{array}{rcl}
[\![C_1 <: C_2]\!] & \stackrel{\text{def}}{=} & \boldsymbol{\lambda}\mathbf{x}_1.\,\lambda\overline{h_2}, k_2.[\![C_1 <: C_2]\!]^{\mathsf{e}}@((\boldsymbol{\lambda}(\overline{\mathbf{h}_1}, \mathbf{k}_1).\,\mathbf{x}_1\,\overline{\mathbf{h}_1}\,\mathbf{k}_1), \overline{h_2}, k_2) \\
 & & (\text{where } |\overline{h_2}| = |C_2.\Sigma| \text{ and } |\overline{\mathbf{h}_1}| = |C_1.\Sigma|)
\end{array}
$$

$\boxed{[\![C_1 <: C_2]\!]^{\mathsf{e}}}$ **for computation subtyping**

$$
\begin{array}{l}
[\![\Sigma_1 \rhd T_1 \,/\, \Box <: \Sigma_2 \rhd T_2 \,/\, \Box]\!]^{\mathsf{e}} \stackrel{\text{def}}{=} \\
\boldsymbol{\lambda}(\mathbf{f}_1, \overline{\mathbf{h}_2}, \mathbf{k}_2).\,\mathsf{let}\, x_1 = \mathbf{f}_1@([\![\Sigma_2 <: \Sigma_1]\!]@(\overline{\mathbf{h}_2}), \mathbf{k}_2)\,\mathsf{in\,return}\,[\![T_1 <: T_2]\!]@(x_1) \quad (\text{where } |\overline{\mathbf{h}_2}| = |\Sigma_2|) \\
[\![\Sigma_1 \rhd T_1 \,/\, \Box <: \Sigma_2 \rhd T_2 \,/\, C_2^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}]\!]^{\mathsf{e}} \stackrel{\text{def}}{=} \\
\boldsymbol{\lambda}(\mathbf{f}_1, \overline{\mathbf{h}_2^{\not\Box}}, \overline{\mathbf{h}_2^{\Box}}, \mathbf{k}_2).\,\mathsf{let}\, x_1 = \mathbf{f}_1@([\![\Sigma_2 <: \Sigma_1]\!]@(\overline{\mathbf{h}_2^{\not\Box}}, \overline{\mathbf{h}_2^{\Box}}), \underline{1})\,\mathsf{in\,let}\, y_2 = \mathbf{k}_2\,([\![T_1 <: T_2]\!]@(x_1))\,\overline{\mathbf{h}_2^{\Box}}\,\mathsf{in\,return}\,[\![C_2^{\mathrm{ini}} <: C_2^{\mathrm{fin}}]\!]@(y_2) \\
\qquad (\text{where } |\overline{\mathbf{h}_2^{\not\Box}}| = |\not\Box(\Sigma_2)| \text{ and } |\overline{\mathbf{h}_2^{\Box}}| = |\Box(\Sigma_2)|) \\
[\![\Sigma \rhd T \,/\, C^{\mathrm{ini}} \Rightarrow C_1^{\mathrm{fin}} <: \Sigma \rhd T \,/\, C^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}]\!]^{\mathsf{e}} \stackrel{\text{def}}{=} \\
\boldsymbol{\lambda}(\mathbf{f}_1, \overline{\mathbf{h}_2}, \mathbf{k}_2).\,\mathsf{let}\, x_1 = \mathbf{f}_1@(\overline{\mathbf{h}_2}, \mathbf{k}_2)\,\mathsf{in\,return}\,[\![C_1^{\mathrm{fin}} <: C_2^{\mathrm{fin}}]\!]@(x_1) \quad (\text{where } |\overline{\mathbf{h}_2}| = |\Sigma|) \\
[\![\Sigma_1 \rhd T_1 \,/\, C_1^{\mathrm{ini}} \Rightarrow C_1^{\mathrm{fin}} <: \Sigma_2 \rhd T_2 \,/\, C_2^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}]\!]^{\mathsf{e}} \stackrel{\text{def}}{=} \\
\boldsymbol{\lambda}(\mathbf{f}_1, \overline{\mathbf{h}_2^{\not\Box}}, \overline{\mathbf{h}_2^{\Box}}, \mathbf{k}_2).\,\mathsf{let}\, x_1 = \mathbf{f}_1@([\![\not\Box(\Sigma_2) <: \not\Box(\Sigma_1)]\!]@(\overline{\mathbf{h}_2^{\not\Box}}), \overline{\mathbf{h}_2^{\Box}}, v)\,\mathsf{in\,return}\,[\![C_1^{\mathrm{fin}} <: C_2^{\mathrm{fin}}]\!]@(x_1) \\
\qquad (\text{where } \Sigma_1 \neq \Sigma_2 \ \lor\ T_1 \neq T_2 \ \lor\ C_1^{\mathrm{ini}} \neq C_2^{\mathrm{ini}} \text{ and } |\overline{\mathbf{h}_2^{\not\Box}}| = |\not\Box(\Sigma_2)| \text{ and } |\overline{\mathbf{h}_2^{\Box}}| = |\overline{h_1^{\Box}}| = |\Box(\Sigma_2)| \text{ and} \\
\qquad v = \lambda y_1, \overline{h_1^{\Box}}.\mathsf{let}\, z_2 = \mathbf{k}_2\,([\![T_1 <: T_2]\!]@(y_1))\,\overline{h_1^{\Box}}\,\mathsf{in\,return}\,[\![C_2^{\mathrm{ini}} <: C_1^{\mathrm{ini}}]\!]@(z_2))
\end{array}
$$

$\boxed{[\![\Sigma_1 <: \Sigma_2]\!]}$ **for operation signature subtyping**

$$
\begin{array}{llcl}
 & [\![\Sigma_1 <: \Sigma_2]\!] & \stackrel{\text{def}}{=} & \boldsymbol{\lambda}\overline{\mathbf{h}_1}.\,\overline{v^{\sigma_i}}^{1 \le i \le m}, \overline{v^{\varsigma_i}}^{1 \le i \le n} \\
\text{with} \quad (\forall i \in [1, m]) & v^{\sigma_i} & \stackrel{\text{def}}{=} & \lambda x_2, k_2.\mathsf{let}\, x_1 = \mathbf{h}_1^{\sigma_i}\,([\![T_{i2}^{\mathrm{par}} <: T_{i1}^{\mathrm{par}}]\!]@(x_2))\,v_i\,\mathsf{in\,return}\,[\![C_{i1}^{\mathrm{fin}} <: C_{i2}^{\mathrm{fin}}]\!]@(x_1) \\
(\forall i \in [1, m]) & v_i & \stackrel{\text{def}}{=} & \lambda y_1, \overline{h_{1j}}^{1 \le j \le |\Box(\Sigma_1)|}.\mathsf{let}\, y_2 = k_2\,([\![T_{i1}^{\mathrm{ari}} <: T_{i2}^{\mathrm{ari}}]\!]@(y_1))\,\overline{w^{\varsigma_j}}^{1 \le j \le n}\,\mathsf{in\,return}\,[\![C_{i2}^{\mathrm{ini}} <: C_{i1}^{\mathrm{ini}}]\!]@(y_2) \\
(\forall i \in [1, n]) & w^{\varsigma_i} & \stackrel{\text{def}}{=} & \lambda z_2.\mathsf{let}\, z_1 = h_1^{\varsigma_i}\,([\![U_{i2}^{\mathrm{par}} <: U_{i1}^{\mathrm{par}}]\!]@(z_2))\,\mathsf{in\,return}\,[\![U_{i1}^{\mathrm{ari}} <: U_{i2}^{\mathrm{ari}}]\!]@(z_1) \\
(\forall i \in [1, n]) & v^{\varsigma_i} & \stackrel{\text{def}}{=} & \lambda x_2.\mathsf{let}\, x_1 = \mathbf{h}_1^{\varsigma_i}\,([\![U_{i2}^{\mathrm{par}} <: U_{i1}^{\mathrm{par}}]\!]@(x_2))\,\mathsf{in\,return}\,[\![U_{i1}^{\mathrm{ari}} <: U_{i2}^{\mathrm{ari}}]\!]@(x_1)
\end{array}
$$

where:

- $\Sigma_1 = \{\sigma_i : T_{i1}^{\mathrm{par}} \rightsquigarrow T_{i1}^{\mathrm{ari}} \,/\, C_{i1}^{\mathrm{ini}} \Rightarrow C_{i1}^{\mathrm{fin}}\}^{1 \le i \le m} \uplus \{\varsigma_i : U_{i1}^{\mathrm{par}} \rightsquigarrow U_{i1}^{\mathrm{ari}} \,/\, \Box\}^{1 \le i \le n} \uplus \Sigma$;
- $\Sigma_2 = \{\sigma_i : T_{i2}^{\mathrm{par}} \rightsquigarrow T_{i2}^{\mathrm{ari}} \,/\, C_{i2}^{\mathrm{ini}} \Rightarrow C_{i2}^{\mathrm{fin}}\}^{1 \le i \le m} \uplus \{\varsigma_i : U_{i2}^{\mathrm{par}} \rightsquigarrow U_{i2}^{\mathrm{ari}} \,/\, \Box\}^{1 \le i \le n}$;
- $|\overline{\mathbf{h}_1}| = |\Sigma_1|$;
- $\overline{\sigma_i}^{1 \le i \le m}$ and $\overline{\varsigma_i}^{1 \le i \le n}$ are ordered;
- the sequence $\overline{\mathbf{h}_1}$ is split into $\overline{\mathbf{h}_1^{\sigma_i}}^{1 \le i \le m}$, $\overline{\mathbf{h}_1^{\varsigma_i}}^{1 \le i \le n}$, and the remaining (which correspond to the implementations of $\overline{\sigma_i}^{1 \le i \le m}$, $\overline{\varsigma_i}^{1 \le i \le n}$, and the operations in Σ, respectively); and
- the sequence $\overline{h_{1j}}^{1 \le j \le |\Box(\Sigma_1)|}$ is split into $\overline{h_1^{\varsigma_i}}^{1 \le i \le n}$ and the remaining (which correspond to $\overline{\varsigma_i}^{1 \le i \le n}$ and the operations in $\Box(\Sigma)$, respectively).

Figure 7: Subtyping coercions.

$\boxed{[\![V]\!]}$ **for values**

$$
\begin{array}{rcl}
[\![x]\!]_{\text{HT\_VAR}} & \stackrel{\text{def}}{=} & x \\
[\![c]\!]_{\text{HT\_CONST}} & \stackrel{\text{def}}{=} & c \\
[\![\underline{\mathsf{n}}]\!]_{\text{HT\_ECONST}} & \stackrel{\text{def}}{=} & \underline{\mathsf{n}} \\
[\![\lambda x.M]\!]_{\text{HT\_ABS}} & \stackrel{\text{def}}{=} & \lambda x.\mathsf{return}\,[\![M]\!] \\
[\![\mathsf{fix}\,x.\,V]\!]_{\text{HT\_FIX}} & \stackrel{\text{def}}{=} & \mathsf{fix}\,x.[\![V]\!] \\
[\![V]\!]_{\mathcal{D}} & \stackrel{\text{def}}{=} & [\![T <: U]\!]@([\![V]\!]_{\mathcal{D}'})
\end{array}
$$

where

$$
\frac{\Gamma \vdash_{\mathcal{D}'} V : T \quad T <: U}{\Gamma \vdash_{\mathcal{D}} V : U}\ (\text{HS\_SUBV})
$$

$\boxed{[\![M]\!]}$ **for thunks** (assume that $\Gamma \vdash M : \Sigma \triangleright T \,/\, A$ and $|\overline{h}| = |\Sigma|$)

$$
[\![M]\!] \stackrel{\text{def}}{=} \lambda\overline{h}, k.[\![M]\!]^{\mathsf{e}}@(\overline{h}, k)
$$

Figure 8: CPS transformation for values and thunk terms.

$\boxed{\llbracket M \rrbracket^{\mathsf{e}}}$ **for terms**

(assume $\Gamma \vdash M : \Sigma \triangleright T \,/\, A$ and $|\overline{\mathbf{h}}| = |\Sigma|$ and $|\overline{\mathbf{h}^{\not\square}}| = |\not\square(\Sigma)|$ and $|\overline{\mathbf{h}^{\square}}| = |\overline{h^{\square}}| = |\square(\Sigma)|$)

$$
\begin{array}{rcl}
\llbracket \mathsf{return}\ V \rrbracket^{\mathsf{e}}_{\text{HT\_RETURN}} & \stackrel{\text{def}}{=} & \boldsymbol{\lambda}(\overline{\mathbf{h}}, \mathbf{k}).\, \mathsf{return}\, \llbracket V \rrbracket \\
\llbracket \mathsf{let}\, x = M \,\mathsf{in}\, N \rrbracket^{\mathsf{e}}_{\text{HT\_LETATM}} & \stackrel{\text{def}}{=} & \boldsymbol{\lambda}(\overline{\mathbf{h}^{\not\square}}, \overline{\mathbf{h}^{\square}}, \mathbf{k}).\, \llbracket M \rrbracket^{\mathsf{e}} @ (\overline{\mathbf{h}^{\not\square}}, \mathbf{h}^{\square}, \lambda x, \overline{h^{\square}}. \llbracket N \rrbracket^{\mathsf{e}} @ (\overline{\mathbf{h}^{\not\square}}, \overline{h^{\square}}, \mathbf{k})) \\
\llbracket \mathsf{let}\, x = M \,\mathsf{in}\, N \rrbracket^{\mathsf{e}}_{\text{HT\_LET}} & \stackrel{\text{def}}{=} & \boldsymbol{\lambda}(\overline{\mathbf{h}}, \mathbf{k}).\, \mathsf{let}\, x = \llbracket M \rrbracket^{\mathsf{e}} @ (\overline{\mathbf{h}}, \underline{1}) \,\mathsf{in}\, \llbracket N \rrbracket^{\mathsf{e}} @ (\overline{\mathbf{h}}, \mathbf{k}) \\
\llbracket V\ W \rrbracket^{\mathsf{e}}_{\text{HT\_APP}} & \stackrel{\text{def}}{=} & \boldsymbol{\lambda}(\overline{\mathbf{h}}, \mathbf{k}).\, \llbracket V \rrbracket\, \llbracket W \rrbracket\, \overline{\mathbf{h}}\, \mathbf{k} \\
\llbracket \mathsf{case}(V; M_1, \cdots, M_n) \rrbracket^{\mathsf{e}}_{\text{HT\_CASE}} & \stackrel{\text{def}}{=} & \boldsymbol{\lambda}(\overline{\mathbf{h}}, \mathbf{k}).\, \mathsf{case}(\llbracket V \rrbracket; \llbracket M_1 \rrbracket^{\mathsf{e}} @ (\overline{\mathbf{h}}, \mathbf{k}), \cdots, \llbracket M_n \rrbracket^{\mathsf{e}} @ (\overline{\mathbf{h}}, \mathbf{k})) \\
\llbracket \sigma_i(V; x.\, M) \rrbracket^{\mathsf{e}}_{\text{HT\_OPATM}} & \stackrel{\text{def}}{=} & \boldsymbol{\lambda}(\overline{\mathbf{h}^{\not\square}}, \overline{\mathbf{h}^{\square}}, \mathbf{k}).\, \mathbf{h}^{\not\square}_i\, \llbracket V \rrbracket\, \lambda x, \overline{h^{\square}}. \llbracket M \rrbracket^{\mathsf{e}} @ (\overline{\mathbf{h}^{\not\square}}, \overline{h^{\square}}, \mathbf{k}) \\
& & \qquad (\text{where } \not\square(\Sigma) = \Sigma_1 \uplus \{\sigma_i : T^{\mathrm{par}}_i \rightsquigarrow T^{\mathrm{ari}}_i \,/\, C^{\mathrm{ini}}_i \Rightarrow C^{\mathrm{fin}}_i\} \uplus \Sigma_2 \text{ and} \\
& & \qquad \Sigma_1 \preceq_{\mathsf{op}} \sigma_i \text{ and } \sigma_i \preceq_{\mathsf{op}} \Sigma_2 \text{ and } |\Sigma_1| = i - 1) \\
\llbracket \sigma_i(V; x.\, M) \rrbracket^{\mathsf{e}}_{\text{HT\_OP}} & \stackrel{\text{def}}{=} & \boldsymbol{\lambda}(\overline{\mathbf{h}^{\not\square}}, \overline{\mathbf{h}^{\square}}, \mathbf{k}).\, \mathsf{let}\, x = \mathbf{h}^{\square}_i\, \llbracket V \rrbracket \,\mathsf{in}\, \llbracket M \rrbracket^{\mathsf{e}} @ (\overline{\mathbf{h}^{\not\square}}, \overline{\mathbf{h}^{\square}}, \mathbf{k}) \\
& & \qquad (\text{where } \square(\Sigma) = \Sigma_1 \uplus \{\sigma_i : T^{\mathrm{par}}_i \rightsquigarrow T^{\mathrm{ari}}_i \,/\, \square\} \uplus \Sigma_2 \text{ and} \\
& & \qquad \Sigma_1 \preceq_{\mathsf{op}} \sigma_i \text{ and } \sigma_i \preceq_{\mathsf{op}} \Sigma_2 \text{ and } |\Sigma_1| = i - 1)
\end{array}
$$

$$
\begin{array}{rcl}
\llbracket \mathsf{with}\, H \,\mathsf{handle}\, M \rrbracket^{\mathsf{e}}_{\text{HT\_HANDLE}} & \stackrel{\text{def}}{=} & \boldsymbol{\lambda}(\overline{\mathbf{h}}, \mathbf{k}).\, \llbracket M \rrbracket^{\mathsf{e}} @ (\overline{w^{\not\square}_i}^{1 \le i \le m}, \overline{w^{\square}_i}^{1 \le i \le n}, \lambda x, \overline{h_i}^{1 \le i \le n}. \mathsf{return}\, \llbracket L \rrbracket)\, \overline{\mathbf{h}}\, \mathbf{k} \text{ , where} \\
H & = & \{\mathsf{return}\, x \mapsto L\} \uplus \{\sigma_i(y_i; k_i) \mapsto M_i\}^{1 \le i \le m} \uplus \{\varsigma_i(z_i) \mapsto N_i\}^{1 \le i \le n} \\
\forall i \in [1, m]. & & \Gamma, y_i : T^{\mathrm{par}}_i, k_i : T^{\mathrm{ari}}_i \to C^{\mathrm{ini}}_i \vdash M_i : C^{\mathrm{fin}}_i \\
\forall i \in [1, n]. & & \Gamma, z_i : U^{\mathrm{par}}_i \vdash N_i : \Sigma_i \triangleright U^{\mathrm{ari}}_i \,/\, \square \\
\forall i \in [1, m].\ w^{\not\square}_i & \stackrel{\text{def}}{=} & \lambda y_i, k'_i. \mathsf{let}\, k_i = \mathsf{return}\, (\lambda y, \overline{h}, k. k'_i\, y\, \overline{v_{ij}}^{1 \le j \le n}\, \overline{h}\, k) \,\mathsf{in}\, \mathsf{return}\, \llbracket M_i \rrbracket \quad (\text{where } |\overline{h}| = |C^{\mathrm{ini}}_i.\Sigma|) \\
\forall i \in [1, m], j \in [1, n].\ v_{ij} & \stackrel{\text{def}}{=} & \lambda z_j. \llbracket \Sigma_j \triangleright U^{\mathrm{ari}}_j \,/\, \square <: C^{\mathrm{ini}}_i.\Sigma \triangleright U^{\mathrm{ari}}_j \,/\, \square \rrbracket^{\mathsf{e}} @ (\llbracket N_j \rrbracket^{\mathsf{e}}, \overline{h}, \underline{1}) \\
\forall i \in [1, n].\ w^{\square}_i & \stackrel{\text{def}}{=} & \lambda z_i. \llbracket \Sigma_i \triangleright U^{\mathrm{ari}}_i \,/\, \square <: \Sigma \triangleright U^{\mathrm{ari}}_i \,/\, \square \rrbracket^{\mathsf{e}} @ (\llbracket N_i \rrbracket^{\mathsf{e}}, \overline{\mathbf{h}}, \underline{1})
\end{array}
$$

$$
\llbracket M \rrbracket^{\mathsf{e}}_{\mathcal{D}} \stackrel{\text{def}}{=} \boldsymbol{\lambda}(\overline{\mathbf{h}}, \mathbf{k}).\, \llbracket C <: \Sigma \triangleright T \,/\, A \rrbracket^{\mathsf{e}} @ (\llbracket M \rrbracket^{\mathsf{e}}_{\mathcal{D}'}, \overline{\mathbf{h}}, \mathbf{k}) \qquad \text{(if condition (*) holds)}
$$

where condition (*) is:

$$
\frac{\Gamma \vdash_{\mathcal{D}'} M : C \qquad C <: \Sigma \triangleright T \,/\, A}{\Gamma \vdash_{\mathcal{D}} M : \Sigma \triangleright T \,/\, A}\ (\text{HS\_SUBC})
$$

Figure 9: CPS transformation for terms.

3 Proofs

3.1 Type Soundness of $\mathsf{HEPCF}^{\mathsf{ATM}}_{\square}$

Lemma 1 (Weakening)**.** Assume that $dom(\Gamma_2) \cap dom(\Gamma_1, \Gamma_3)$ is empty.

- If $\Gamma_1, \Gamma_3 \vdash V : T$, then $\Gamma_1, \Gamma_2, \Gamma_3 \vdash V : T$.
- If $\Gamma_1, \Gamma_3 \vdash M : C$, then $\Gamma_1, \Gamma_2, \Gamma_3 \vdash M : C$.

Proof. Straightforward by mutual induction on the typing derivations. □

Lemma 2 (Value Substitution)**.** Assume that $\Gamma_1 \vdash W : U$.

- If $\Gamma_1, x : U, \Gamma_2 \vdash V : T$, then $\Gamma_1, \Gamma_2 \vdash V[W/x] : T$.
- If $\Gamma_1, x : U, \Gamma_2 \vdash M : C$, then $\Gamma_1, \Gamma_2 \vdash M[W/x] : C$.

Proof. Straightforward by mutual induction on the typing derivations. The case for (HT_Var) rests on Lemma 1. □

Lemma 3 (Inversion of Subtyping)**.**

- If $T <: B$, then $T = B$.
- If $T <: \mathsf{n}$, then $T = \mathsf{n}$.
- If $T <: T' \to C'$, then $T = T'' \to C''$ for some T'' and C'' such that $T' <: T''$ and $C'' <: C'$.
- If $C <: D$, then $D.\Sigma <: C.\Sigma$ and $C.T <: D.T$ and $C.A <: D.A$. Furthermore, if $C.A \neq \square$, then $\square(C.\Sigma) <: \square(D.\Sigma)$.
- If $\Sigma_1 <: \Sigma_2$ and $\sigma : T_2^{\mathrm{par}} \rightsquigarrow T_2^{\mathrm{ari}} \,/\, A_2 \in \Sigma_2$, then there exist some T_1^{par}, T_1^{ari}, and A_1 such that
  - $\sigma : T_1^{\mathrm{par}} \rightsquigarrow T_1^{\mathrm{ari}} \,/\, A_1 \in \Sigma_1$,
  - $T_2^{\mathrm{par}} <: T_1^{\mathrm{par}}$,
  - $T_1^{\mathrm{ari}} <: T_2^{\mathrm{ari}}$, and
  - $A_1 <: A_2$.
- If $A <: \square$, then $A = \square$.
- If $\square <: C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$, then $C^{\mathrm{ini}} <: C^{\mathrm{fin}}$.
- If $C_1^{\mathrm{ini}} \Rightarrow C_1^{\mathrm{fin}} <: C_2^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}$, then $C_2^{\mathrm{ini}} <: C_1^{\mathrm{ini}}$ and $C_1^{\mathrm{fin}} <: C_2^{\mathrm{fin}}$.

Proof. Straightforward by case analysis on the subtyping derivations. □

Lemma 4 (Reflexivity of Subtyping)**.**

- For any T, $T <: T$.
- For any C, $C <: C$.
- For any Σ, $\Sigma <: \Sigma$.
- For any A, $A <: A$.

Proof. Straightforward by mutual induction on the structures of T, C, Σ, and A. □

Lemma 5 (Transitivity of Subtyping)**.**

- If $T_1 <: T_2$ and $T_2 <: T_3$, then $T_1 <: T_3$.
- If $C_1 <: C_2$ and $C_2 <: C_3$, then $C_1 <: C_3$.

- If $\Sigma_1 <: \Sigma_2$ and $\Sigma_2 <: \Sigma_3$, then $\Sigma_1 <: \Sigma_3$.
- If $A_1 <: A_2$ and $A_2 <: A_3$, then $A_1 <: A_3$.

Proof. Straightforward by induction on the total sizes of the tuples (T_1, T_2, T_3), (C_1, C_2, C_3), $(\Sigma_1, \Sigma_2, \Sigma_3)$, and (A_1, A_2, A_3) with case analysis on the subtyping derivations. The only interesting cases are the case of computation subtyping and the case that $A_1 <: A_2$ is derived by (HS_AnsEmb) and $A_2 <: A_3$ is derived by (HS_AnsMod).

For the former, assume that $C_1 <: C_2$ and $C_2 <: C_3$. By Lemma 3,

- $C_2.\Sigma <: C_1.\Sigma$,
- $C_1.T <: C_2.T$,
- $C_1.A <: C_2.A$,
- $C_1.A \neq \square \Longrightarrow \square(C_1.\Sigma) <: \square(C_2.\Sigma)$,
- $C_3.\Sigma <: C_2.\Sigma$,
- $C_2.T <: C_3.T$,
- $C_2.A <: C_3.A$, and
- $C_2.A \neq \square \Longrightarrow \square(C_2.\Sigma) <: \square(C_3.\Sigma)$.

By the IHs, we have

- $C_3.\Sigma <: C_1.\Sigma$,
- $C_1.T <: C_3.T$, and
- $C_1.A <: C_3.A$.

Finally, we show that
$$C_1.A \neq \square \Longrightarrow \square(C_1.\Sigma) <: \square(C_3.\Sigma) .$$
Assume that $C_1.A \neq \square$. By Lemma 3, $C_2.A \neq \square$. Therefore, by the assumptions, $\square(C_1.\Sigma) <: \square(C_2.\Sigma)$ and $\square(C_2.\Sigma) <: \square(C_3.\Sigma)$. Thus, by the IH, $\square(C_1.\Sigma) <: \square(C_3.\Sigma)$.

For the latter, assume that

- $A_1 = \square$,
- $A_2 = C_2^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}$,
- $A_3 = C_3^{\mathrm{ini}} \Rightarrow C_3^{\mathrm{fin}}$,
- $C_2^{\mathrm{ini}} <: C_2^{\mathrm{fin}}$,
- $C_3^{\mathrm{ini}} <: C_2^{\mathrm{ini}}$, and
- $C_2^{\mathrm{fin}} <: C_3^{\mathrm{fin}}$

for some C_2^{ini}, C_2^{fin}, C_3^{ini}, and C_3^{fin}. By the IH on the subtyping derivations $C_3^{\mathrm{ini}} <: C_2^{\mathrm{ini}}$ and $C_2^{\mathrm{ini}} <: C_2^{\mathrm{fin}}$, we have $C_3^{\mathrm{ini}} <: C_2^{\mathrm{fin}}$. By the IH on the subtyping derivations $C_3^{\mathrm{ini}} <: C_2^{\mathrm{fin}}$ and $C_2^{\mathrm{fin}} <: C_3^{\mathrm{fin}}$, we have $C_3^{\mathrm{ini}} <: C_3^{\mathrm{fin}}$. Thus, by (HS_AmsEmb), $\square <: C_3^{\mathrm{ini}} \Rightarrow C_3^{\mathrm{fin}}$. Because $A_1 = \square$, we have the conclusion. $\square$

Lemma 6 (Asymmetry of Subtyping)**.**

- If $T <: U$ and $U <: T$, then $T = U$.
- If $C <: D$ and $D <: C$, then $C = D$.
- If $\Sigma_1 <: \Sigma_2$ and $\Sigma_2 <: \Sigma_1$, then $\Sigma_1 = \Sigma_2$.
- If $A_1 <: A_2$ and $A_2 <: A_1$, then $A_1 = A_2$.

Proof. Straightforward by mutual induction on the subtyping derivations. The case for computation subtyping rests on Lemma 4. □

Lemma 7 (Canonical Forms)**.** Assume that $\emptyset \vdash V : T$.

- If $T = B$, then $V = c$ for some c such that $ty(c) = B$.
- If $T = \mathsf{n}$, then $V = \underline{\mathsf{i}}$ for some i such that $0 < i \leq n$.
- If $T = T' \rightarrow C'$, then $V = \lambda x.M$ for some x and M, or $V = \mathsf{fix}\, x.\, V'$ for some x and V'.

Proof. Straightforward by induction on the typing derivation. Note that:

- the case for (HT_CONST) rests on Assumption 1, which states that, for any c, $ty(c) = B$ for some B; and
- the case for (HT_SUBV) rests on Lemma 3.

□

Lemma 8 (Inversion of Operation Calls)**.** If $\Gamma \vdash \sigma(V; x.\, M) : C$, then one of the following holds:

- there exist some T^{par}, T^{ari}, Σ, T, and A such that
  - $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, \square \in \Sigma$,
  - $\Gamma \vdash V : T^{\mathrm{par}}$,
  - $\Gamma, x : T^{\mathrm{ari}} \vdash M : \Sigma \triangleright T \,/\, A$, and
  - $\Sigma \triangleright T \,/\, A <: C$;

  or
- there exist some T^{par}, T^{ari}, C^{ini}, C^{fin}, Σ, T, and C' such that
  - $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}} \in \Sigma$,
  - $\Gamma \vdash V : T^{\mathrm{par}}$,
  - $\Gamma, x : T^{\mathrm{ari}} \vdash M : \Sigma \triangleright T \,/\, C' \Rightarrow C^{\mathrm{ini}}$, and
  - $\Sigma \triangleright T \,/\, C' \Rightarrow C^{\mathrm{fin}} <: C$.

Proof. Straightforward by induction on the typing derivation with Lemmas 4 and 5. □

Lemma 9 (Progress)**.** If $\emptyset \vdash M : C$, then one of the following holds:

- $M = \mathsf{return}\, V$ for some V;
- $M = \sigma(V; x.\, M')$ for some σ, V, x, and M'; or
- $M \longrightarrow M'$ for some M'.

Proof. By induction on the typing derivation. We proceed with case analysis on the typing rule applied last to derive $\emptyset \vdash M : C$.

Case (HT_RETURN): Obvious.

Case (HT_SUBC): By the IH.

Case (HT_LET): We are given

$$\frac{\emptyset \vdash M_1 : \Sigma \triangleright T_1 \,/\, \square \quad x : T_1 \vdash M_2 : \Sigma \triangleright T_2 \,/\, A_2}{\emptyset \vdash \mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2 : \Sigma \triangleright T_2 \,/\, A}$$

for some x, M_1, M_2, Σ, T_1, T_2, and A_2 such that $M = (\mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2)$ and $C = \Sigma \triangleright T_2 \,/\, A_2$. By case analysis on the result of the IH on $\emptyset \vdash M_1 : \Sigma \triangleright T_1 \,/\, \square$.

Case $\exists\, V_1.\ M_1 = \mathsf{return}\ V_1$: By (HE_LetV).

Case $\exists\, \sigma, V_1, y, M_1'.\ M_1 = \sigma(V_1; y.\, M_1')$: By (HE_LetOp).

Case $\exists\, M_1'.\ M_1 \longrightarrow M_1'$: By (HE_LetE).

Case (HT_LetATM): We are given

$$\frac{\emptyset \vdash M_1 : \Sigma \triangleright T_1 \,/\, C_0 \Rightarrow C_1 \quad x : T_1 \vdash M_2 : \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_0}{\emptyset \vdash \mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2 : \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_1}$$

for some x, M_1, M_2, Σ, T_1, T_2, C_1, C_2, and C_0 such that $M = (\mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2)$ and $C = \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_1$. By case analysis on the result of the IH on $\emptyset \vdash M_1 : \Sigma \triangleright T_1 \,/\, C_0 \Rightarrow C_1$.

Case $\exists\, V_1.\ M_1 = \mathsf{return}\ V_1$: By (HE_LetV).

Case $\exists\, \sigma, V_1, y, M_1'.\ M_1 = \sigma(V_1; y.\, M_1')$: By (HE_LetOp).

Case $\exists\, M_1'.\ M_1 \longrightarrow M_1'$: By (HE_LetE).

Case (HT_App): We are given

$$\frac{\emptyset \vdash V_1 : T \to C \quad \emptyset \vdash V_2 : T}{\emptyset \vdash V_1\ V_2 : C}$$

for some V_1, V_2, and T such that $M = V_1\ V_2$. By case analysis on the result of applying Lemma 7 to $\emptyset \vdash V_1 : T \to C$.

Case $\exists\, x, M_1.\ V_1 = \lambda x. M_1$: By (HE_Beta).

Case $\exists\, x, V_1'.\ V_1 = \mathsf{fix}\, x.\, V_1'$: By (HE_Fix).

Case (HT_Case): We are given

$$\frac{\emptyset \vdash V : \mathsf{n} \quad \forall\, i \in [1, n].\ \emptyset \vdash M_i : C}{\emptyset \vdash \mathsf{case}(V; M_1, \cdots, M_n) : C}$$

for some V, n, $M_1, \cdots, M_n$ such that $M = \mathsf{case}(V; M_1, \cdots, M_n)$. By Lemma 7, $V = \underline{\mathrm{i}}$ for some i such that $0 < i \leq n$. Thus, we have the conclusion by (HE_Case).

Case (HT_Op) and (HT_OpATM): Obvious.

Case (HT_Handle): We are given

$$\frac{\begin{array}{c} H = \{\mathsf{return}\, x \mapsto L\} \uplus \{\sigma_i(y_i; k_i) \mapsto M_i\}^{1 \leq i \leq m} \uplus \{\varsigma_i(z_i) \mapsto N_i\}^{1 \leq i \leq n} \\ \Sigma = \{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, C_i^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{fin}}\}^{1 \leq i \leq m} \uplus \{\varsigma_i : U_i^{\mathrm{par}} \rightsquigarrow U_i^{\mathrm{ari}} \,/\, \Box\}^{1 \leq i \leq n} \\ \emptyset \vdash M' : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C \qquad x : T \vdash L : C^{\mathrm{ini}} \qquad \forall\, i \in [1, m].\ y_i : T_i^{\mathrm{par}}, k_i : T_i^{\mathrm{ari}} \to C_i^{\mathrm{ini}} \vdash M_i : C_i^{\mathrm{fin}} \\ \forall\, i \in [1, n].\ z_i : U_i^{\mathrm{par}} \vdash N_i : \Sigma_i \triangleright U_i^{\mathrm{ari}} \,/\, \Box \qquad \forall\, C' \in \{\overline{C_i^{\mathrm{ini}}}^{1 \leq i \leq m}, C\}.\ \forall\, i \in [1, n].\ C'.\Sigma <: \Sigma_i \end{array}}{\emptyset \vdash \mathsf{with}\, H \,\mathsf{handle}\, M' : C}$$

for some H, M', x, L, Σ, $\sigma_1, \cdots, \sigma_m$, $y_1 \cdots, y_m$ $k_1 \cdots, k_m$, $M_1, \cdots, M_m$, $T_1^{\mathrm{par}}, \cdots, T_m^{\mathrm{par}}$, $T_1^{\mathrm{ari}}, \cdots, T_m^{\mathrm{ari}}$, $C_1^{\mathrm{ini}}, \cdots, C_m^{\mathrm{ini}}$, $C_1^{\mathrm{fin}}, \cdots, C_m^{\mathrm{fin}}$, $\varsigma_1, \cdots, \varsigma_n$, $z_1, \cdots, z_n$, $N_1, \cdots, N_n$, $U_1^{\mathrm{par}}, \cdots, U_n^{\mathrm{par}}$, $U_1^{\mathrm{ari}}, \cdots, U_n^{\mathrm{ari}}$, $\Sigma_1, \cdots, \Sigma_n$, T, and C^{ini} such that $M = \mathsf{with}\, H \,\mathsf{handle}\, M'$. By case analysis on the result of the IH on $\emptyset \vdash M' : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C$.

Case $\exists\, V'.\ M' = \mathsf{return}\ V'$: By (HE_HandleV).

Case $\exists\, \sigma, V', y, M''.\ M' = \sigma(V'; y.\, M'')$: By Lemma 8 with $\emptyset \vdash \sigma(V'; y.\, M'') : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C$, there exists some C' such that $\sigma \in \mathit{dom}(C'.\Sigma)$ and $C' <: \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C$. By Lemma 3, $\sigma \in \mathit{dom}(\Sigma)$. Thus, $\sigma(y_i; k_i) \mapsto M_i \in H$ for some $i \in [1, m]$, or $\sigma(z_i) \mapsto N_i \in H$ for some $i \in [1, n]$. The conclusion is derived by (HE_HandleOp) in the former case, or (HE_HandleOPTail) in the latter case.

Case $\exists\, M''.\ M' \longrightarrow M''$: By (HE_HandleE).

$\square$

Lemma 10 (Inversion of Return Values)**.** If $\Gamma \vdash \mathsf{return}\, V : C$, then $\Gamma \vdash V : T$ and $\Sigma \triangleright T \,/\, \square <: C$ for some Σ and T.

Proof. Straightforward by induction on the typing derivation with Lemmas 4 and 5. $\square$

Lemma 11 (Strengthening Typing Contexts)**.** Assume that $T_1 <: T_2$.

- If $\Gamma_1, x : T_2, \Gamma_2 \vdash V : T$, then $\Gamma_1, x : T_1, \Gamma_2 \vdash V : T$.
- If $\Gamma_1, x : T_2, \Gamma_2 \vdash M : C$, then $\Gamma_1, x : T_1, \Gamma_2 \vdash M : C$.

Proof. Straightforward by mutual induction on the typing derivations. $\square$

Lemma 12 (Inversion of Lambda Abstractions)**.** If $\Gamma \vdash \lambda x.M : T$, then $\Gamma, x : T' \vdash M : C'$ and $T' \to C' <: T$ for some T' and C'.

Proof. Straightforward by induction on the typing derivation with Lemmas 4 and 5. $\square$

Lemma 13 (Inversion of Fixed Points)**.** If $\Gamma \vdash \mathsf{fix}\, x. V : T$, then $\Gamma, x : T' \to C' \vdash V : T' \to C'$ and $T' \to C' <: T$ for some T' and C'.

Proof. Straightforward by induction on the typing derivation with Lemmas 4 and 5. $\square$

Lemma 14 (Subject Reduction)**.** If $\Gamma \vdash M : C$ and $M \longrightarrow M'$, then $\Gamma \vdash M' : C$.

Proof. By induction on the typing derivation.

Case (HT_Return): We have $M = \mathsf{return}\, V$ for some V, but there is a contradiction because there is no evaluation rule applicable to $\mathsf{return}\, V$.

Case (HT_SubC): By the IH and (HT_SubC).

Case (HT_Let): We are given

$$\frac{\Gamma \vdash M_1 : \Sigma \triangleright T_1 \,/\, \square \quad \Gamma, x : T_1 \vdash M_2 : \Sigma \triangleright T_2 \,/\, A}{\Gamma \vdash \mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2 : \Sigma \triangleright T_2 \,/\, A}$$

for some x, M_1, M_2, Σ, T_1, T_2, and A such that $M = (\mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2)$ and $C = \Sigma \triangleright T_2 \,/\, A$. We have $\mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2 \longrightarrow M'$. By case analysis on the evaluation rule applied last to derive it.

Case (HE_LetV): We are given

$$\mathsf{let}\, x = \mathsf{return}\, V_1 \,\mathsf{in}\, M_2 \longrightarrow M_2[V_1/x]$$

for some V_1 such that $M_1 = \mathsf{return}\, V_1$ and $M' = M_2[V_1/x]$. By Lemma 10 with $\Gamma \vdash \mathsf{return}\, V_1 : \Sigma \triangleright T_1 \,/\, \square$, there exist some Σ'_1 and T'_1 such that $\Gamma \vdash V_1 : T'_1$ and $\Sigma' \triangleright T'_1 \,/\, \square <: \Sigma \triangleright T_1 \,/\, \square$. By Lemma 3, we have $T'_1 <: T_1$. Thus, by (HT_SubV), $\Gamma \vdash V_1 : T_1$, and by Lemma 2, we have the conclusion $\Gamma \vdash M_2[V_1/x] : \Sigma \triangleright T_2 \,/\, A$.

Case (HE_LetOp): We are given

$$\mathsf{let}\, x = \sigma(V_1; y.\, M'_1) \,\mathsf{in}\, M_2 \longrightarrow \sigma(V_1; y.\, \mathsf{let}\, x = M'_1 \,\mathsf{in}\, M_2)$$

for some σ, V_1, y, and M'_1 such that $M_1 = \sigma(V_1; y.\, M'_1)$ and $M' = \sigma(V_1; y.\, \mathsf{let}\, x = M'_1 \,\mathsf{in}\, M_2)$ and $y \notin \mathit{fv}(M_2)$. Without loss of generality, we can assume that $y \notin \mathit{dom}(\Gamma) \cup \{x\}$. By Lemma 8 with $\Gamma \vdash \sigma(V_1; y.\, M'_1) : \Sigma \triangleright T_1 \,/\, \square$, and Lemma 3,

- $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, \square \in \Sigma'$,
- $\Gamma \vdash V_1 : T^{\mathrm{par}}$,
- $\Gamma, y : T^{\mathrm{ari}} \vdash M'_1 : \Sigma' \triangleright T'_1 \,/\, A'$, and
- $\Sigma' \triangleright T'_1 \,/\, A' <: \Sigma \triangleright T_1 \,/\, \square$ (that is, $\Sigma <: \Sigma'$ and $A' = \square$)

for some T^{par}, T^{ari}, Σ', T'_1, and A' (note that the case that the ATM of σ given by Σ' is not $\square$ is contradictory by Lemma 3). By (HT_SubC), $\Gamma, y : T^{\mathrm{ari}} \vdash M'_1 : \Sigma \triangleright T_1 \,/\, \square$. By Lemma 1 with $\Gamma, x : T_1 \vdash M_2 : \Sigma \triangleright T_2 \,/\, A$, we have $\Gamma, y : T^{\mathrm{ari}}, x : T_1 \vdash M_2 : \Sigma \triangleright T_2 \,/\, A$. By (HT_Let),

$$\Gamma, y : T^{\mathrm{ari}} \vdash \mathsf{let}\, x = M'_1 \,\mathsf{in}\, M_2 : \Sigma \triangleright T_2 \,/\, A \ . \tag{1}$$

Because $\Sigma <: \Sigma'$ and $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, \square \in \Sigma'$, Lemma 3 implies that there exist some U^{par} and U^{ari} such that

- $\sigma : U^{\mathrm{par}} \rightsquigarrow U^{\mathrm{ari}} \,/\, \square \in \Sigma$,
- $T^{\mathrm{par}} <: U^{\mathrm{par}}$, and
- $U^{\mathrm{ari}} <: T^{\mathrm{ari}}$.

By Lemma 11 with derivation (1) and $U^{\mathrm{ari}} <: T^{\mathrm{ari}}$,

$$\Gamma, y : U^{\mathrm{ari}} \vdash \mathsf{let}\, x = M'_1 \,\mathsf{in}\, M_2 : \Sigma \triangleright T_2 \,/\, A \ . \tag{2}$$

Then, we can derive the conclusion as follows:

$$\dfrac{\sigma : U^{\mathrm{par}} \rightsquigarrow U^{\mathrm{ari}} \,/\, \square \in \Sigma \qquad \dfrac{\Gamma \vdash V_1 : T^{\mathrm{par}} \qquad T^{\mathrm{par}} <: U^{\mathrm{par}}}{\Gamma \vdash V_1 : U^{\mathrm{par}}}\,(\text{HT\_SubV}) \qquad \text{derivation (2)}}{\Gamma \vdash \sigma(V_1; y.\, \mathsf{let}\, x = M'_1 \,\mathsf{in}\, M_2) : \Sigma \triangleright T_2 \,/\, A}\,(\text{HT\_Op})$$

Case (HE_LetE): We are given

$$M_1 \longrightarrow M'_1$$

for some M'_1 such that $M' = (\mathsf{let}\, x = M'_1 \,\mathsf{in}\, M_2)$. By the IH, $\Gamma \vdash M'_1 : \Sigma \triangleright T_1 \,/\, \square$. Therefore, by (HT_Let), we have the conclusion

$$\Gamma \vdash \mathsf{let}\, x = M'_1 \,\mathsf{in}\, M_2 : \Sigma \triangleright T_2 \,/\, A \ .$$

Case (HT_LetATM): We are given

$$\dfrac{\Gamma \vdash M_1 : \Sigma \triangleright T_1 \,/\, C_0 \Rightarrow C_1 \qquad \Gamma, x : T_1 \vdash M_2 : \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_0}{\Gamma \vdash \mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2 : \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_1}$$

for some x, M_1, M_2, Σ, T_1, T_2, C_1, C_2, and C_0 such that $M = (\mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2)$ and $C = \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_1$. We have $\mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2 \longrightarrow M'$. By case analysis on the evaluation rule applied last to derive it.

Case (HE_LetV): We are given

$$\mathsf{let}\, x = \mathsf{return}\ V_1 \,\mathsf{in}\, M_2 \longrightarrow M_2[V_1/x]$$

for some V_1 such that $M_1 = \mathsf{return}\ V_1$ and $M' = M_2[V_1/x]$. By Lemma 10 with $\Gamma \vdash \mathsf{return}\ V_1 : \Sigma \triangleright T_1 \,/\, C_0 \Rightarrow C_1$, there exist some Σ'_1 and T'_1 such that $\Gamma \vdash V_1 : T'_1$ and $\Sigma' \triangleright T'_1 \,/\, \square <: \Sigma \triangleright T_1 \,/\, C_0 \Rightarrow C_1$. By Lemma 3 with $\Sigma' \triangleright T'_1 \,/\, \square <: \Sigma \triangleright T_1 \,/\, C_0 \Rightarrow C_1$, we have $T'_1 <: T_1$ and $\square <: C_0 \Rightarrow C_1$, that is, $C_0 <: C_1$. By (HT_SubV), $\Gamma \vdash V_1 : T_1$, and by Lemma 2, we have $\Gamma \vdash M_2[V_1/x] : \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_0$. By Lemma 4, we have

$$\dfrac{\Sigma <: \Sigma \qquad T_2 <: T_2 \qquad \dfrac{C_2 <: C_2 \qquad C_0 <: C_1}{C_2 \Rightarrow C_0 <: C_2 \Rightarrow C_1}\,(\text{HS\_AnsMod}) \qquad \square(\Sigma) <: \square(\Sigma)}{\Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_0 <: \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_1}\,(\text{HS\_Comp})$$

Thus, by (HT_SubC), we have the conclusion $\Gamma \vdash M_2[V_1/x] : \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_1$.

Case (HE_LetOp): We are given

$$\mathsf{let}\, x = \sigma(V_1; y.\, M'_1) \,\mathsf{in}\, M_2 \longrightarrow \sigma(V_1; y.\, \mathsf{let}\, x = M'_1 \,\mathsf{in}\, M_2)$$

for some σ, V_1, y, and M'_1 such that $M_1 = \sigma(V_1; y.\, M'_1)$ and $M' = \sigma(V_1; y.\, \mathsf{let}\, x = M'_1 \,\mathsf{in}\, M_2)$ and $y \notin \mathit{fv}(M_2)$. Without loss of generality, we can assume that $y \notin \mathit{dom}(\Gamma) \cup \{x\}$. By case analysis on the result of applying Lemma 8 to $\Gamma \vdash \sigma(V_1; y.\, M'_1) : \Sigma \triangleright T_1 \,/\, C_0 \Rightarrow C_1$.

Case 1: We are given

- $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, \square \in \Sigma'$,
- $\Gamma \vdash V_1 : T^{\mathrm{par}}$,
- $\Gamma, y : T^{\mathrm{ari}} \vdash M_1' : \Sigma' \triangleright T' \,/\, A'$, and
- $\Sigma' \triangleright T' \,/\, A' <: \Sigma \triangleright T_1 \,/\, C_0 \Rightarrow C_1$

for some T^{par}, T^{ari}, Σ', T', and A'. By (HT_SubC), $\Gamma, y : T^{\mathrm{ari}} \vdash M_1' : \Sigma \triangleright T_1 \,/\, C_0 \Rightarrow C_1$. By Lemma 1 with $\Gamma, x : T_1 \vdash M_2 : \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_0$, we have $\Gamma, y : T^{\mathrm{ari}}, x : T_1 \vdash M_2 : \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_0$. By (HT_LetATM),

$$\Gamma, y : T^{\mathrm{ari}} \vdash \mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2 : \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_1 \ . \tag{3}$$

By Lemma 3 with $\Sigma' \triangleright T' \,/\, A' <: \Sigma \triangleright T_1 \,/\, C_0 \Rightarrow C_1$, we have $\Sigma <: \Sigma'$. Again by Lemma 3 with $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, \square \in \Sigma'$, we have

- $\sigma : U^{\mathrm{par}} \rightsquigarrow U^{\mathrm{ari}} \,/\, A \in \Sigma$,
- $T^{\mathrm{par}} <: U^{\mathrm{par}}$,
- $U^{\mathrm{ari}} <: T^{\mathrm{ari}}$, and
- $A <: \square$ (that is, $A = \square$)

for some U^{par}, U^{ari}, and A. By Lemma 11 with derivation (3) and $U^{\mathrm{ari}} <: T^{\mathrm{ari}}$,

$$\Gamma, y : U^{\mathrm{ari}} \vdash \mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2 : \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_1 \ . \tag{4}$$

Then, we can derive the conclusion as follows:

$$\dfrac{\sigma : U^{\mathrm{par}} \rightsquigarrow U^{\mathrm{ari}} \,/\, \square \in \Sigma \qquad \dfrac{\Gamma \vdash V_1 : T^{\mathrm{par}} \qquad T^{\mathrm{par}} <: U^{\mathrm{par}}}{\Gamma \vdash V_1 : U^{\mathrm{par}}}\,(\text{HT\_SubV}) \qquad \text{derivation (4)}}{\Gamma \vdash \sigma(V_1; y.\, \mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2) : \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_1}\,(\text{HT\_Op})$$

Case 2: We are given

- $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}} \in \Sigma'$,
- $\Gamma \vdash V_1 : T^{\mathrm{par}}$,
- $\Gamma, y : T^{\mathrm{ari}} \vdash M_1' : \Sigma' \triangleright T' \,/\, C' \Rightarrow C^{\mathrm{ini}}$, and
- $\Sigma' \triangleright T' \,/\, C' \Rightarrow C^{\mathrm{fin}} <: \Sigma \triangleright T_1 \,/\, C_0 \Rightarrow C_1$

for some T^{par}, T^{ari}, C^{ini}, C^{fin}, Σ', T', and C'. By Lemma 3 with $\Sigma' \triangleright T' \,/\, C' \Rightarrow C^{\mathrm{fin}} <: \Sigma \triangleright T_1 \,/\, C_0 \Rightarrow C_1$, we have

- $\Sigma <: \Sigma'$,
- $T' <: T_1$,
- $C_0 <: C'$,
- $C^{\mathrm{fin}} <: C_1$, and
- $\square(\Sigma') <: \square(\Sigma)$.

By Lemma 3 with $\Sigma <: \Sigma'$ and $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}} \in \Sigma'$, we have

- $\sigma : U^{\mathrm{par}} \rightsquigarrow U^{\mathrm{ari}} \,/\, A \in \Sigma$,
- $T^{\mathrm{par}} <: U^{\mathrm{par}}$,
- $U^{\mathrm{ari}} <: T^{\mathrm{ari}}$, and
- $A <: C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$

for some U^{par}, U^{ari}, and A. Because $C^{\mathrm{ini}} <: C^{\mathrm{ini}}$ by Lemma 4, we have

$$\dfrac{\Sigma <: \Sigma' \qquad T' <: T_1 \qquad \dfrac{C_0 <: C' \qquad C^{\mathrm{ini}} <: C^{\mathrm{ini}}}{C' \Rightarrow C^{\mathrm{ini}} <: C_0 \Rightarrow C^{\mathrm{ini}}}\,(\text{HS\_AnsMod}) \qquad \square(\Sigma') <: \square(\Sigma)}{\Sigma' \triangleright T' \,/\, C' \Rightarrow C^{\mathrm{ini}} <: \Sigma \triangleright T_1 \,/\, C_0 \Rightarrow C^{\mathrm{ini}}}\,(\text{HS\_Comp})$$

Thus, by (HT_SubC) with $\Gamma, y : T^{\mathrm{ari}} \vdash M_1' : \Sigma' \triangleright T' \,/\, C' \Rightarrow C^{\mathrm{ini}}$, we have

$$\Gamma, y : T^{\mathrm{ari}} \vdash M_1' : \Sigma \triangleright T_1 \,/\, C_0 \Rightarrow C^{\mathrm{ini}} \ .$$

By Lemma 1 with $\Gamma, x : T_1 \vdash M_2 : \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_0$, we have

$$\Gamma, y : T^{\mathrm{ari}}, x : T_1 \vdash M_2 : \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_0 \ .$$

Thus, by (HT_LetATM),

$$\Gamma, y : T^{\mathrm{ari}} \vdash \mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2 : \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C^{\mathrm{ini}} \ .$$

By Lemma 11 with $U^{\mathrm{ari}} <: T^{\mathrm{ari}}$,

$$\Gamma, y : U^{\mathrm{ari}} \vdash \mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2 : \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C^{\mathrm{ini}} \ . \tag{5}$$

We proceed by case analysis on A.

Case $A = \square$: Because $\square <: C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$, we have $C^{\mathrm{ini}} <: C^{\mathrm{fin}}$ by Lemma 3. Because

$$\Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C^{\mathrm{ini}} <: \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_1$$

by (HS_Comp), (HS_AnsMod), Lemmas 4 and 5, $C^{\mathrm{ini}} <: C^{\mathrm{fin}}$, and $C^{\mathrm{fin}} <: C_1$, we have

$$\Gamma, y : U^{\mathrm{ari}} \vdash \mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2 : \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_1 \tag{6}$$

by (HT_SubC) with derivation (5). Then, we derive the conclusion as follows:

$$\dfrac{\sigma : U^{\mathrm{par}} \rightsquigarrow U^{\mathrm{ari}} \,/\, \square \in \Sigma \qquad \dfrac{\Gamma \vdash V_1 : T^{\mathrm{par}} \quad T^{\mathrm{par}} <: U^{\mathrm{par}}}{\Gamma \vdash V_1 : U^{\mathrm{par}}}\,(\text{HT\_SubV}) \qquad \text{derivation (6)}}{\Gamma \vdash \sigma(V_1; y.\, \mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2) : \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_1}\,(\text{HT\_Op})$$

Case $\exists\, D^{\mathrm{ini}}, D^{\mathrm{fin}}.\ A = D^{\mathrm{ini}} \Rightarrow D^{\mathrm{fin}}$: Because $D^{\mathrm{ini}} \Rightarrow D^{\mathrm{fin}} <: C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$, we have $C^{\mathrm{ini}} <: D^{\mathrm{ini}}$ and $D^{\mathrm{fin}} <: C^{\mathrm{fin}}$ by Lemma 3. Because

$$\Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C^{\mathrm{ini}} <: \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow D^{\mathrm{ini}}$$

by (HS_Comp), (HS_AnsMod), Lemma 4, and $C^{\mathrm{ini}} <: D^{\mathrm{ini}}$, we have the following derivation by (HT_SubC) with derivation (5):

$$\Gamma, y : U^{\mathrm{ari}} \vdash \mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2 : \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow D^{\mathrm{ini}} \ . \tag{7}$$

Then, we have the following typing derivation:

$$\dfrac{\sigma : U^{\mathrm{par}} \rightsquigarrow U^{\mathrm{ari}} \,/\, D^{\mathrm{ini}} \Rightarrow D^{\mathrm{fin}} \in \Sigma \qquad \dfrac{\Gamma \vdash V_1 : T^{\mathrm{par}} \quad T^{\mathrm{par}} <: U^{\mathrm{par}}}{\Gamma \vdash V_1 : U^{\mathrm{par}}}\,(\text{HT\_SubV}) \qquad \text{derivation (7)}}{\Gamma \vdash \sigma(V_1; y.\, \mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2) : \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow D^{\mathrm{fin}}}\,(\text{HT\_OpATM})$$

Because $\Sigma \triangleright T_2 \,/\, C_2 \Rightarrow D^{\mathrm{fin}} <: \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_1$ by (HS_Comp), (HS_AnsMod), Lemmas 4 and 5, $D^{\mathrm{fin}} <: C^{\mathrm{fin}}$, and $C^{\mathrm{fin}} <: C_1$, we have the conclusion

$$\Gamma \vdash \sigma(V_1; y.\, \mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2) : \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_1$$

by (HT_SubC).

Case (HE_LetE): We are given

$$M_1 \longrightarrow M_1'$$

for some M_1' such that $M' = (\mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2)$. By the IH, $\Gamma \vdash M_1' : \Sigma \triangleright T_1 \,/\, C_0 \Rightarrow C_1$. Therefore, by (HT_LetATM), we have the conclusion

$$\Gamma \vdash \mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2 : \Sigma \triangleright T_2 \,/\, C_2 \Rightarrow C_1 \ .$$

Case (HT_App): We are given

$$\frac{\Gamma \vdash V_1 : T \rightarrow C \quad \Gamma \vdash V_2 : T}{\Gamma \vdash V_1\ V_2 : C}$$

for some V_1, V_2, and T such that $M = V_1\ V_2$. We have $V_1\ V_2 \longrightarrow M'$. By case analysis on the evaluation rule applied last to derive it.

Case (HE_Beta): We are given

$$(\lambda x.M_1)\ V_2 \longrightarrow M_1[V_2/x]$$

for some x and M_1 such that $V_1 = \lambda x.M_1$ and $M' = M_1[V_2/x]$. By Lemmas 12 and 3 with $\Gamma \vdash \lambda x.M_1 : T \rightarrow C$, there exist some T' and C' such that

- $\Gamma, x : T' \vdash M_1 : C'$,
- $T <: T'$, and
- $C' <: C$.

Because $\Gamma \vdash V_2 : T'$ by (HT_SubV), we have $\Gamma \vdash M_1[V_2/x] : C'$ by Lemma 2. By (HT_SubC) with $C' <: C$, we have the conclusion $\Gamma \vdash M_1[V_2/x] : C$.

Case (HE_Fix): We are given

$$(\mathsf{fix}\, x. V_1')\ V_2 \longrightarrow V_1'[\mathsf{fix}\, x. V_1'/x]\ V_2$$

for some x and V_1' such that $V_1 = \mathsf{fix}\, x. V_1'$ and $M' = V_1'[\mathsf{fix}\, x. V_1'/x]\ V_2$. By Lemma 13 with $\Gamma \vdash \mathsf{fix}\, x. V_1' : T \rightarrow C$, there exist some T' and C' such that

- $\Gamma, x : T' \rightarrow C' \vdash V_1' : T' \rightarrow C'$ and
- $T' \rightarrow C' <: T \rightarrow C$.

Because $\Gamma \vdash \mathsf{fix}\, x. V_1' : T' \rightarrow C'$ by (HT_Fix), we have $\Gamma \vdash V_1'[\mathsf{fix}\, x. V_1'/x] : T' \rightarrow C'$ by Lemma 2. By (HT_SubV) with $T' \rightarrow C' <: T \rightarrow C$, we have $\Gamma \vdash V_1'[\mathsf{fix}\, x. V_1'/x] : T \rightarrow C$. Therefore, by (HT_App), we have the conclusion

$$\Gamma \vdash V_1'[\mathsf{fix}\, x. V_1'/x]\ V_2 : C\ .$$

Case (HT_Case): We are given

$$\frac{\Gamma \vdash V : \mathsf{n} \quad \forall\, i \in [1, n].\ \Gamma \vdash M_i : C}{\Gamma \vdash \mathsf{case}(V; M_1, \cdots, M_n) : C}$$

for some V, n, $M_1, \cdots, M_n$ such that $M = \mathsf{case}(V; M_1, \cdots, M_n)$. Because $\mathsf{case}(V; M_1, \cdots, M_n) \longrightarrow M'$, we have $V = \underline{\mathsf{i}}$ and $M' = M_i$ for some i such that $0 < i \leq n$. Because $\Gamma \vdash M_i : C$, we have the conclusion.

Case (HT_Op) and (HT_OpATM): We have $M = \sigma(V; x. M'')$ for some σ, V, x, and M'', but there is a contradiction because there is no evaluation rule applicable to $\sigma(V; x. M'')$.

Case (HT_Handle): We are given

$$\frac{\begin{array}{c} H = \{\mathsf{return}\, x \mapsto L\} \uplus \{\sigma_i(y_i; k_i) \mapsto M_i\}^{1 \leq i \leq m} \uplus \{\varsigma_i(z_i) \mapsto N_i\}^{1 \leq i \leq n} \\ \Sigma = \{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, C_i^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{fin}}\}^{1 \leq i \leq m} \uplus \{\varsigma_i : U_i^{\mathrm{par}} \rightsquigarrow U_i^{\mathrm{ari}} \,/\, \square\}^{1 \leq i \leq n} \\ \Gamma \vdash M_0 : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C \qquad \Gamma, x : T \vdash L : C^{\mathrm{ini}} \qquad \forall\, i \in [1, m].\ \Gamma, y_i : T_i^{\mathrm{par}}, k_i : T_i^{\mathrm{ari}} \rightarrow C_i^{\mathrm{ini}} \vdash M_i : C_i^{\mathrm{fin}} \\ \forall\, i \in [1, n].\ \Gamma, z_i : U_i^{\mathrm{par}} \vdash N_i : \Sigma_i \triangleright U_i^{\mathrm{ari}} \,/\, \square \qquad \forall\, C' \in \{\overline{C_i^{\mathrm{ini}}}^{1 \leq i \leq m}, C\}.\ \forall\, i \in [1, n].\ C'.\Sigma <: \Sigma_i \end{array}}{\Gamma \vdash \mathsf{with}\, H\, \mathsf{handle}\, M_0 : C}$$

for some H, M_0, x, L, Σ, $\sigma_1, \cdots, \sigma_m$, $y_1 \cdots, y_m$ $k_1 \cdots, k_m$, $M_1, \cdots, M_m$, $T_1^{\mathrm{par}}, \cdots, T_m^{\mathrm{par}}$, $T_1^{\mathrm{ari}}, \cdots, T_m^{\mathrm{ari}}$, $C_1^{\mathrm{ini}}, \cdots, C_m^{\mathrm{ini}}$, $C_1^{\mathrm{fin}}, \cdots, C_m^{\mathrm{fin}}$, $\varsigma_1, \cdots, \varsigma_n$, $z_1, \cdots, z_n$, $N_1, \cdots, N_n$, $U_1^{\mathrm{par}}, \cdots, U_n^{\mathrm{par}}$, $U_1^{\mathrm{ari}}, \cdots, U_n^{\mathrm{ari}}$, $\Sigma_1, \cdots, \Sigma_n$, T, and C^{ini} such that $M = \mathsf{with}\, H\, \mathsf{handle}\, M_0$. We have $\mathsf{with}\, H\, \mathsf{handle}\, M_0 \longrightarrow M'$. By case analysis on the evaluation rule applied last to derive it.

Case (HE_HandleV): We are given

$$\textsf{with}\ H\ \textsf{handle}\ \textsf{return}\ V \longrightarrow L[V/x]$$

for some V such that $M_0 = \textsf{return}\ V$ and $M' = L[V/x]$. By Lemma 10 with $\Gamma \vdash \textsf{return}\ V : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C$, we have

- $\Gamma \vdash V : T'$ and
- $\Sigma' \triangleright T' \,/\, \square <: \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C$

for some Σ' and T'. By Lemma 3, $T' <: T$ and $C^{\mathrm{ini}} <: C$. By (HT_SubV), $\Gamma \vdash V : T$. By Lemma 2 with $\Gamma, x : T \vdash L : C^{\mathrm{ini}}$, we have $\Gamma \vdash L[V/x] : C^{\mathrm{ini}}$. By (HT_SubC) with $C^{\mathrm{ini}} <: C$, we have the conclusion $\Gamma \vdash L[V/x] : C$.

Case (HE_HandleOp): We are given

$$\textsf{with}\ H\ \textsf{handle}\ \sigma_i(V; y.\, M_0') \longrightarrow M_i[V/y_i][\lambda y.\textsf{with}\ H\ \textsf{handle}\ M_0'/k_i]$$

for some $i \in [1, m]$, V, y, and M_0' such that $M_0 = \sigma_i(V; y.\, M_0')$ and $M' = M_i[V/y_i][\lambda y.\textsf{with}\ H\ \textsf{handle}\ M_0'/k_i]$. By case analysis on the result of applying Lemma 8 to $\Gamma \vdash \sigma_i(V; y.\, M_0') : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C$.

Case 1: We are given

- $\sigma_i : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, \square \in \Sigma'$,
- $\Gamma \vdash V : T^{\mathrm{par}}$,
- $\Gamma, y : T^{\mathrm{ari}} \vdash M_0' : \Sigma' \triangleright T' \,/\, A'$, and
- $\Sigma' \triangleright T' \,/\, A' <: \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C$

for some T^{par}, T^{ari}, Σ', T', and A'. By Lemma 3 , $\Sigma <: \Sigma'$. Because $\sigma_i : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, \square \in \Sigma'$, we have $\sigma_i : U^{\mathrm{par}} \rightsquigarrow U^{\mathrm{ari}} \,/\, \square \in \Sigma$ for some U^{par} and U^{ari} by Lemma 3. However, it is contradictory with the definition of Σ.

Case 2: We are given

- $\sigma_i : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, D^{\mathrm{ini}} \Rightarrow D^{\mathrm{fin}} \in \Sigma'$,
- $\Gamma \vdash V : T^{\mathrm{par}}$,
- $\Gamma, y : T^{\mathrm{ari}} \vdash M_0' : \Sigma' \triangleright T' \,/\, C' \Rightarrow D^{\mathrm{ini}}$, and
- $\Sigma' \triangleright T' \,/\, C' \Rightarrow D^{\mathrm{fin}} <: \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C$

for some T^{par}, T^{ari}, D^{ini}, D^{fin}, Σ', T', and C'. By Lemma 3 with $\Sigma' \triangleright T' \,/\, C' \Rightarrow D^{\mathrm{fin}} <: \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C$, we have

- $\Sigma <: \Sigma'$,
- $T' <: T$,
- $C^{\mathrm{ini}} <: C'$,
- $D^{\mathrm{fin}} <: C$, and
- $\square(\Sigma') <: \square(\Sigma)$.

By Lemma 3 with $\Sigma <: \Sigma'$ and $\sigma_i : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, D^{\mathrm{ini}} \Rightarrow D^{\mathrm{fin}} \in \Sigma'$, we have

- $\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, C_i^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{fin}} \in \Sigma$,
- $T^{\mathrm{par}} <: T_i^{\mathrm{par}}$,
- $T_i^{\mathrm{ari}} <: T^{\mathrm{ari}}$,
- $D^{\mathrm{ini}} <: C_i^{\mathrm{ini}}$, and
- $C_i^{\mathrm{fin}} <: D^{\mathrm{fin}}$.

By (HT_SubV) with $\Gamma \vdash V : T^{\mathrm{par}}$ and $T^{\mathrm{par}} <: T_i^{\mathrm{par}}$, we have $\Gamma \vdash V : T_i^{\mathrm{par}}$. Because $\Gamma, y_i : T_i^{\mathrm{par}}, k_i : T_i^{\mathrm{ari}} \to C_i^{\mathrm{ini}} \vdash M_i : C_i^{\mathrm{fin}}$, we have

$$\Gamma, k_i : T_i^{\mathrm{ari}} \to C_i^{\mathrm{ini}} \vdash M_i[V/y_i] : C_i^{\mathrm{fin}} \tag{8}$$

by Lemma 2. Because

$$\dfrac{\Sigma <: \Sigma' \qquad T' <: T \qquad \dfrac{C^{\mathrm{ini}} <: C' \qquad D^{\mathrm{ini}} <: C_i^{\mathrm{ini}}}{C' \Rightarrow D^{\mathrm{ini}} <: C^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{ini}}}\ (\text{HS\_AnsMod}) \qquad \square(\Sigma') <: \square(\Sigma)}{\Sigma' \triangleright T' \,/\, C' \Rightarrow D^{\mathrm{ini}} <: \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{ini}}}\ (\text{HS\_Comp})$$

we have

$$\Gamma, y : T^{\mathrm{ari}} \vdash M_0' : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{ini}}$$

by (HT_SubC) with $\Gamma, y : T^{\mathrm{ari}} \vdash M_0' : \Sigma' \triangleright T' \,/\, C' \Rightarrow D^{\mathrm{ini}}$. By Lemma 11 with $T_i^{\mathrm{ari}} <: T^{\mathrm{ari}}$, we have

$$\Gamma, y : T_i^{\mathrm{ari}} \vdash M_0' : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{ini}} \ .$$

Thus, by (HT_Handle) with

- $H = \{\mathsf{return}\, x \mapsto L\} \uplus \{\sigma_j(y_j; k_j) \mapsto M_j\}^{1 \le j \le m} \uplus \{\varsigma_j(z_j) \mapsto N_j\}^{1 \le j \le n}$,
- $\Sigma = \{\sigma_j : T_j^{\mathrm{par}} \rightsquigarrow T_j^{\mathrm{ari}} \,/\, C_j^{\mathrm{ini}} \Rightarrow C_j^{\mathrm{fin}}\}^{1 \le j \le m} \uplus \{\varsigma_j : U_j^{\mathrm{par}} \rightsquigarrow U_j^{\mathrm{ari}} \,/\, \square\}^{1 \le j \le n}$,
- $\Gamma, y : T_i^{\mathrm{ari}}, x : T \vdash L : C^{\mathrm{ini}}$ by Lemma 1,
- $\forall\, j \in [1, m].\ \Gamma, y : T_i^{\mathrm{ari}}, y_j : T_j^{\mathrm{par}}, k_j : T_j^{\mathrm{ari}} \to C_j^{\mathrm{ini}} \vdash M_j : C_j^{\mathrm{fin}}$ by Lemma 1,
- $\forall\, j \in [1, n].\ \Gamma, y : T_i^{\mathrm{ari}}, z_j : U_j^{\mathrm{par}} \vdash N_j : \Sigma_j \triangleright U_j^{\mathrm{ari}} \,/\, \square$ by Lemma 1,
- $\forall\, C'' \in \{\overline{C_j^{\mathrm{ini}}}^{1 \le j \le m}, C_i^{\mathrm{ini}}\}.\ \forall\, j \in [1, n].\ C''.\Sigma <: \Sigma_j$

(these are derived by the premise of the derivation of $\Gamma \vdash \mathsf{with}\, H\, \mathsf{handle}\, M_0 : C$), we have

$$\Gamma, y : T_i^{\mathrm{ari}} \vdash \mathsf{with}\, H\, \mathsf{handle}\, M_0' : C_i^{\mathrm{ini}} \ .$$

By (HT_Abs),

$$\Gamma \vdash \lambda y.\mathsf{with}\, H\, \mathsf{handle}\, M_0' : T_i^{\mathrm{ari}} \to C_i^{\mathrm{ini}} \ .$$

By Lemma 2 with derivation (8),

$$\Gamma \vdash M_i[V/y_i][\lambda y.\mathsf{with}\, H\, \mathsf{handle}\, M_0'/k_i] : C_i^{\mathrm{fin}} \ .$$

Because $C_i^{\mathrm{fin}} <: D^{\mathrm{fin}}$ and $D^{\mathrm{fin}} <: C$, we have the conclusion

$$\Gamma \vdash M_i[V/y_i][\lambda y.\mathsf{with}\, H\, \mathsf{handle}\, M_0'/k_i] : C \ .$$

by (HT_SubC) and Lemma 5.

Case (HE_HandleOpTail): We are given

$$\mathsf{with}\, H\, \mathsf{handle}\, \varsigma_i(V; y.\, M_0') \longrightarrow \mathsf{let}\, y = N_i[V/z_i]\, \mathsf{in}\, \mathsf{with}\, H\, \mathsf{handle}\, M_0'$$

for some $i \in [1, n]$, V, y, and M_0' such that $M_0 = \varsigma_i(V; y.\, M_0')$ and $M' = \mathsf{let}\, y = N_i[V/z_i]\, \mathsf{in}\, \mathsf{with}\, H\, \mathsf{handle}\, M_0'$. By case analysis on the result of applying Lemma 8 to $\Gamma \vdash \varsigma_i(V; y.\, M_0') : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C$.

Case 1: We are given

- $\varsigma_i : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, \square \in \Sigma'$,
- $\Gamma \vdash V : T^{\mathrm{par}}$,
- $\Gamma, y : T^{\mathrm{ari}} \vdash M_0' : \Sigma' \triangleright T' \,/\, A'$, and
- $\Sigma' \triangleright T' \,/\, A' <: \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C$

for some T^{par}, T^{ari}, Σ', T', and A'. By Lemma 3 , $\Sigma <: \Sigma'$. Because $\varsigma_i : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, \square \in \Sigma'$ and $\varsigma_i : U_i^{\mathrm{par}} \rightsquigarrow U_i^{\mathrm{ari}} \,/\, \square \in \Sigma$, we have

- $T^{\mathrm{par}} <: U_i^{\mathrm{par}}$ and
- $U_i^{\mathrm{ari}} <: T^{\mathrm{ari}}$

by Lemma 3. Because

- $\Gamma \vdash V : U_i^{\mathrm{par}}$ by (HT_SubV) with $\Gamma \vdash V : T^{\mathrm{par}}$ and $T^{\mathrm{par}} <: U_i^{\mathrm{par}}$, and
- $\Gamma, z_i : U_i^{\mathrm{par}} \vdash N_i : \Sigma_i \triangleright U_i^{\mathrm{ari}} \,/\, \square$,

we have

$$\Gamma \vdash N_i[V/z_i] : \Sigma_i \triangleright U_i^{\mathrm{ari}} \,/\, \square$$

by Lemma 2. Because

$$\dfrac{C.\Sigma <: \Sigma_i \qquad U_i^{\mathrm{ari}} <: U_i^{\mathrm{ari}} \qquad \dfrac{}{\square <: \square}\ (\text{HS\_AnsBox})}{\Sigma_i \triangleright U_i^{\mathrm{ari}} \,/\, \square <: C.\Sigma \triangleright U_i^{\mathrm{ari}} \,/\, \square}\ (\text{HS\_Comp})$$

by Lemma 4, we have

$$\Gamma \vdash N_i[V/z_i] : C.\Sigma \triangleright U_i^{\mathrm{ari}} \,/\, \Box \tag{9}$$

by (HT_SubC). Because

- $\Gamma, y : T^{\mathrm{ari}} \vdash M_0' : \Sigma' \triangleright T' \,/\, A'$,
- $\Sigma' \triangleright T' \,/\, A' <: \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C$, and
- $U_i^{\mathrm{ari}} <: T^{\mathrm{ari}}$,

we have

$$\Gamma, y : U_i^{\mathrm{ari}} \vdash M_0' : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C$$

by (HT_SubC) and Lemma 11. Thus, by (HT_Handle) with

- $H = \{\mathsf{return}\, x \mapsto L\} \uplus \{\sigma_j(y_j; k_j) \mapsto M_j\}^{1 \le j \le m} \uplus \{\varsigma_j(z_j) \mapsto N_j\}^{1 \le j \le n}$,
- $\Sigma = \{\sigma_j : T_j^{\mathrm{par}} \rightsquigarrow T_j^{\mathrm{ari}} \,/\, C_j^{\mathrm{ini}} \Rightarrow C_j^{\mathrm{fin}}\}^{1 \le j \le m} \uplus \{\varsigma_j : U_j^{\mathrm{par}} \rightsquigarrow U_j^{\mathrm{ari}} \,/\, \Box\}^{1 \le j \le n}$,
- $\Gamma, y : U_i^{\mathrm{ari}}, x : T \vdash L : C^{\mathrm{ini}}$ by Lemma 1,
- $\forall j \in [1, m].\ \Gamma, y : U_i^{\mathrm{ari}}, y_j : T_j^{\mathrm{par}}, k_j : T_j^{\mathrm{ari}} \to C_j^{\mathrm{ini}} \vdash M_j : C_j^{\mathrm{fin}}$ by Lemma 1,
- $\forall j \in [1, n].\ \Gamma, y : U_i^{\mathrm{ari}}, z_j : U_j^{\mathrm{par}} \vdash N_j : \Sigma_j \triangleright U_j^{\mathrm{ari}} \,/\, \Box$ by Lemma 1,
- $\forall C' \in \{\overline{C_j^{\mathrm{ini}}}^{1 \le j \le m}, C\}.\ \forall j \in [1, n].\ C'.\Sigma <: \Sigma_j$

(these are derived by the premise of the derivation of $\Gamma \vdash \mathsf{with}\, H\, \mathsf{handle}\, M_0 : C$), we have

$$\Gamma, y : U_i^{\mathrm{ari}} \vdash \mathsf{with}\, H\, \mathsf{handle}\, M_0' : C \ . \tag{10}$$

By (HT_Let) with derivations (9) and (10), we have the conclusion

$$\Gamma \vdash \mathsf{let}\, y = N_i[V/z_i]\, \mathsf{in}\, \mathsf{with}\, H\, \mathsf{handle}\, M_0' : C \ .$$

Case 2: We are given

- $\varsigma_i : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, D^{\mathrm{ini}} \Rightarrow D^{\mathrm{fin}} \in \Sigma'$,
- $\Gamma \vdash V : T^{\mathrm{par}}$,
- $\Gamma, y : T^{\mathrm{ari}} \vdash M_0' : \Sigma' \triangleright T' \,/\, C' \Rightarrow D^{\mathrm{ini}}$, and
- $\Sigma' \triangleright T' \,/\, C' \Rightarrow D^{\mathrm{fin}} <: \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C$

for some T^{par}, T^{ari}, D^{ini}, D^{fin}, Σ', T', and C'. By Lemma 3 with $\Sigma' \triangleright T' \,/\, C' \Rightarrow D^{\mathrm{fin}} <: \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C$, we have

- $\Sigma <: \Sigma'$,
- $T' <: T$,
- $C^{\mathrm{ini}} <: C'$,
- $D^{\mathrm{fin}} <: C$, and
- $\Box(\Sigma') <: \Box(\Sigma)$.

By Lemma 3 with $\Sigma <: \Sigma'$ and $\varsigma_i : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, D^{\mathrm{ini}} \Rightarrow D^{\mathrm{fin}} \in \Sigma'$, we have

- $\varsigma_i : U_i^{\mathrm{par}} \rightsquigarrow U_i^{\mathrm{ari}} \,/\, \Box \in \Sigma$,
- $T^{\mathrm{par}} <: U_i^{\mathrm{par}}$,
- $U_i^{\mathrm{ari}} <: T^{\mathrm{ari}}$, and
- $D^{\mathrm{ini}} <: D^{\mathrm{fin}}$.

By (HT_SubV) with $\Gamma \vdash V : T^{\mathrm{par}}$ and $T^{\mathrm{par}} <: U_i^{\mathrm{par}}$, we have $\Gamma \vdash V : U_i^{\mathrm{par}}$. Because $\Gamma, z_i : U_i^{\mathrm{par}} \vdash N_i : \Sigma_i \triangleright U_i^{\mathrm{ari}} \,/\, \Box$, we have

$$\Gamma \vdash N_i[V/z_i] : \Sigma_i \triangleright U_i^{\mathrm{ari}} \,/\, \Box$$

by Lemma 2. Because

$$\dfrac{C.\Sigma <: \Sigma_i \qquad U_i^{\mathrm{ari}} <: U_i^{\mathrm{ari}} \qquad \dfrac{}{\Box <: \Box}\ (\text{HS\_AnsBox})}{\Sigma_i \triangleright U_i^{\mathrm{ari}} \,/\, \Box <: C.\Sigma \triangleright U_i^{\mathrm{ari}} \,/\, \Box}\ (\text{HS\_Comp})$$

by Lemma 4, we have

$$\Gamma \vdash N_i[V/z_i] : C.\Sigma \triangleright U_i^{\mathrm{ari}} \,/\, \square \tag{11}$$

by (HT_SubC). Because $D^{\mathrm{ini}} <: C$ by Lemma 5 with $D^{\mathrm{ini}} <: D^{\mathrm{fin}}$ and $D^{\mathrm{fin}} <: C$, we have

$$\dfrac{\Sigma <: \Sigma' \qquad T' <: T \qquad \dfrac{C^{\mathrm{ini}} <: C' \qquad D^{\mathrm{ini}} <: C}{C' \Rightarrow D^{\mathrm{ini}} <: C^{\mathrm{ini}} \Rightarrow C}\,(\text{HS\_AnsMod}) \qquad \square(\Sigma') <: \square(\Sigma)}{\Sigma' \triangleright T' \,/\, C' \Rightarrow D^{\mathrm{ini}} <: \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C}\,(\text{HS\_Comp})$$

Thus, by (HT_SubC) with $\Gamma, y : T^{\mathrm{ari}} \vdash M'_0 : \Sigma' \triangleright T' \,/\, C' \Rightarrow D^{\mathrm{ini}}$, and Lemma 11 with $U_i^{\mathrm{ari}} <: T^{\mathrm{ari}}$, we have

$$\Gamma, y : U_i^{\mathrm{ari}} \vdash M'_0 : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C\ .$$

Thus, by (HT_Handle) with

- $H = \{\mathsf{return}\, x \mapsto L\} \uplus \{\sigma_j(y_j; k_j) \mapsto M_j\}^{1 \le j \le m} \uplus \{\varsigma_j(z_j) \mapsto N_j\}^{1 \le j \le n}$,
- $\Sigma = \{\sigma_j : T_j^{\mathrm{par}} \rightsquigarrow T_j^{\mathrm{ari}} \,/\, C_j^{\mathrm{ini}} \Rightarrow C_j^{\mathrm{fin}}\}^{1 \le j \le m} \uplus \{\varsigma_j : U_j^{\mathrm{par}} \rightsquigarrow U_j^{\mathrm{ari}} \,/\, \square\}^{1 \le j \le n}$,
- $\Gamma, y : U_i^{\mathrm{ari}}, x : T \vdash L : C^{\mathrm{ini}}$ by Lemma 1,
- $\forall\, j \in [1, m].\ \Gamma, y : U_i^{\mathrm{ari}}, y_j : T_j^{\mathrm{par}}, k_j : T_j^{\mathrm{ari}} \to C_j^{\mathrm{ini}} \vdash M_j : C_j^{\mathrm{fin}}$ by Lemma 1,
- $\forall\, j \in [1, n].\ \Gamma, y : U_i^{\mathrm{ari}}, z_j : U_j^{\mathrm{par}} \vdash N_j : \Sigma_j \triangleright U_j^{\mathrm{ari}} \,/\, \square$ by Lemma 1,
- $\forall\, C'' \in \{\overline{C_j^{\mathrm{ini}}}^{1 \le j \le m}, C\}.\ \forall\, j \in [1, n].\ C''.\Sigma <: \Sigma_j$

(these are derived by the premise of the derivation of $\Gamma \vdash \mathsf{with}\, H\, \mathsf{handle}\, M_0 : C$), we have

$$\Gamma, y : U_i^{\mathrm{ari}} \vdash \mathsf{with}\, H\, \mathsf{handle}\, M'_0 : C\ . \tag{12}$$

By (HT_Let) with derivations (11) and (12), we have the conclusion

$$\Gamma \vdash \mathsf{let}\, y = N_i[V/z_i]\, \mathsf{in}\, \mathsf{with}\, H\, \mathsf{handle}\, M'_0 : C\ .$$

Case (HE_HandleE): We are given $M_0 \longrightarrow M'_0$ for some M'_0 such that $M' = \mathsf{with}\, H\, \mathsf{handle}\, M'_0$. By the IH, $\Gamma \vdash M'_0 : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C$. By (HT_Handle), we have the conclusion

$$\Gamma \vdash \mathsf{with}\, H\, \mathsf{handle}\, M'_0 : C\ .$$

□

Theorem 1 (Type Soundness)**.** *Assume that* $\Sigma = \{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, \square\}^{1 \le i \le n}$*. If* $\emptyset \vdash M : \Sigma \triangleright T \,/\, A$ *and* $M \longrightarrow^* N$ *and* $N \not\longrightarrow$*, then either of the following holds:*

- $N = \mathsf{return}\, V$ *and* $\emptyset \vdash V : T$ *for some* V*; or*
- $N = \sigma_i(V; x.\, L)$ *and* $\emptyset \vdash V : T_i^{\mathrm{par}}$ *and* $x : T_i^{\mathrm{ari}} \vdash L : \Sigma \triangleright T \,/\, A$ *for some* $i \in [1, n]$*,* V*,* x*, and* L*.*

3.2 Type Soundness of EPCF

Lemma 15 (Weakening)**.** Assume that $dom(\Delta_2) \cap dom(\Delta_1, \Delta_3)$ is empty.

- If $\Xi \parallel \Delta_1, \Delta_3 \vdash v : \tau$, then $\Xi \parallel \Delta_1, \Delta_2, \Delta_3 \vdash v : \tau$.
- If $\Xi \parallel \Delta_1, \Delta_3 \vdash e : \tau$, then $\Xi \parallel \Delta_1, \Delta_2, \Delta_3 \vdash e : \tau$.

Proof. Straightforward by mutual induction on the typing derivations. □

Lemma 16 (Value Substitution)**.** Assume that $\Xi \parallel \Delta_1 \vdash v_0 : \tau_0$.

- If $\Xi \parallel \Delta_1, x : \tau_0, \Delta_2 \vdash v : \tau$, then $\Xi \parallel \Delta_1, \Delta_2 \vdash v[v_0/x] : \tau$.
- If $\Xi \parallel \Delta_1, x : \tau_0, \Delta_2 \vdash e : \tau$, then $\Xi \parallel \Delta_1, \Delta_2 \vdash e[v_0/x] : \tau$.

Proof. Straightforward by mutual induction on the typing derivations. The case for (T_VAR) rests on Lemma 15. $\square$

Lemma 17 (Canonical Forms)**.** Assume that $\Xi \parallel \emptyset \vdash v : \tau$.

- If $\tau = B$, then $v = c$ for some c such that $ty(c) = B$.
- If $\tau = \mathsf{n}$, then $v = \underline{\mathsf{i}}$ for some i such that $0 < i \leq n$.
- If $\tau = \tau_1 \rightarrow \tau_2$, then $v = \lambda x.e$ for some x and e, or $v = \mathsf{fix}\, x.v'$ for some x and v'.

Proof. Straightforward by case analysis on the typing derivation. Note that, for any c, $ty(c) = B$ for some B by Assumption 1. $\square$

Lemma 18 (Progress)**.** If $\Xi \parallel \emptyset \vdash e : \tau$, then one of the following holds:

- $e = \mathsf{return}\, v$ for some v;
- $e = \sigma(v; x.\, e')$ for some σ, v, x, and e'; or
- $e \longrightarrow e'$ for some e'.

Proof. By induction on the typing derivation applied last to derive $\Xi \parallel \emptyset \vdash e : \tau$.

Case (T_RETURN): Obvious.

Case (T_LET): We are given

$$\frac{\Xi \parallel \emptyset \vdash e_1 : \tau_1 \quad \Xi \parallel x : \tau_1 \vdash e_2 : \tau}{\Xi \parallel \emptyset \vdash \mathsf{let}\, x = e_1 \,\mathsf{in}\, e_2 : \tau}$$

for some x, e_1, e_2, and τ_1 such that $e = (\mathsf{let}\, x = e_1 \,\mathsf{in}\, e_2)$. By case analysis on the result of the IH on $\Xi \parallel \emptyset \vdash e_1 : \tau_1$.

Case $\exists\, v_1.\ e_1 = \mathsf{return}\, v_1$: By (E_LETV).
Case $\exists\, \sigma, v_1, y, e_1'.\ e_1 = \sigma(v_1; y.\, e_1')$: By (E_LETOP).
Case $\exists\, e_1'.\ e_1 \longrightarrow e_1'$: By (E_LETE).

Case (T_APP): We are given

$$\frac{\Xi \parallel \emptyset \vdash v_1 : \tau' \rightarrow \tau \quad \Xi \parallel \emptyset \vdash v_2 : \tau'}{\Xi \parallel \emptyset \vdash v_1\, v_2 : \tau}$$

for some v_1, v_2, and τ' such that $e = v_1\, v_2$. By case analysis on the result of applying Lemma 17 to $\Xi \parallel \emptyset \vdash v_1 : \tau' \rightarrow \tau$.

Case $\exists\, x, e_1.\ v_1 = \lambda x.e_1$: By (E_BETA).
Case $\exists\, x, v_1'.\ v_1 = \mathsf{fix}\, x.v_1'$: By (E_FIX).

Case (T_CASE): We are given

$$\frac{\Xi \parallel \emptyset \vdash v : \mathsf{n} \quad \forall\, i \in [1, n].\ \Xi \parallel \emptyset \vdash e_i : \tau}{\Xi \parallel \emptyset \vdash \mathsf{case}(v; e_1, \cdots, e_n) : \tau}$$

for some v, n, $e_1, \cdots, e_n$ such that $e = \mathsf{case}(v; e_1, \cdots, e_n)$. By Lemma 17, $v = \underline{\mathsf{i}}$ for some i such that $0 < i \leq n$. Thus, we have the conclusion by (E_CASE).

Case (T_OP): Obvious.

$\square$

Lemma 19 (Subject Reduction)**.** If $\Xi \parallel \Delta \vdash e : \tau$ and $e \longrightarrow e'$, then $\Xi \parallel \Delta \vdash e' : \tau$.

Proof. By induction on the typing derivation.

Case (T_Return): We have $e = \mathsf{return}\, v$ for some v, but there is a contradiction because there is no evaluation rule applicable to $\mathsf{return}\, v$.

Case (T_Let): We are given

$$\frac{\Xi \parallel \Delta \vdash e_1 : \tau_1 \quad \Xi \parallel \Delta, x : \tau_1 \vdash e_2 : \tau}{\Xi \parallel \Delta \vdash \mathsf{let}\, x = e_1 \,\mathsf{in}\, e_2 : \tau}$$

for some x, e_1, e_2, and τ_1 such that $e = (\mathsf{let}\, x = e_1 \,\mathsf{in}\, e_2)$. We have $\mathsf{let}\, x = e_1 \,\mathsf{in}\, e_2 \longrightarrow e'$. By case analysis on the evaluation rule applied last to derive it.

Case (E_LetV): We are given

$$\mathsf{let}\, x = \mathsf{return}\, v_1 \,\mathsf{in}\, e_2 \longrightarrow e_2[v_1/x]$$

for some v_1 such that $e_1 = \mathsf{return}\, v_1$ and $e' = e_2[v_1/x]$. Because $\Xi \parallel \Delta \vdash \mathsf{return}\, v_1 : \tau_1$, its inversion implies $\Xi \parallel \Delta \vdash v_1 : \tau_1$. By Lemma 16, we have the conclusion $\Xi \parallel \Delta \vdash e_2[v_1/x] : \tau$.

Case (E_LetOp): We are given

$$\mathsf{let}\, x = \sigma(v_1; y.\, e_1') \,\mathsf{in}\, e_2 \longrightarrow \sigma(v_1; y.\, \mathsf{let}\, x = e_1' \,\mathsf{in}\, e_2)$$

for some σ, v_1, y, and e_1' such that $e_1 = \sigma(v_1; y.\, e_1')$ and $e' = \sigma(v_1; y.\, \mathsf{let}\, x = e_1' \,\mathsf{in}\, e_2)$ and $y \notin fv(e_2)$. Because $\Xi \parallel \Delta \vdash \sigma(v_1; y.\, e_1') : \tau_1$, its inversion implies

- $\sigma : B \rightsquigarrow E \in \Xi$,
- $\Xi \parallel \Delta \vdash v_1 : B$, and
- $\Xi \parallel \Delta, y : E \vdash e_1' : \tau_1$

for some B and E. By Lemma 15, $\Xi \parallel \Delta, y : E, x : \tau_1 \vdash e_2 : \tau$. By (T_Let),

$$\Xi \parallel \Delta, y : E \vdash \mathsf{let}\, x = e_1' \,\mathsf{in}\, e_2 : \tau \ .$$

By (T_Op), we have the conclusion

$$\Xi \parallel \Delta \vdash \sigma(v_1; y.\, \mathsf{let}\, x = e_1' \,\mathsf{in}\, e_2) : \tau \ .$$

Case (E_LetE): We are given

$$e_1 \longrightarrow e_1'$$

for some e_1' such that $e' = (\mathsf{let}\, x = e_1' \,\mathsf{in}\, e_2)$. By the IH, $\Xi \parallel \Delta \vdash e_1' : \tau_1$. Therefore, by (T_Let), we have the conclusion

$$\Xi \parallel \Delta \vdash \mathsf{let}\, x = e_1' \,\mathsf{in}\, e_2 : \tau.$$

Case (T_App): We are given

$$\frac{\Xi \parallel \Delta \vdash v_1 : \tau' \to \tau \quad \Xi \parallel \Delta \vdash v_2 : \tau'}{\Xi \parallel \Delta \vdash v_1\, v_2 : \tau}$$

for some v_1, v_2, and τ' such that $e = v_1\, v_2$. We have $v_1\, v_2 \longrightarrow e'$. By case analysis on the evaluation rule applied last to derive it.

Case (E_Beta): We are given

$$(\lambda x.e_1)\, v_2 \longrightarrow e_1[v_2/x]$$

for some x and e_1 such that $v_1 = \lambda x.e_1$ and $e' = e_1[v_2/x]$. By the inversion of $\Xi \parallel \Delta \vdash \lambda x.e_1 : \tau' \to \tau$, we have $\Xi \parallel \Delta, x : \tau' \vdash e_1 : \tau$. Because $\Xi \parallel \Delta \vdash v_2 : \tau'$, we have the conclusion $\Xi \parallel \Delta \vdash e_1[v_2/x] : \tau$ by Lemma 16.

Case (E_Fix): We are given

$$(\mathsf{fix}\,x.v_1')\,v_2 \;\longrightarrow\; v_1'[\mathsf{fix}\,x.v_1'/x]\,v_2$$

for some x and v_1' such that $v_1 = \mathsf{fix}\,x.v_1'$ and $e' = v_1'[\mathsf{fix}\,x.v_1'/x]\,v_2$. By the inversion of $\Xi \parallel \Delta \vdash \mathsf{fix}\,x.v_1' : \tau' \to \tau$, we have $\Xi \parallel \Delta, x : \tau' \to \tau \vdash v_1' : \tau' \to \tau$. By Lemma 16, $\Xi \parallel \Delta \vdash v_1'[\mathsf{fix}\,x.v_1'/x] : \tau' \to \tau$. Therefore, by (T_App), we have the conclusion

$$\Xi \parallel \Delta \vdash v_1'[\mathsf{fix}\,x.v_1'/x]\,v_2 : \tau\ .$$

Case (T_Case): We are given

$$\frac{\Xi \parallel \Delta \vdash v : \mathsf{n} \quad \forall\, i \in [1, n].\ \Xi \parallel \Delta \vdash e_i : \tau}{\Xi \parallel \Delta \vdash \mathsf{case}(v; e_1, \cdots, e_n) : \tau}$$

for some v, n, $e_1, \cdots, e_n$ such that $e = \mathsf{case}(v; e_1, \cdots, e_n)$. Because $\mathsf{case}(v; e_1, \cdots, e_n) \longrightarrow e'$, we have $v = \underline{i}$ and $e' = e_i$ for some i such that $0 < i \le n$. Because $\Xi \parallel \Delta \vdash e_i : \tau$, we have the conclusion.

Case (T_Op): We have $e = \sigma(v; x.\, e'')$ for some σ, v, x, and e'', but there is a contradiction because there is no evaluation rule applicable to $\sigma(v; x.\, e'')$.

$\Box$

3.3 Type Preservation

Lemma 20 (Asymmetry of Pure Signatures)**.** If $\Sigma_1 <: \Sigma_2$ and $\Box(\Sigma_2) <: \Box(\Sigma_1)$, then $\Box(\Sigma_1) = \Box(\Sigma_2)$.

Proof. Let $\sigma : T_1^{\mathrm{par}} \leadsto T_1^{\mathrm{ari}} \,/\, \Box \in \Box(\Sigma_1)$. Because $\Box(\Sigma_2) <: \Box(\Sigma_1)$, there exist some T_2^{par} and T_2^{ari} such that $\sigma : T_2^{\mathrm{par}} \leadsto T_2^{\mathrm{ari}} \,/\, \Box \in \Box(\Sigma_2)$ and $T_1^{\mathrm{par}} <: T_2^{\mathrm{par}}$ and $T_2^{\mathrm{ari}} <: T_1^{\mathrm{ari}}$ by Lemma 3. By Lemma 3 with $\Sigma_1 <: \Sigma_2$, $T_2^{\mathrm{par}} <: T_1^{\mathrm{par}}$ and $T_1^{\mathrm{ari}} <: T_2^{\mathrm{ari}}$. By Lemma 6, $T_1^{\mathrm{par}} = T_2^{\mathrm{par}}$ and $T_1^{\mathrm{ari}} = T_2^{\mathrm{ari}}$. Thus, $\sigma : T_1^{\mathrm{par}} \leadsto T_1^{\mathrm{ari}} \,/\, \Box \in \Box(\Sigma_2)$.

To show the converse, let $\sigma : T_2^{\mathrm{par}} \leadsto T_2^{\mathrm{ari}} \,/\, \Box \in \Box(\Sigma_2)$. Because $\Sigma_1 <: \Sigma_2$, there exist some T_1^{par} and T_1^{ari} such that $\sigma : T_1^{\mathrm{par}} \leadsto T_1^{\mathrm{ari}} \,/\, \Box \in \Sigma_1$ and $T_2^{\mathrm{par}} <: T_1^{\mathrm{par}}$ and $T_1^{\mathrm{ari}} <: T_2^{\mathrm{ari}}$ by Lemma 3. By Lemma 3 with $\Box(\Sigma_2) <: \Box(\Sigma_1)$, $T_1^{\mathrm{par}} <: T_2^{\mathrm{par}}$ and $T_2^{\mathrm{ari}} <: T_1^{\mathrm{ari}}$. By Lemma 6, $T_1^{\mathrm{par}} = T_2^{\mathrm{par}}$ and $T_1^{\mathrm{ari}} = T_2^{\mathrm{ari}}$. Thus, $\sigma : T_2^{\mathrm{par}} \leadsto T_2^{\mathrm{ari}} \,/\, \Box \in \Box(\Sigma_1)$. $\Box$

Definition 36 (Partial Order on EPCF Typing Contexts)**.** We write $\Delta_1 \preceq \Delta_2$ if $dom(\Delta_1) \subseteq dom(\Delta_2)$ and, for any $x \in dom(\Delta_1)$, $\Delta_1(x) = \Delta_2(x)$.

Definition 37 (Typing of Effect Handlers)**.** Let $\Sigma = \{\sigma_i : T_i^{\mathrm{par}} \leadsto T_i^{\mathrm{ari}} \,/\, C_i^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{fin}}\}^{1 \le i \le m} \uplus \{\varsigma_i : U_i^{\mathrm{par}} \leadsto U_i^{\mathrm{ari}} \,/\, \Box\}$ where $\sigma_1, \cdots, \sigma_m$ and $\varsigma_1, \cdots, \varsigma_n$ are ordered, respectively.

For variable sequences $\overline{h^{\boxslash}} = h_1^{\boxslash}, \cdots, h_m^{\boxslash}$ and $\overline{h^{\Box}} = h_1^{\Box}, \cdots, h_n^{\Box}$, we write $\overline{h^{\boxslash}}; \overline{h^{\Box}} : \Sigma$ to denote the typing context that:

- for each $i \in [1, m]$, assigns to the variable $h_i^{\boxslash}$ the type $\llbracket T_i^{\mathrm{par}} \rrbracket \to (\llbracket T_i^{\mathrm{ari}} \rrbracket \to \overline{(\llbracket U_j^{\mathrm{par}} \rrbracket \to \llbracket U_j^{\mathrm{ari}} \rrbracket) \to}^{1 \le j \le n} \llbracket C_i^{\mathrm{ini}} \rrbracket) \to \llbracket C_i^{\mathrm{fin}} \rrbracket$; and
- for each $i \in [1, n]$, assigns to the variable $h_i^{\Box}$ the type $\llbracket U_i^{\mathrm{par}} \rrbracket \to \llbracket U_i^{\mathrm{ari}} \rrbracket$.

Given a variable sequence $\overline{h} = \overline{h^{\boxslash}}, \overline{h^{\Box}}$, we simply write $\overline{h} : \Sigma$ to denote $\overline{h^{\boxslash}}; \overline{h^{\Box}} : \Sigma$.

For value sequences $\overline{v^{\boxslash}} = v_1^{\boxslash}, \cdots, v_m^{\boxslash}$ and $\overline{v^{\Box}} = v_1^{\Box}, \cdots, v_n^{\Box}$, we write $\Xi \parallel \Delta \vdash \overline{v^{\boxslash}}; \overline{v^{\Box}} : \Sigma$ if:

- for each $i \in [1, m]$, $\Xi \parallel \Delta \vdash v_i^{\boxslash} : \llbracket T_i^{\mathrm{par}} \rrbracket \to (\llbracket T_i^{\mathrm{ari}} \rrbracket \to \overline{(\llbracket U_j^{\mathrm{par}} \rrbracket \to \llbracket U_j^{\mathrm{ari}} \rrbracket) \to}^{1 \le j \le n} \llbracket C_i^{\mathrm{ini}} \rrbracket) \to \llbracket C_i^{\mathrm{fin}} \rrbracket$ holds; and
- for each $i \in [1, n]$, $\Xi \parallel \Delta \vdash v_i^{\Box} : \llbracket U_i^{\mathrm{par}} \rrbracket \to \llbracket U_i^{\mathrm{ari}} \rrbracket$ holds.

Given a value sequence $\overline{v} = \overline{v^{\boxslash}}, \overline{v^{\Box}}$, we simply write $\Xi \parallel \Delta \vdash \overline{v} : \Sigma$ to denote $\Xi \parallel \Delta \vdash \overline{v^{\boxslash}}; \overline{v^{\Box}} : \Sigma$.

Lemma 21 (Typing Applications)**.**

1. If $\Xi \parallel \Delta \vdash v_1 : \tau_1 \to \tau_2$ and $\Xi \parallel \Delta \vdash v_2 : \tau_1$, then $\Xi \parallel \Delta \vdash v_1\,v_2 : \tau_2$.

2. If $\Xi \parallel \Delta \vdash v : \tau_1 \to \tau_2$ and $\Xi \parallel \Delta \vdash e : \tau_1$, then $\Xi \parallel \Delta \vdash v\ e : \tau_2$.

3. If $\Xi \parallel \Delta \vdash e_1 : \tau_1 \to \tau_2$ and $\Xi \parallel \Delta \vdash e_2 : \tau_1$, then $\Xi \parallel \Delta \vdash e_1\ e_2 : \tau_2$.

Proof.

1. By (T_App).

2. It suffices to show that
$$\Xi \parallel \Delta \vdash \mathsf{let}\, x = e\, \mathsf{in}\, v\, x : \tau_2$$
for some fresh x. By Lemma 15, $\Xi \parallel \Delta, x : \tau_1 \vdash v : \tau_1 \to \tau_2$. Then, we can derive the conclusion as follows.
$$\dfrac{\Xi \parallel \Delta \vdash e : \tau_1 \qquad \dfrac{\Xi \parallel \Delta, x : \tau_1 \vdash v : \tau_1 \to \tau_2 \qquad \dfrac{(\text{T\_Var})}{\Xi \parallel \Delta, x : \tau_1 \vdash x : \tau_1}}{\Xi \parallel \Delta, x : \tau_1 \vdash v\, x : \tau_2}\,(\text{T\_App})}{\Xi \parallel \Delta \vdash \mathsf{let}\, x = e\, \mathsf{in}\, v\, x : \tau_2}\,(\text{T\_Let})$$

3. It suffices to show that
$$\Xi \parallel \Delta \vdash \mathsf{let}\, x = e_1\, \mathsf{in}\, x\, e_2 : \tau_2$$
for some fresh x. By Lemma 15, $\Xi \parallel \Delta, x : \tau_1 \to \tau_2 \vdash e_2 : \tau_1$. Then, we can derive the conclusion as follows:
$$\dfrac{\Xi \parallel \Delta \vdash e_1 : \tau_1 \to \tau_2 \qquad \dfrac{\dfrac{(\text{T\_Var})}{\Xi \parallel \Delta, x : \tau_1 \to \tau_2 \vdash x : \tau_1 \to \tau_2} \qquad \Xi \parallel \Delta, x : \tau_1 \to \tau_2 \vdash e_2 : \tau_1}{\Xi \parallel \Delta, x : \tau_1 \to \tau_2 \vdash x\, e_2 : \tau_2}\,\text{case (2)}}{\Xi \parallel \Delta \vdash \mathsf{let}\, x = e_1\, \mathsf{in}\, x\, e_2 : \tau_2}\,(\text{T\_Let})$$

□

Definition 38 (Types of the Static Lambda Calculus)**.** We write:

- $\mathsf{term}[\,\Xi \parallel \Delta \vdash \tau\,]$ for the set of EPCF terms e such that $\Xi \parallel \Delta \vdash e : \tau$;
- $\mathsf{val}[\,\Xi \parallel \Delta \vdash \tau\,]$ for the set of EPCF values v such that $\Xi \parallel \Delta \vdash v : \tau$; and
- $\mathsf{vals}[\,\Xi \parallel \Delta \vdash \Sigma\,]$ for the set of sequences of EPCF values $\overline{v}$ such that $\Xi \parallel \Delta \vdash \overline{v} : \Sigma$.

We also define $\mathsf{comp}[\,\Xi \parallel \Delta \vdash C\,]$ depending on $C.A$, as follows:

$$\begin{array}{rcl}\mathsf{comp}[\,\Xi \parallel \Delta \vdash \Sigma \triangleright T \,/\, \square\,] & \stackrel{\text{def}}{=} & \mathsf{vals}[\,\Xi \parallel \Delta \vdash \Sigma\,] \times \mathsf{val}[\,\Xi \parallel \Delta \vdash 1\,] \to \mathsf{term}[\,\Xi \parallel \Delta \vdash [\![T]\!]\,] \\ \mathsf{comp}[\,\Xi \parallel \Delta \vdash \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}\,] & \stackrel{\text{def}}{=} & \mathsf{vals}[\,\Xi \parallel \Delta \vdash \Sigma\,] \times (\mathsf{val}[\,\Xi \parallel \Delta \vdash [\![T]\!] \to [\![\square(\Sigma)]\!][\,[\![C^{\mathrm{ini}}]\!]\,]\,]) \to \mathsf{term}[\,\Xi \parallel \Delta \vdash [\![C^{\mathrm{fin}}]\!]\,]\end{array}$$

Lemma 22 (CPS Transformation of Subtyping)**.**

1. If $T_1 <: T_2$, then $[\![T_1 <: T_2]\!] : \mathsf{val}[\,\Xi \parallel \Delta \vdash [\![T_1]\!]\,] \to \mathsf{val}[\,\Xi \parallel \Delta \vdash [\![T_2]\!]\,]$ for any Ξ and Δ.

2. If $C_1 <: C_2$, then $[\![C_1 <: C_2]\!] : \mathsf{val}[\,\Xi \parallel \Delta \vdash [\![C_1]\!]\,] \to \mathsf{val}[\,\Xi \parallel \Delta \vdash [\![C_2]\!]\,]$ for any Ξ and Δ.

3. If $C_1 <: \Sigma_2 \triangleright T_2 \,/\, \square$, then $[\![C_1 <: \Sigma_2 \triangleright T_2 \,/\, \square]\!]^{\mathsf{e}} : \mathsf{comp}[\,\Xi \parallel \Delta \vdash C_1\,] \times \mathsf{vals}[\,\Xi \parallel \Delta \vdash \Sigma_2\,] \times \mathsf{val}[\,\Xi \parallel \Delta \vdash 1\,] \to \mathsf{term}[\,\Xi \parallel \Delta \vdash [\![T_2]\!]\,]$ for any Ξ and Δ.

4. If $C_1 <: \Sigma_2 \triangleright T_2 \,/\, C_2^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}$, then $[\![C_1 <: \Sigma_2 \triangleright T_2 \,/\, \square]\!]^{\mathsf{e}} : \mathsf{comp}[\,\Xi \parallel \Delta \vdash C_1\,] \times \mathsf{vals}[\,\Xi \parallel \Delta \vdash \Sigma_2\,] \times (\mathsf{val}[\,\Xi \parallel \Delta \vdash [\![T_2]\!] \to [\![\square(\Sigma_2)]\!][\,[\![C_2^{\mathrm{ini}}]\!]\,]\,]) \to \mathsf{term}[\,\Xi \parallel \Delta \vdash [\![C_2^{\mathrm{fin}}]\!]\,]$ for any Ξ and Δ.

5. If $\Sigma_1 <: \Sigma_2$, then $[\![\Sigma_1 <: \Sigma_2]\!] : \mathsf{vals}[\,\Xi \parallel \Delta \vdash \Sigma_1\,] \to \mathsf{vals}[\,\Xi \parallel \Delta \vdash \Sigma_2\,]$ for any Ξ and Δ.

Proof. By mutual induction on the total sizes of the pairs (T_1, T_2), (C_1, C_2), $(C_1, \Sigma_2 \triangleright T_2 \,/\, \square)$, $(C_1, \Sigma_2 \triangleright T_2 \,/\, C_2^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}})$, and (Σ_1, Σ_2). Note that case (2) relies on cases (3) and (4), but cases (3) and (4) rely on the IHs, so there is no circularity.

1. Assume that $T_1 <: T_2$ is derived. By case analysis on the subtyping rule applied last to derive it.

   Case (HS_BASE) and (HS_ENUM): Obvious.

   Case (HS_FUN): We are given

   $$\frac{T_2' <: T_1' \qquad C_1' <: C_2'}{T_1' \to C_1' <: T_2' \to C_2'}$$

   for some T_1', T_2', C_1', and C_2' such that $T_1 = T_1' \to C_1'$ and $T_2 = T_2' \to C_2'$. We are given

   $$\llbracket T_1' \to C_1' <: T_2' \to C_2' \rrbracket = \boldsymbol{\lambda}\mathbf{x}_1.\, \lambda y_2.\mathsf{let}\, z_1 = \mathbf{x}_1\, (\llbracket T_2' <: T_1' \rrbracket @(y_2))\, \mathsf{in\ return}\, \llbracket C_1' <: C_2' \rrbracket @(z_1)$$

   for some fresh y_2 and z_1.
   Assume that Ξ, Δ, and v_1 such that $\Xi \parallel \Delta \vdash v_1 : \llbracket T_1' \to C_1' \rrbracket$ are given. Then, because $\llbracket T_2' \to C_2' \rrbracket = \llbracket T_2' \rrbracket \to \llbracket C_2' \rrbracket$, it suffices to show that

   $$\Xi \parallel \Delta \vdash \lambda y_2.\mathsf{let}\, z_1 = v_1\, (\llbracket T_2' <: T_1' \rrbracket @(y_2))\, \mathsf{in\ return}\, \llbracket C_1' <: C_2' \rrbracket @(z_1) : \llbracket T_2' \rrbracket \to \llbracket C_2' \rrbracket\ ,$$

   which is derived by

   $$\dfrac{\dfrac{\Xi \parallel \Delta, y_2 : \llbracket T_2' \rrbracket \vdash v_1\, (\llbracket T_2' <: T_1' \rrbracket @(y_2)) : \llbracket C_1' \rrbracket \qquad \dfrac{\dfrac{}{\Xi \parallel \Delta, y_2 : \llbracket T_2' \rrbracket, z_1 : \llbracket C_1' \rrbracket \vdash z_1 : \llbracket C_1' \rrbracket}\ (\textsc{T\_Var})}{\Xi \parallel \Delta, y_2 : \llbracket T_2' \rrbracket, z_1 : \llbracket C_1' \rrbracket \vdash \mathsf{return}\, \llbracket C_1' <: C_2' \rrbracket @(z_1) : \llbracket C_2' \rrbracket}\ \textsc{By the IH}}{\Xi \parallel \Delta, y_2 : \llbracket T_2' \rrbracket \vdash \mathsf{let}\, z_1 = v_1\, (\llbracket T_2' <: T_1' \rrbracket @(y_2))\, \mathsf{in\ return}\, \llbracket C_1' <: C_2' \rrbracket @(z_1) : \llbracket C_2' \rrbracket}\ (\textsc{T\_Let})}{\Xi \parallel \Delta \vdash \lambda y_2.\mathsf{let}\, z_1 = v_1\, (\llbracket T_2' <: T_1' \rrbracket @(y_2))\, \mathsf{in\ return}\, \llbracket C_1' <: C_2' \rrbracket @(z_1) : \llbracket T_2' \rrbracket \to \llbracket C_2' \rrbracket}\ (\textsc{T\_Abs})$$

   with

   $$\dfrac{\dfrac{\Xi \parallel \Delta \vdash v_1 : \llbracket T_1' \to C_1' \rrbracket \quad \llbracket T_1' \to C_1' \rrbracket = \llbracket T_1' \rrbracket \to \llbracket C_1' \rrbracket}{\Xi \parallel \Delta, y_2 : \llbracket T_2' \rrbracket \vdash v_1 : \llbracket T_1' \rrbracket \to \llbracket C_1' \rrbracket}\ \textsc{Lemma 15} \qquad \dfrac{\dfrac{}{\Xi \parallel \Delta, y_2 : \llbracket T_2' \rrbracket \vdash y_2 : \llbracket T_2' \rrbracket}\ (\textsc{T\_Var})}{\Xi \parallel \Delta, y_2 : \llbracket T_2' \rrbracket \vdash \llbracket T_2' <: T_1' \rrbracket @(y_2) : \llbracket T_1' \rrbracket}\ \textsc{By the IH}}{\Xi \parallel \Delta, y_2 : \llbracket T_2' \rrbracket \vdash v_1\, (\llbracket T_2' <: T_1' \rrbracket @(y_2)) : \llbracket C_1' \rrbracket}\ (\textsc{T\_App}).$$

2. Assume that $C_1 <: C_2$ is derived. We are given

   $$\boldsymbol{\lambda}\mathbf{x}_1.\, \lambda \overline{h_2}, k_2.\llbracket C_1 <: C_2 \rrbracket^{\mathsf{e}} @((\boldsymbol{\lambda}(\overline{\mathbf{h}_1}, \mathbf{k}_1).\, \mathbf{x}_1\, \overline{\mathbf{h}_1}\, \mathbf{k}_1), \overline{h_2}, k_2)$$

   where $|\overline{\mathbf{h}_1}| = |C_1.\Sigma|$ and $|\overline{h_2}| = |C_2.\Sigma|$ for some fresh variable sequence $\overline{h_2}$ and fresh variable k_2. Assume that Ξ, Δ, and v_1 such that $\Xi \parallel \Delta \vdash v_1 : \llbracket C_1 \rrbracket$ are given. Then, it suffices to show that

   $$\Xi \parallel \Delta \vdash \lambda \overline{h_2}, k_2.\llbracket C_1 <: C_2 \rrbracket^{\mathsf{e}} @((\boldsymbol{\lambda}(\overline{\mathbf{h}_1}, \mathbf{k}_1).\, v_1\, \overline{\mathbf{h}_1}\, \mathbf{k}_1), \overline{h_2}, k_2) : \llbracket C_2 \rrbracket\ .$$

   First, we show that, for any Δ' such that $dom(\Delta') \cap dom(\Delta) = \emptyset$,

   $$\boldsymbol{\lambda}(\overline{\mathbf{h}_1}, \mathbf{k}_1).\, v_1\, \overline{\mathbf{h}_1}\, \mathbf{k}_1 : \mathsf{comp}[\, \Xi \parallel \Delta, \Delta' \vdash C_1\,] \tag{13}$$

   by case analysis on $C_1.A$.

   Case $C_1.A = \square$: It suffices to show that

   $$\boldsymbol{\lambda}(\overline{\mathbf{h}_1}, \mathbf{k}_1).\, v_1\, \overline{\mathbf{h}_1}\, \mathbf{k}_1 : \mathsf{vals}[\, \Xi \parallel \Delta, \Delta' \vdash C_1.\Sigma\,] \times \mathsf{val}[\, \Xi \parallel \Delta, \Delta' \vdash 1\,] \to \mathsf{term}[\, \Xi \parallel \Delta, \Delta' \vdash \llbracket C_1.T \rrbracket\,],$$

   which is obvious by $\Xi \parallel \Delta \vdash v_1 : \llbracket C_1 \rrbracket$ and Lemmas 15 and 21.
   Case $\exists\, C_1^{\mathrm{ini}}, C_1^{\mathrm{fin}}.\ C_1.A = C_1^{\mathrm{ini}} \Rightarrow C_1^{\mathrm{fin}}$: It suffices to show that

   $$\boldsymbol{\lambda}(\overline{\mathbf{h}_1}, \mathbf{k}_1).\, v_1\, \overline{\mathbf{h}_1}\, \mathbf{k}_1 : \mathsf{vals}[\, \Xi \parallel \Delta, \Delta' \vdash C_1.\Sigma\,] \times (\mathsf{val}[\, \Xi \parallel \Delta, \Delta' \vdash \llbracket C_1.T \rrbracket \to \llbracket \square(C_1.\Sigma) \rrbracket [\, \llbracket C_1^{\mathrm{ini}} \rrbracket\,]\,]) \to \mathsf{term}[\, \Xi \parallel \Delta, \Delta' \vdash \llbracket C_1^{\mathrm{fin}} \rrbracket\,]\ ,$$

   which is derived by $\Xi \parallel \Delta \vdash v_1 : \llbracket C_1 \rrbracket$ and Lemmas 15 and 21.

Thus, we have formula (13).

Next, we proceed by case analysis on $C_2.A$.

Case $C_2.A = \square$: It suffices to show that

$$\Xi \parallel \Delta \vdash \lambda \overline{h_2}, k_2. \llbracket C_1 <: C_2 \rrbracket^{\mathsf{e}} @((\boldsymbol{\lambda}(\overline{\mathbf{h}_1}, \mathbf{k}_1).\, v_1\, \overline{\mathbf{h}_1}\, \mathbf{k}_1), \overline{h_2}, k_2) : \llbracket C_2.\Sigma \rrbracket [\, 1 \to \llbracket C_2.T \rrbracket \,] \;.$$

By (T_Abs) and (T_Return), it suffices to show that

$$\Xi \parallel \Delta, \overline{h_2} \;:\; C_2.\Sigma, k_2 : 1 \vdash \llbracket C_1 <: C_2 \rrbracket^{\mathsf{e}} @((\boldsymbol{\lambda}(\overline{\mathbf{h}_1}, \mathbf{k}_1).\, v_1\, \overline{\mathbf{h}_1}\, \mathbf{k}_1), \overline{h_2}, k_2) : \llbracket C_2.T \rrbracket \;,$$

which is derived by case (3) with

- formula (13),
- $\overline{h_2} \in \mathsf{vals}[\, \Xi \parallel \Delta, \overline{h_2} \;:\; C_2.\Sigma, k_2 : 1 \vdash C_2.\Sigma \,]$ by (T_Var), and
- $k_2 \in \mathsf{val}[\, \Xi \parallel \Delta, \overline{h_2} \;:\; C_2.\Sigma, k_2 : 1 \vdash 1 \,]$ by (T_Var).

Case $\exists\, C_2^{\mathrm{ini}}, C_2^{\mathrm{fin}}.\ C_2.A = C_2^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}$: It suffices to show that

$$\Xi \parallel \Delta \vdash \lambda \overline{h_2}, k_2. \llbracket C_1 <: C_2 \rrbracket^{\mathsf{e}} @((\boldsymbol{\lambda}(\overline{\mathbf{h}_1}, \mathbf{k}_1).\, v_1\, \overline{\mathbf{h}_1}\, \mathbf{k}_1), \overline{h_2}, k_2) : \llbracket C_2.\Sigma \rrbracket [\, (\llbracket C_2.T \rrbracket \to \llbracket \square(C_2.\Sigma) \rrbracket [\, \llbracket C_2^{\mathrm{ini}} \rrbracket \,]) \to \llbracket C_2^{\mathrm{fin}} \rrbracket \,] \;.$$

Let $\Delta' = \Delta, h_2 \;:\; C_2.\Sigma, k_2 : \llbracket C_2.T \rrbracket \to \llbracket \square(C_2.\Sigma) \rrbracket [\, \llbracket C_2^{\mathrm{ini}} \rrbracket \,]$. By (T_Abs) and (T_Return), it suffices to show that

$$\Xi \parallel \Delta' \vdash \llbracket C_1 <: C_2 \rrbracket^{\mathsf{e}} @((\boldsymbol{\lambda}(\overline{\mathbf{h}_1}, \mathbf{k}_1).\, v_1\, \overline{\mathbf{h}_1}\, \mathbf{k}_1), \overline{h_2}, k_2) : \llbracket C_2^{\mathrm{fin}} \rrbracket \;,$$

which is derived by case (4) with

- formula (13),
- $\overline{h_2} \in \mathsf{vals}[\, \Xi \parallel \Delta' \vdash C_2.\Sigma \,]$ by (T_Var), and
- $k_2 \in \mathsf{val}[\, \Xi \parallel \Delta' \vdash \llbracket C_2.T \rrbracket \to \llbracket \square(C_2.\Sigma) \rrbracket [\, \llbracket C_2^{\mathrm{ini}} \rrbracket \,] \,]$ by (T_Var).

3. Assume that $C_1 <: \Sigma_2 \triangleright T_2 \,/\, \square$ is derived. By inversion and Lemma 3, $\Sigma_2 <: C_1.\Sigma$ and $C_1.T <: T_2$ and $C_1.A = \square$. By the definition of the CPS transformation,

$$\llbracket C_1 <: \Sigma_2 \triangleright T_2 \,/\, \square \rrbracket^{\mathsf{e}} = \boldsymbol{\lambda}(\mathbf{f}_1, \overline{\mathbf{h}_2}, \mathbf{k}_2).\, \mathsf{let}\, x_1 = \mathbf{f}_1 @(\llbracket \Sigma_2 <: C_1.\Sigma \rrbracket @(\overline{\mathbf{h}_2}), \mathbf{k}_2)\, \mathsf{in\ return}\, \llbracket C_1.T <: T_2 \rrbracket @(x_1)$$

where $|\overline{\mathbf{h}_2}| = |\Sigma_2|$ for some fresh variable x_1. Assume that Ξ, Δ, $\mathbf{t}_1$, $\overline{v_2^{\mathsf{h}}}$, and v^{k_2} such that

- $\mathbf{t}_1 \in \mathsf{comp}[\, \Xi \parallel \Delta \vdash C_1 \,] = \mathsf{vals}[\, \Xi \parallel \Delta \vdash C_1.\Sigma \,] \times \mathsf{val}[\, \Xi \parallel \Delta \vdash 1 \,] \to \mathsf{term}[\, \Xi \parallel \Delta \vdash \llbracket C_1.T \rrbracket \,]$,
- $\Xi \parallel \Delta \vdash \overline{v_2^{\mathsf{h}}} : \Sigma_2$, and
- $\Xi \parallel \Delta \vdash v^{\mathsf{k}_2} : 1$

are given. Then, it suffices to show that

$$\Xi \parallel \Delta \vdash \mathsf{let}\, x_1 = \mathbf{t}_1 @(\llbracket \Sigma_2 <: C_1.\Sigma \rrbracket @(\overline{v_2^{\mathsf{h}}}), v^{\mathsf{k}_2})\, \mathsf{in\ return}\, \llbracket C_1.T <: T_2 \rrbracket @(x_1) : \llbracket T_2 \rrbracket \;.$$

Because $\overline{v_2^{\mathsf{h}}} \in \mathsf{vals}[\, \Xi \parallel \Delta \vdash \Sigma_2 \,]$, we have $\llbracket \Sigma_2 <: C_1.\Sigma \rrbracket @(\overline{v_2^{\mathsf{h}}}) \in \mathsf{vals}[\, \Xi \parallel \Delta \vdash C_1.\Sigma \,]$ by the IH on $(\Sigma_2, C_1.\Sigma)$. Because $v^{\mathsf{k}_2} \in \mathsf{val}[\, \Xi \parallel \Delta \vdash 1 \,]$, we have $\mathbf{t}_1 @(\llbracket \Sigma_2 <: C_1.\Sigma \rrbracket @(\overline{v_2^{\mathsf{h}}}), v^{\mathsf{k}_2}) \in \mathsf{term}[\, \Xi \parallel \Delta \vdash \llbracket C_1.T \rrbracket \,]$. Thus, the conclusion is derived as follows:

$$\dfrac{\Xi \parallel \Delta \vdash \mathbf{t}_1 @(\llbracket \Sigma_2 <: C_1.\Sigma \rrbracket @(\overline{v_2^{\mathsf{h}}}), v^{\mathsf{k}_2}) : \llbracket C_1.T \rrbracket \qquad \dfrac{\dfrac{\dfrac{\text{(T\_Var)}}{\Xi \parallel \Delta, x_1 : \llbracket C_1.T \rrbracket \vdash x_1 : \llbracket C_1.T \rrbracket}}{\Xi \parallel \Delta, x_1 : \llbracket C_1.T \rrbracket \vdash \llbracket C_1.T <: T_2 \rrbracket @(x_1) : \llbracket T_2 \rrbracket}\ \text{By the IH}}{\Xi \parallel \Delta, x_1 : \llbracket C_1.T \rrbracket \vdash \mathsf{return}\, \llbracket C_1.T <: T_2 \rrbracket @(x_1) : \llbracket T_2 \rrbracket}\ \text{(T\_Return)}}{\Xi \parallel \Delta \vdash \mathsf{let}\, x_1 = \mathbf{t}_1 @(\llbracket \Sigma_2 <: C_1.\Sigma \rrbracket @(\overline{v_2^{\mathsf{h}}}), v^{\mathsf{k}_2})\, \mathsf{in\ return}\, \llbracket C_1.T <: T_2 \rrbracket @(x_1) : \llbracket T_2 \rrbracket}\ \text{(T\_Let)}$$

4. Assume that $C_1 <: \Sigma_2 \triangleright T_2 \,/\, C_2^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}$ is derived. By inversion, $\Sigma_2 <: C_1.\Sigma$ and $C_1.T <: T_2$ and $C_1.A <: C_2^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}$. By case analysis on the subtyping rule applied last to derive $C_1.A <: C_2^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}$.

Case (HS_AnsBox): Contradictory.

Case (HS_AnsEmb): We are given $C_1.A = \Box$ and $C_2^{\mathrm{ini}} <: C_2^{\mathrm{fin}}$. By the definition of the CPS transformation,

$$\begin{array}{rl} & [\![C_1 <: \Sigma_2 \triangleright T_2 \,/\, C_2^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}]\!]^{\mathsf{e}} \\ = & \boldsymbol{\lambda}(\mathbf{f}_1, \overline{\mathbf{h}_2^{\not\Box}}, \overline{\mathbf{h}_2^{\Box}}, \mathbf{k}_2).\, \mathsf{let}\, x_1 = \mathbf{f}_1 @ ([\![\Sigma_2 <: C_1.\Sigma]\!] @ (\overline{\mathbf{h}_2^{\not\Box}}, \overline{\mathbf{h}_2^{\Box}}), \underline{1})\, \mathsf{in}\, \mathsf{let}\, y_2 = \mathbf{k}_2\, ([\![C_1.T <: T_2]\!] @ (x_1))\, \overline{\mathbf{h}_2^{\Box}}\, \mathsf{in}\, \mathsf{return}\, [\![C_2^{\mathrm{ini}} <: C_2^{\mathrm{fin}}]\!] @ (y_2) \;. \end{array}$$

where $|\overline{\mathbf{h}_2^{\not\Box}}| = |\not\Box(\Sigma_2)|$ and $|\overline{\mathbf{h}_2^{\Box}}| = |\Box(\Sigma_2)|$ for some fresh variables x_1 and y_2.

Assume that Ξ, Δ, $\mathbf{t}_1$, $\overline{v_2^{\mathsf{h}\not\Box}}$, $\overline{v_2^{\mathsf{h}\Box}}$, and v^{k_2} such that

- $\mathbf{t}_1 \in \mathsf{comp}[\, \Xi \parallel \Delta \vdash C_1 \,] = \mathsf{vals}[\, \Xi \parallel \Delta \vdash C_1.\Sigma \,] \times \mathsf{val}[\, \Xi \parallel \Delta \vdash 1 \,] \to \mathsf{term}[\, \Xi \parallel \Delta \vdash [\![C_1.T]\!] \,]$,
- $\Xi \parallel \Delta \vdash \overline{v_2^{\mathsf{h}\not\Box}}; \overline{v_2^{\mathsf{h}\Box}} : \Sigma_2$, and
- $\Xi \parallel \Delta \vdash v^{\mathsf{k}_2} : [\![T_2]\!] \to [\![\Box(\Sigma_2)]\!][\, [\![C_2^{\mathrm{ini}}]\!] \,]$.

Then, it suffices to show that

$$\Xi \parallel \Delta \vdash \mathsf{let}\, x_1 = \mathbf{t}_1 @ ([\![\Sigma_2 <: C_1.\Sigma]\!] @ (\overline{v_2^{\mathsf{h}\not\Box}}, \overline{v_2^{\mathsf{h}\Box}}), \underline{1})\, \mathsf{in}\, \mathsf{let}\, y_2 = v^{\mathsf{k}_2}\, ([\![C_1.T <: T_2]\!] @ (x_1))\, \overline{v_2^{\mathsf{h}\Box}}\, \mathsf{in}\, \mathsf{return}\, [\![C_2^{\mathrm{ini}} <: C_2^{\mathrm{fin}}]\!] @ (y_2) : [\![C_2^{\mathrm{fin}}]\!] \;.$$

By the IH on $(\Sigma_2, C_1.\Sigma)$ with $\Xi \parallel \Delta \vdash \overline{v_2^{\mathsf{h}\not\Box}}; \overline{v_2^{\mathsf{h}\Box}} : \Sigma_2$, we have $[\![\Sigma_2 <: C_1.\Sigma]\!] @ (\overline{v_2^{\mathsf{h}\not\Box}}, \overline{v_2^{\mathsf{h}\Box}}) \in \mathsf{vals}[\, \Xi \parallel \Delta \vdash C_1.\Sigma \,]$. Because $\underline{1} \in \mathsf{val}[\, \Xi \parallel \Delta \vdash 1 \,]$ by (T_EConst), we have

$$\Xi \parallel \Delta \vdash \mathbf{t}_1 @ ([\![\Sigma_2 <: C_1.\Sigma]\!] @ (\overline{v_2^{\mathsf{h}\not\Box}}, \overline{v_2^{\mathsf{h}\Box}}), \underline{1}) : [\![C_1.T]\!] \;.$$

By (T_Let), it suffices to show that

$$\Xi \parallel \Delta, x_1 : [\![C_1.T]\!] \vdash \mathsf{let}\, y_2 = v^{\mathsf{k}_2}\, ([\![C_1.T <: T_2]\!] @ (x_1))\, \overline{v_2^{\mathsf{h}\Box}}\, \mathsf{in}\, \mathsf{return}\, [\![C_2^{\mathrm{ini}} <: C_2^{\mathrm{fin}}]\!] @ (y_2) : [\![C_2^{\mathrm{fin}}]\!] \;.$$

By the IH on $(C_1.T, T_2)$ with $x_1 \in \mathsf{val}[\, \Xi \parallel \Delta, x_1 : [\![C_1.T]\!] \vdash [\![C_1.T]\!] \,]$ by (T_Var), we have $\Xi \parallel \Delta, x_1 : [\![C_1.T]\!] \vdash [\![C_1.T <: T_2]\!] @ (x_1) : [\![T_2]\!]$. By Lemmas 15 and 21 with $\Xi \parallel \Delta \vdash v^{\mathsf{k}_2} : [\![T_2]\!] \to [\![\Box(\Sigma_2)]\!][\, [\![C_2^{\mathrm{ini}}]\!] \,]$ and $\Xi \parallel \Delta \vdash \overline{v_2^{\mathsf{h}\Box}} : \Box(\Sigma_2)$, we have

$$\Xi \parallel \Delta, x_1 : [\![C_1.T]\!] \vdash v^{\mathsf{k}_2}\, ([\![C_1.T <: T_2]\!] @ (x_1))\, \overline{v_2^{\mathsf{h}\Box}} : [\![C_2^{\mathrm{ini}}]\!] \;.$$

By (T_Let) and (T_Return), it suffices to show that

$$\Xi \parallel \Delta, x_1 : [\![C_1.T]\!], y_2 : [\![C_2^{\mathrm{ini}}]\!] \vdash [\![C_2^{\mathrm{ini}} <: C_2^{\mathrm{fin}}]\!] @ (y_2) : [\![C_2^{\mathrm{fin}}]\!] \;,$$

which is derived by the IH on $(C_2^{\mathrm{ini}}, C_2^{\mathrm{fin}})$ and $y_2 \in \mathsf{val}[\, \Xi \parallel \Delta, x_1 : [\![C_1.T]\!], y_2 : [\![C_2^{\mathrm{ini}}]\!] \vdash [\![C_2^{\mathrm{ini}}]\!] \,]$ by (T_Var).

Case (HS_AnsMod): We are given

$$\frac{C_2^{\mathrm{ini}} <: C_1^{\mathrm{ini}} \qquad C_1^{\mathrm{fin}} <: C_2^{\mathrm{fin}}}{C_1^{\mathrm{ini}} \Rightarrow C_1^{\mathrm{fin}} <: C_2^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}}$$

for some C_1^{ini} and C_2^{fin} such that $C_1.A = C_1^{\mathrm{ini}} \Rightarrow C_1^{\mathrm{fin}}$.

We consider two cases as follows:

Case $C_1.\Sigma = \Sigma_2$ and $C_1.T = T_2$ and $C_1^{\mathrm{ini}} = C_2^{\mathrm{ini}}$: By the definition of the CPS transformation,

$$\begin{array}{rl} & [\![C_1 <: \Sigma_2 \triangleright T_2 \,/\, C_2^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}]\!]^{\mathsf{e}} \\ = & \boldsymbol{\lambda}(\mathbf{f}_1, \overline{\mathbf{h}_2}, \mathbf{k}_2).\, \mathsf{let}\, x_1 = \mathbf{f}_1 @ (\overline{\mathbf{h}_2}, \mathbf{k}_2)\, \mathsf{in}\, \mathsf{return}\, [\![C_1^{\mathrm{fin}} <: C_2^{\mathrm{fin}}]\!] @ (x_1) \end{array}$$

where $|\overline{\mathbf{h}_2}| = |\Sigma_2|$ for some fresh variable x_1. Assume that Ξ, Δ, $\mathbf{t}_1$, $\overline{v_2^{\mathsf{h}}}$, and v^{k_2} such that

- $\mathbf{t}_1 \in \mathsf{comp}[\, \Xi \parallel \Delta \vdash C_1 \,] = \mathsf{vals}[\, \Xi \parallel \Delta \vdash C_1.\Sigma \,] \times (\mathsf{val}[\, \Xi \parallel \Delta \vdash [\![C_1.T]\!] \to [\![\Box(C_1.\Sigma)]\!][\, [\![C_1^{\mathrm{ini}}]\!] \,] \,]) \to \mathsf{term}[\, \Xi \parallel \Delta \vdash [\![C_1^{\mathrm{fin}}]\!] \,]$,
- $\Xi \parallel \Delta \vdash \overline{v_2^{\mathsf{h}}} : \Sigma_2$, and
- $\Xi \parallel \Delta \vdash v^{\mathsf{k}_2} : [\![T_2]\!] \to [\![\Box(\Sigma_2)]\!][\, [\![C_2^{\mathrm{ini}}]\!] \,]$

are given. Then, it suffices to show that

$$\Xi \parallel \Delta \vdash \mathsf{let}\, x_1 = \mathbf{t}_1 @(\overline{v_2^{\mathsf{h}}}, v^{\mathsf{k}_2})\, \mathsf{in\ return}\, [\![C_1^{\mathrm{fin}} <: C_2^{\mathrm{fin}}]\!]@(x_1) : [\![C_2^{\mathrm{fin}}]\!] \ .$$

Because

- $\mathbf{t}_1 \in \mathsf{vals}[\, \Xi \parallel \Delta \vdash C_1.\Sigma\,] \times (\mathsf{val}[\, \Xi \parallel \Delta \vdash [\![C_1.T]\!] \to [\![\Box(C_1.\Sigma)]\!][\, [\![C_1^{\mathrm{ini}}]\!]\,]\,]) \to \mathsf{term}[\, \Xi \parallel \Delta \vdash [\![C_1^{\mathrm{fin}}]\!]\,]$,
- $\Xi \parallel \Delta \vdash \overline{v_2^{\mathsf{h}}} : C_1.\Sigma$ by $C_1.\Sigma = \Sigma_2$, and
- $\Xi \parallel \Delta \vdash v^{\mathsf{k}_2} : [\![C_1.T]\!] \to [\![\Box(C_1.\Sigma)]\!][\, [\![C_1^{\mathrm{ini}}]\!]\,]$ by $C_1.\Sigma = \Sigma_2$ and $C_1.T = T_2$ and $C_1^{\mathrm{ini}} = C_2^{\mathrm{ini}}$,

we have

$$\Xi \parallel \Delta \vdash \mathbf{t}_1 @(\overline{v_2^{\mathsf{h}}}, v^{\mathsf{k}_2}) : [\![C_1^{\mathrm{fin}}]\!] \ .$$

By (T_Let) and (T_Return), it suffices to show that

$$\Xi \parallel \Delta, x_1 : [\![C_1^{\mathrm{fin}}]\!] \vdash [\![C_1^{\mathrm{fin}} <: C_2^{\mathrm{fin}}]\!]@(x_1) : [\![C_2^{\mathrm{fin}}]\!] \ ,$$

which is shown by the IH on $(C_1^{\mathrm{fin}}, C_2^{\mathrm{fin}})$ with $\Xi \parallel \Delta, x_1 : [\![C_1^{\mathrm{fin}}]\!] \vdash x_1 : [\![C_1^{\mathrm{fin}}]\!]$ by (T_Var).

Case $C_1.\Sigma \neq \Sigma_2$ or $C_1.T \neq T_2$ or $C_1^{\mathrm{ini}} \neq C_2^{\mathrm{ini}}$: By Lemmas 3 and 20, $\Box(C_1.\Sigma) = \Box(\Sigma_2)$. By the definition of the CPS transformation,

$$\begin{array}{rl} & [\![C_1 <: \Sigma_2 \triangleright T_2 \,/\, C_2^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}]\!]^{\mathsf{e}} \\ = & \boldsymbol{\lambda}(\mathbf{f}_1, \overline{\mathbf{h}_2^{\not\Box}}, \overline{\mathbf{h}_2^{\Box}}, \mathbf{k}_2).\, \mathsf{let}\, x_1 = \mathbf{f}_1 @([\![\not\Box(\Sigma_2) <: \not\Box(C_1.\Sigma)]\!]@(\overline{\mathbf{h}_2^{\not\Box}}), \overline{\mathbf{h}_2^{\Box}}, v)\, \mathsf{in\ return}\, [\![C_1^{\mathrm{fin}} <: C_2^{\mathrm{fin}}]\!]@(x_1) \end{array}$$

where $|\overline{\mathbf{h}_2^{\not\Box}}| = |\not\Box(\Sigma_2)|$ and $|\overline{\mathbf{h}_2^{\Box}}| = |\Box(\Sigma_2)|$ and

$$v = \lambda y_1, \overline{h_1^{\Box}}.\mathsf{let}\, z_2 = \mathbf{k}_2\, ([\![C_1.T <: T_2]\!]@(y_1))\, \overline{h_1^{\Box}}\, \mathsf{in\ return}\, [\![C_2^{\mathrm{ini}} <: C_1^{\mathrm{ini}}]\!]@(z_2) \ .$$

for some fresh variables x_1, y_2, and z_2 and fresh variable sequence $\overline{h_1^{\Box}}$ such that $|\overline{h_1^{\Box}}| = |\Box(\Sigma_2)|$. Assume that Ξ, Δ, $\mathbf{t}_1$, $\overline{v_2^{\mathsf{h}\not\Box}}$, $\overline{v_2^{\mathsf{h}\Box}}$, and v^{k_2} such that

- $\mathbf{t}_1 \in \mathsf{comp}[\, \Xi \parallel \Delta \vdash C_1\,] = \mathsf{vals}[\, \Xi \parallel \Delta \vdash C_1.\Sigma\,] \times (\mathsf{val}[\, \Xi \parallel \Delta \vdash [\![C_1.T]\!] \to [\![\Box(C_1.\Sigma)]\!][\, [\![C_1^{\mathrm{ini}}]\!]\,]\,]) \to \mathsf{term}[\, \Xi \parallel \Delta \vdash [\![C_1^{\mathrm{fin}}]\!]\,]$,
- $\Xi \parallel \Delta \vdash \overline{v_2^{\mathsf{h}\not\Box}}; \overline{v_2^{\mathsf{h}\Box}} : \Sigma_2$, and
- $\Xi \parallel \Delta \vdash v^{\mathsf{k}_2} : [\![T_2]\!] \to [\![\Box(\Sigma_2)]\!][\, [\![C_2^{\mathrm{ini}}]\!]\,]$

are given. Then, it suffices to show that

$$\Xi \parallel \Delta \vdash \mathsf{let}\, x_1 = \mathbf{t}_1 @([\![\not\Box(\Sigma_2) <: \not\Box(C_1.\Sigma)]\!]@(\overline{v_2^{\mathsf{h}\not\Box}}), \overline{v_2^{\mathsf{h}\Box}}, v^{\mathsf{k}_1})\, \mathsf{in\ return}\, [\![C_1^{\mathrm{fin}} <: C_2^{\mathrm{fin}}]\!]@(x_1) : [\![C_2^{\mathrm{fin}}]\!] \quad (14)$$

where

$$v^{\mathsf{k}_1} = \lambda y_1, \overline{h_1^{\Box}}.\mathsf{let}\, z_2 = v^{\mathsf{k}_2}\, ([\![C_1.T <: T_2]\!]@(y_1))\, \overline{h_1^{\Box}}\, \mathsf{in\ return}\, [\![C_2^{\mathrm{ini}} <: C_1^{\mathrm{ini}}]\!]@(z_2) \ .$$

First, we show that

$$\Xi \parallel \Delta \vdash v^{\mathsf{k}_1} : [\![C_1.T]\!] \to [\![\Box(C_1.\Sigma)]\!][\, [\![C_1^{\mathrm{ini}}]\!]\,] \ . \quad (15)$$

Let $\Delta' = \Delta, y_1 : [\![C_1.T]\!], \overline{h_1^{\Box}} : \Box(C_1.\Sigma)$. By (T_Abs) and (T_Return), it suffices to show that

$$\Xi \parallel \Delta' \vdash \mathsf{let}\, z_2 = v^{\mathsf{k}_2}\, ([\![C_1.T <: T_2]\!]@(y_1))\, \overline{h_1^{\Box}}\, \mathsf{in\ return}\, [\![C_2^{\mathrm{ini}} <: C_1^{\mathrm{ini}}]\!]@(z_2) : [\![C_1^{\mathrm{ini}}]\!] \ .$$

By Lemma 21 with

- $\Xi \parallel \Delta' \vdash v^{\mathsf{k}_2} : [\![T_2]\!] \to [\![\Box(\Sigma_2)]\!][\, [\![C_2^{\mathrm{ini}}]\!]\,]$ by Lemma 15 with $\Xi \parallel \Delta \vdash v^{\mathsf{k}_2} : [\![T_2]\!] \to [\![\Box(\Sigma_2)]\!][\, [\![C_2^{\mathrm{ini}}]\!]\,]$,
- $\Xi \parallel \Delta' \vdash [\![C_1.T <: T_2]\!]@(y_1) : [\![T_2]\!]$ by the IH on $(C_1.T, T_2)$ with $\Xi \parallel \Delta' \vdash y_1 : [\![C_1.T]\!]$ by (T_Var),
- $\Xi \parallel \Delta' \vdash \overline{h_1^{\Box}} : \Box(C_1.\Sigma)$ by (T_Var), and
- $\Box(C_1.\Sigma) = \Box(\Sigma_2)$,

we have

$$\Xi \parallel \Delta' \vdash v^{\mathsf{k}_2}\,(\llbracket C_1.T <: T_2 \rrbracket @(y_1))\,\overline{h_1^{\Box}} : \llbracket C_2^{\mathrm{ini}} \rrbracket \,.$$

By (T_Let) and (T_Return), it suffices to show that

$$\Xi \parallel \Delta', z_2 : \llbracket C_2^{\mathrm{ini}} \rrbracket \vdash \llbracket C_2^{\mathrm{ini}} <: C_1^{\mathrm{ini}} \rrbracket @(z_2) : \llbracket C_1^{\mathrm{ini}} \rrbracket \,,$$

which is derived by the IH on $(C_2^{\mathrm{ini}}, C_1^{\mathrm{ini}})$ and $\Xi \parallel \Delta', z_2 : \llbracket C_2^{\mathrm{ini}} \rrbracket \vdash z_2 : \llbracket C_2^{\mathrm{ini}} \rrbracket$ by (T_Var).
Now, we derive judgment (14). Because

- $\mathbf{t}_1 \in \mathsf{vals}[\,\Xi \parallel \Delta \vdash C_1.\Sigma\,] \times (\mathsf{val}[\,\Xi \parallel \Delta \vdash \llbracket C_1.T \rrbracket \to \llbracket \Box(C_1.\Sigma) \rrbracket [\,\llbracket C_1^{\mathrm{ini}} \rrbracket\,]\,]) \to \mathsf{term}[\,\Xi \parallel \Delta \vdash \llbracket C_1^{\mathrm{fin}} \rrbracket\,]$,
- $\Xi \parallel \Delta \vdash \llbracket \boxslash(\Sigma_2) <: \boxslash(C_1.\Sigma) \rrbracket @(\overline{v_2^{\mathsf{h}\boxslash}}) : \boxslash(C_1.\Sigma)$, by the IH on $(\boxslash(\Sigma_2), \boxslash(C_1.\Sigma))$ with $\Xi \parallel \Delta \vdash \overline{v_2^{\mathsf{h}\boxslash}} : \boxslash(\Sigma_2)$
- $\Xi \parallel \Delta \vdash \overline{v_2^{\mathsf{h}\Box}} : \Box(C_1.\Sigma)$ because $\Xi \parallel \Delta \vdash \overline{v_2^{\mathsf{h}\Box}} : \Box(\Sigma_2)$ and $\Box(\Sigma_2) = \Box(C_1.\Sigma)$, and
- derivation (15),

we have

$$\Xi \parallel \Delta \vdash \mathbf{t}_1 @(\llbracket \boxslash(\Sigma_2) <: \boxslash(C_1.\Sigma) \rrbracket @(\overline{v_2^{\mathsf{h}\boxslash}}), \overline{v_2^{\mathsf{h}\Box}}, v^{\mathsf{k}_1}) : \llbracket C_1^{\mathrm{fin}} \rrbracket \,.$$

By (T_Let) and (T_Return), it suffices to show that

$$\Xi \parallel \Delta, x_1 : \llbracket C_1^{\mathrm{fin}} \rrbracket \vdash \llbracket C_1^{\mathrm{fin}} <: C_2^{\mathrm{fin}} \rrbracket @(x_1) : \llbracket C_2^{\mathrm{fin}} \rrbracket \,,$$

which is shown by the IH on $(C_1^{\mathrm{fin}}, C_2^{\mathrm{fin}})$ with $\Xi \parallel \Delta, x_1 : \llbracket C_1^{\mathrm{fin}} \rrbracket \vdash x_1 : \llbracket C_1^{\mathrm{fin}} \rrbracket$ by (T_Var).

5. Assume that $\Sigma_1 <: \Sigma_2$ is derived. By inversion, we can suppose that

   - $\Sigma_1 = \Sigma_1' \uplus \Sigma$,
   - $\Sigma_1' = \{\sigma_i : T_{i1}^{\mathrm{par}} \rightsquigarrow T_{i1}^{\mathrm{ari}} \,/\, C_{i1}^{\mathrm{ini}} \Rightarrow C_{i1}^{\mathrm{fin}}\}^{1 \le i \le m} \uplus \{\varsigma_i : U_{i1}^{\mathrm{par}} \rightsquigarrow U_{i1}^{\mathrm{ari}} \,/\, \Box\}^{1 \le i \le n}$,
   - $\Sigma_2 = \{\sigma_i : T_{i2}^{\mathrm{par}} \rightsquigarrow T_{i2}^{\mathrm{ari}} \,/\, C_{i2}^{\mathrm{ini}} \Rightarrow C_{i2}^{\mathrm{fin}}\}^{1 \le i \le m} \uplus \{\varsigma_i : U_{i2}^{\mathrm{par}} \rightsquigarrow U_{i2}^{\mathrm{ari}} \,/\, \Box\}^{1 \le i \le n}$,
   - $\forall i \in [1,m].\ T_{i2}^{\mathrm{par}} <: T_{i1}^{\mathrm{par}}$,
   - $\forall i \in [1,m].\ T_{i1}^{\mathrm{ari}} <: T_{i2}^{\mathrm{ari}}$,
   - $\forall i \in [1,m].\ C_{i2}^{\mathrm{ini}} <: C_{i1}^{\mathrm{ini}}$,
   - $\forall i \in [1,m].\ C_{i1}^{\mathrm{fin}} <: C_{i2}^{\mathrm{fin}}$,
   - $\forall i \in [1,n].\ U_{i2}^{\mathrm{par}} <: U_{i1}^{\mathrm{par}}$, and
   - $\forall i \in [1,n].\ U_{i1}^{\mathrm{ari}} <: U_{i2}^{\mathrm{ari}}$.

   Assume that Ξ, Δ, $\overline{v_{1i}^{\mathsf{h}\boxslash}}^{1 \le i \le m}$, and $\overline{v_{1i}^{\mathsf{h}\Box}}^{1 \le i \le n}$ such that $\Xi \parallel \Delta \vdash \overline{v_{1i}^{\mathsf{h}\boxslash}}^{1 \le i \le m} ; \overline{v_{1i}^{\mathsf{h}\Box}}^{1 \le i \le n} : \Sigma_1'$ are given. Then, it suffices to show that

   $$\Xi \parallel \Delta \vdash \overline{v_{2i}^{\mathsf{h}\boxslash}}^{1 \le i \le m} ; \overline{v_{2i}^{\mathsf{h}\Box}}^{1 \le i \le n} : \Sigma_2$$

   where:

   - $\forall i \in [1,m].\ v_{2i}^{\mathsf{h}\boxslash} = \lambda x_2, k_2.\mathsf{let}\, x_1 = v_{1i}^{\mathsf{h}\boxslash}\,(\llbracket T_{i2}^{\mathrm{par}} <: T_{i1}^{\mathrm{par}} \rrbracket @(x_2))\, v_i\, \mathsf{in\ return}\, \llbracket C_{i1}^{\mathrm{fin}} <: C_{i2}^{\mathrm{fin}} \rrbracket @(x_1)$,
   - $\forall i \in [1,m].\ v_i = \lambda y_1, \overline{h_{1j}}^{1 \le j \le |\Box(\Sigma_1)|}.\mathsf{let}\, y_2 = k_2\,(\llbracket T_{i1}^{\mathrm{ari}} <: T_{i2}^{\mathrm{ari}} \rrbracket @(y_1))\, \overline{w_j}^{1 \le j \le n}\, \mathsf{in\ return}\, \llbracket C_{i2}^{\mathrm{ini}} <: C_{i1}^{\mathrm{ini}} \rrbracket @(y_2)$,
   - $\forall i \in [1,n].\ w_i = \lambda z_2.\mathsf{let}\, z_1 = h^{\varsigma_i}\,(\llbracket U_{i2}^{\mathrm{par}} <: U_{i1}^{\mathrm{par}} \rrbracket @(z_2))\, \mathsf{in\ return}\, \llbracket U_{i1}^{\mathrm{ari}} <: U_{i2}^{\mathrm{ari}} \rrbracket @(z_1)$ (where h^{ς_i} is a variable in the sequence $\overline{h_{1j}}^{1 \le j \le |\Box(\Sigma_1)|}$ that corresponds to ς_i in Σ_1), and
   - $\forall i \in [1,n].\ v_{2i}^{\mathsf{h}\Box} = \lambda x_2.\mathsf{let}\, x_1 = v_{2i}^{\mathsf{h}\Box}\,(\llbracket U_{i2}^{\mathrm{par}} <: U_{i1}^{\mathrm{par}} \rrbracket @(x_2))\, \mathsf{in\ return}\, \llbracket U_{i1}^{\mathrm{ari}} <: U_{i2}^{\mathrm{ari}} \rrbracket @(x_1)$.

   First, we have

   $$\forall i \in [1,n].\ \Xi \parallel \Delta, h^{\varsigma_i} : \llbracket U_{i1}^{\mathrm{par}} \rrbracket \to \llbracket U_{i1}^{\mathrm{ari}} \rrbracket \vdash w_i : \llbracket U_{i2}^{\mathrm{par}} \rrbracket \to \llbracket U_{i2}^{\mathrm{ari}} \rrbracket \,,$$

   which is derived by (T_Abs), (T_Let), (T_Var), (T_App), (T_Return), and the IHs on $(\llbracket U_{i2}^{\mathrm{par}} \rrbracket, \llbracket U_{i1}^{\mathrm{par}} \rrbracket)$ and $(\llbracket U_{i1}^{\mathrm{ari}} \rrbracket, \llbracket U_{i2}^{\mathrm{ari}} \rrbracket)$. Thus,

   $$\forall i \in [1,m].\ \Xi \parallel \Delta, k_2 : \llbracket T_{i2}^{\mathrm{ari}} \rrbracket \to \llbracket \Box(\Sigma_2) \rrbracket [\,\llbracket C_{i2}^{\mathrm{ini}} \rrbracket\,] \vdash v_i : \llbracket T_{i1}^{\mathrm{ari}} \rrbracket \to \llbracket \Box(\Sigma_1) \rrbracket [\,\llbracket C_{i1}^{\mathrm{ini}} \rrbracket\,]$$

by (T_Abs), (T_Return), (T_Let), (T_Var), Lemmas 15 and 21, and the IHs on (T^{ari}_{i1}, T^{ari}_{i2}) and (C^{ini}_{i2}, C^{ini}_{i1}). Furthermore, then

$$\forall\, i \in [1, m].\ \Xi \parallel \Delta \vdash v^{\mathsf{h}\not\square}_{2i} : [\![T^{\mathrm{par}}_{i2}]\!] \rightarrow ([\![T^{\mathrm{ari}}_{i2}]\!] \rightarrow [\![\square(\Sigma_2)]\!][\, [\![C^{\mathrm{ini}}_{i2}]\!] \,]) \rightarrow [\![C^{\mathrm{fin}}_{i2}]\!]$$

by (T_Abs), (T_Return), (T_Let), (T_Var), Lemmas 15 and 21, the IHs on (T^{par}_{i2}, T^{par}_{i1}) and (C^{fin}_{i1}, C^{fin}_{i2}), and $\Xi \parallel \Delta \vdash v^{\mathsf{h}\not\square}_{1i} : [\![T^{\mathrm{par}}_{i1}]\!] \rightarrow ([\![T^{\mathrm{ari}}_{i1}]\!] \rightarrow [\![\square(\Sigma_1)]\!][\, [\![C^{\mathrm{ini}}_{i1}]\!] \,]) \rightarrow [\![C^{\mathrm{fin}}_{1i}]\!]$.

We also have

$$\forall\, i \in [1, n].\ \Xi \parallel \Delta \vdash v^{\mathsf{h}\square}_{2i} : [\![U^{\mathrm{par}}_{i2}]\!] \rightarrow [\![U^{\mathrm{ari}}_{i2}]\!]$$

by (T_Abs), (T_Let), (T_Var), (T_Return), Lemma 15, the IHs on (U^{par}_{i2}, U^{par}_{i1}) and (U^{ari}_{i1}, U^{ari}_{i2}), and $\Xi \parallel \Delta \vdash v^{\mathsf{h}\square}_{1i} : [\![U^{\mathrm{par}}_{i1}]\!] \rightarrow [\![U^{\mathrm{ari}}_{i1}]\!]$. Therefore, we have the conclusion.

$\square$

Lemma 23 (Type Preservation of the CPS Transformation)**.** Assume that $[\![\Gamma]\!] \preceq \Delta$.

1. If $\Gamma \vdash_{\mathcal{D}} V : T$, then $\Xi \parallel \Delta \vdash [\![V]\!]_{\mathcal{D}} : [\![T]\!]$ for any Ξ.
2. If $\Gamma \vdash_{\mathcal{D}} M : C$, then $\Xi \parallel \Delta \vdash [\![M]\!]_{\mathcal{D}} : [\![C]\!]$ for any Ξ.
3. If $\Gamma \vdash_{\mathcal{D}} M : C$, then $[\![M]\!]^{\mathsf{e}}_{\mathcal{D}} : \mathsf{comp}[\, \Xi \parallel \Delta \vdash C \,]$ for any Ξ.

Proof. By mutual induction on the typing derivations. We often omit $\mathcal{D}$ in the proof because it is clear from the context.

- Assume that $\Gamma \vdash V : T$ is given. By case analysis on the typing rule applied last to derive it.

  Case (HT_Var): Obvious by (T_Var).

  Case (HT_Const): Obvious by (T_Const). Note that $ty(c)$ is a base type by Assumption 1.

  Case (HT_EConst): Obvious by (T_EConst).

  Case (HT_Abs): We are given $\Gamma \vdash \lambda x.M : T' \rightarrow C'$ for some x, M, T', and C' such that $V = \lambda x.M$ and $T = T' \rightarrow C'$. By inversion, $\Gamma, x : T' \vdash M : C'$. Because $[\![\Gamma]\!] \preceq \Delta$, we have $[\![\Gamma]\!], x : [\![T']\!] \preceq \Delta, x : [\![T']\!]$. Therefore, by the IH, $\Xi \parallel \Delta, x : [\![T']\!] \vdash [\![M]\!] : [\![C']\!]$. By (T_Return) and (T_Abs), $\Xi \parallel \Delta \vdash \lambda x.\mathsf{return}\, [\![M]\!] : [\![T']\!] \rightarrow [\![C']\!]$. By the definition of the CPS transformation, we have the conclusion.

  Case (HT_Fix): We are given $\Gamma \vdash \mathsf{fix}\, x.V' : T' \rightarrow C'$ for some x, V', T', and C' such that $V = \mathsf{fix}\, x.V'$ and $T = T' \rightarrow C'$. By inversion, $\Gamma, x : T' \rightarrow C' \vdash V' : T' \rightarrow C'$. Because $[\![\Gamma]\!] \preceq \Delta$, we have $[\![\Gamma]\!], x : [\![T']\!] \rightarrow [\![C']\!] \preceq \Delta, x : [\![T']\!] \rightarrow [\![C']\!]$. Therefore, by the IH, $\Xi \parallel \Delta, x : [\![T']\!] \rightarrow [\![C']\!] \vdash [\![V']\!] : [\![T']\!] \rightarrow [\![C']\!]$. By (T_Fix), $\Xi \parallel \Delta \vdash \mathsf{fix}\, x.[\![V']\!] : [\![T']\!] \rightarrow [\![C']\!]$. By the definition of the CPS transformation, we have the conclusion.

  Case (HT_SubV): By inversion, we are given $\Gamma \vdash_{\mathcal{D}'} V : U$ for some U and $\mathcal{D}'$ such that $U <: T$. By the definition of the CPS transformation, it suffices to show that

  $$\Xi \parallel \Delta \vdash [\![U <: T]\!]\, [\![V]\!]_{\mathcal{D}'} : [\![T]\!] \ ,$$

  which is derived by (T_App), Lemma 22, and the IH.

- Assume that $\Gamma \vdash M : C$ is given. Let $\overline{h}$ be a fresh variable sequence such that $|\overline{h}| = |C.\Sigma|$. By case analysis on $C.A$.

  Case $C.A = \square$: By the definition of the CPS transformation, it suffices to show that

  $$\Xi \parallel \Delta, \overline{h} \,:\, C.\Sigma, k : \mathsf{1} \vdash [\![M]\!]^{\mathsf{e}} @ (\overline{h}, k) : [\![C.T]\!]$$

  for some fresh variable k. It is shown by case (3) with
  - $\Xi \parallel \Delta, \overline{h} \,:\, C.\Sigma, k : \mathsf{1} \vdash \overline{h} : C.\Sigma$ and
  - $\Xi \parallel \Delta, \overline{h} \,:\, C.\Sigma, k : \mathsf{1} \vdash k : \mathsf{1}$

by (T_Var).

Case $\exists\, C^{\mathrm{ini}}, C^{\mathrm{fin}}.\ C.A = C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$: By the definition of the CPS transformation, it suffices to show that

$$\Xi \parallel \Delta, \overline{h} : C.\Sigma, k : [\![C.T]\!] \to [\![\Box(C.\Sigma)]\!][\,[\![C^{\mathrm{ini}}]\!]\,] \vdash [\![M]\!]^{\mathsf{e}}@(\overline{h}, k) : [\![C^{\mathrm{fin}}]\!] ,$$

which is shown by case (3) with

- $\Xi \parallel \Delta, \overline{h} : C.\Sigma, k : [\![C.T]\!] \to [\![C^{\mathrm{ini}}]\!] \vdash \overline{h} : C.\Sigma$ and
- $\Xi \parallel \Delta, \overline{h} : C.\Sigma, k : [\![C.T]\!] \to [\![\Box(C.\Sigma)]\!][\,[\![C^{\mathrm{ini}}]\!]\,] \vdash k : [\![C.T]\!] \to [\![\Box(C.\Sigma)]\!][\,[\![C^{\mathrm{ini}}]\!]\,]$

derived by (T_Var).

- Assume that $\Gamma \vdash M : C$ and Ξ and Δ such that $[\![\Gamma]\!] \preceq \Delta$ are given. Let
  - $\overline{v^{\mathsf{h}}} \in \mathsf{vals}[\,\Xi \parallel \Delta \vdash C.\Sigma\,]$,
  - $(\overline{v^{\mathsf{h}\not\Box}}, \overline{v^{\mathsf{h}\Box}}) = \mathit{split}(\overline{v^{\mathsf{h}}}, C.\Sigma)$, and
  - $C.\Sigma = \{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, C_i^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{fin}}\}^{1 \le i \le m} \uplus \{\varsigma_i : U_i^{\mathrm{par}} \rightsquigarrow U_i^{\mathrm{ari}} \,/\, \Box\}^{1 \le i \le n}$

  for some $\sigma_1, \cdots, \sigma_m$, $T_1^{\mathrm{par}}, \cdots, T_m^{\mathrm{par}}$, $T_1^{\mathrm{ari}}, \cdots, T_m^{\mathrm{ari}}$, $C_1^{\mathrm{ini}}, \cdots, C_m^{\mathrm{ini}}$, and $C_1^{\mathrm{fin}}, \cdots, C_m^{\mathrm{fin}}$, $\varsigma_1, \cdots, \varsigma_n$, $U_1^{\mathrm{par}}, \cdots, U_n^{\mathrm{par}}$, and $U_1^{\mathrm{ari}}, \cdots, U_n^{\mathrm{ari}}$ such that the sequences $\sigma_1, \cdots, \sigma_m$ and $\varsigma_1, \cdots, \varsigma_n$ are ordered in terms of $\preceq_{\mathsf{op}}$. We proceed by case analysis on the typing rule applied last to derive $\Gamma \vdash M : C$.

  Case (HT_Return): We are given

$$\frac{\Gamma \vdash V : C.T}{\Gamma \vdash \mathsf{return}\ V : C.\Sigma \triangleright C.T \,/\, \Box}$$

  and $C.A = \Box$ for some V such that $M = \mathsf{return}\ V$. Then, it suffices to show that

$$\Xi \parallel \Delta \vdash \mathsf{return}\, [\![V]\!] : [\![C.T]\!] ,$$

  which is derived by

$$\dfrac{\dfrac{}{\Xi \parallel \Delta \vdash [\![V]\!] : [\![C.T]\!]}\ \textsc{By the IH}}{\Xi \parallel \Delta \vdash \mathsf{return}\, [\![V]\!] : [\![C.T]\!]}\ (\text{T\_Return})$$

  Case (HT_Let): We are given

$$\frac{\Gamma \vdash M_1 : C.\Sigma \triangleright T_1 \,/\, \Box \qquad \Gamma, x : T_1 \vdash M_2 : C}{\Gamma \vdash \mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2 : C}$$

  for some x, M_1, M_2, and T_1 such that $M = (\mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2)$. Without loss of generality, we can assume that $x \notin \mathit{dom}(\Delta)$. By the IH with

  - $\Gamma \vdash M_1 : C.\Sigma \triangleright T_1 \,/\, \Box$,
  - $\Xi \parallel \Delta \vdash \overline{v^{\mathsf{h}}} : C.\Sigma$, and
  - $\Xi \parallel \Delta \vdash \underline{1} : 1$ by (T_EConst),

  we have

$$\Xi \parallel \Delta \vdash [\![M_1]\!]^{\mathsf{e}}@(\overline{v^{\mathsf{h}}}, \underline{1}) : [\![T_1]\!] . \tag{16}$$

  We proceed by case analysis on $C.A$.

  Case $C.A = \Box$: Assume that v^{k} such that $\Xi \parallel \Delta \vdash v^{\mathsf{k}} : 1$ is given. By the definition, it suffices to show that

$$\Xi \parallel \Delta \vdash \mathsf{let}\, x = [\![M_1]\!]^{\mathsf{e}}@(\overline{v^{\mathsf{h}}}, \underline{1}) \,\mathsf{in}\, [\![M_2]\!]^{\mathsf{e}}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) : [\![C.T]\!] .$$

  By (T_Let) with derivation (16), it suffices to show that

$$\Xi \parallel \Delta, x : [\![T_1]\!] \vdash [\![M_2]\!]^{\mathsf{e}}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) : [\![C.T]\!] ,$$

  which is derived by the IH with

– $\Gamma, x : T_1 \vdash M_2 : C$,
– $[\![\Gamma, x : T_1]\!] \preceq \Delta, x : [\![T_1]\!]$,
– $\Xi \parallel \Delta, x : [\![T_1]\!] \vdash \overline{v^{\mathsf{h}}} : C.\Sigma$ by Lemma 15, and
– $\Xi \parallel \Delta, x : [\![T_1]\!] \vdash v^{\mathsf{k}} : 1$ by Lemma 15.

Case $\exists\, C^{\mathrm{ini}}, C^{\mathrm{fin}}.\ C.A = C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$: Assume that v^{k} such that $\Xi \parallel \Delta \vdash v^{\mathsf{k}} : [\![C.T]\!] \to [\![\Box(C.\Sigma)]\!][\,[\![C^{\mathrm{ini}}]\!]\,]$ is given. By the definition, it suffices to show that

$$\Xi \parallel \Delta \vdash \mathsf{let}\, x = [\![M_1]\!]^{\mathsf{e}} @(\overline{v^{\mathsf{h}}}, \underline{1})\, \mathsf{in}\, [\![M_2]\!]^{\mathsf{e}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) : [\![C^{\mathrm{fin}}]\!] \ .$$

By (T_Let) with derivation (16), it suffices to show that

$$\Xi \parallel \Delta, x : [\![T_1]\!] \vdash [\![M_2]\!]^{\mathsf{e}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) : [\![C^{\mathrm{fin}}]\!] \ ,$$

which is derived by the IH with
– $\Gamma, x : T_1 \vdash M_2 : C$,
– $[\![\Gamma, x : T_1]\!] \preceq \Delta, x : [\![T_1]\!]$,
– $\Xi \parallel \Delta, x : [\![T_1]\!] \vdash \overline{v^{\mathsf{h}}} : C.\Sigma$ by Lemma 15, and
– $\Xi \parallel \Delta, x : [\![T_1]\!] \vdash v^{\mathsf{k}} : [\![C.T]\!] \to [\![\Box(C.\Sigma)]\!][\,[\![C^{\mathrm{ini}}]\!]\,]$ by Lemma 15.

Case (HT_LetATM): We are given

$$\frac{\Gamma \vdash M_1 : C.\Sigma \triangleright T_1 \,/\, C' \Rightarrow C^{\mathrm{fin}} \quad \Gamma, x : T_1 \vdash M_2 : C.\Sigma \triangleright C.T \,/\, C^{\mathrm{ini}} \Rightarrow C'}{\Gamma \vdash \mathsf{let}\, x = M_1\, \mathsf{in}\, M_2 : C.\Sigma \triangleright C.T \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}}$$

for some x, M_1, M_2, T_1, C', C^{ini}, and C^{fin} such that $M = (\mathsf{let}\, x = M_1\, \mathsf{in}\, M_2)$ and $C.A = C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$. Without loss of generality, we can assume that $x \notin \mathit{dom}(\Delta)$. Assume that v^{k} such that $\Xi \parallel \Delta \vdash v^{\mathsf{k}} : [\![C.T]\!] \to [\![\Box(C.\Sigma)]\!][\,[\![C^{\mathrm{ini}}]\!]\,]$ is given. Let $\overline{h^{\Box}}$ be a sequence of fresh variables such that $|\overline{h^{\Box}}| = |\Box(\Sigma)|$. By the definition, it suffices to show that

$$\Xi \parallel \Delta \vdash [\![M_1]\!]^{\mathsf{e}} @(\overline{v^{\mathsf{h}}}, \lambda x, \overline{h^{\Box}}.[\![M_2]\!]^{\mathsf{e}} @(\overline{v^{\mathsf{h}\not\Box}}, \overline{h^{\Box}}, v^{\mathsf{k}})) : [\![C^{\mathrm{fin}}]\!] \ .$$

Let $\Delta' = \Delta, x : [\![T_1]\!], \overline{h^{\Box}} : \Box(C.\Sigma)$. By the IH with
– $\Gamma, x : T_1 \vdash M_2 : C.\Sigma \triangleright C.T \,/\, C^{\mathrm{ini}} \Rightarrow C'$,
– $[\![\Gamma, x : T_1]\!] \preceq \Delta'$,
– $\Xi \parallel \Delta' \vdash \overline{v^{\mathsf{h}\not\Box}}; \overline{h^{\Box}} : C.\Sigma$ by Lemma 15 and (T_Var), and
– $\Xi \parallel \Delta' \vdash v^{\mathsf{k}} : [\![C.T]\!] \to [\![\Box(C.\Sigma)]\!][\,[\![C^{\mathrm{ini}}]\!]\,]$ by Lemma 15,

we have

$$\Xi \parallel \Delta' \vdash [\![M_2]\!]^{\mathsf{e}} @(\overline{v^{\mathsf{h}\not\Box}}, \overline{h^{\Box}}, v^{\mathsf{k}}) : [\![C']\!] \ .$$

By (T_Abs) and (T_Return),

$$\Xi \parallel \Delta \vdash \lambda x, \overline{h^{\Box}}.[\![M_2]\!]^{\mathsf{e}} @(\overline{v^{\mathsf{h}\not\Box}}, \overline{h^{\Box}}, v^{\mathsf{k}}) : [\![T_1]\!] \to [\![\Box(C.\Sigma)]\!][\,[\![C']\!]\,] \ . \tag{17}$$

By the IH with
– $\Gamma \vdash M_1 : C.\Sigma \triangleright T_1 \,/\, C' \Rightarrow C^{\mathrm{fin}}$,
– $[\![\Gamma]\!] \preceq \Delta$,
– $\Xi \parallel \Delta \vdash \overline{v^{\mathsf{h}}} : C.\Sigma$, and
– derivation (17),

we have the conclusion.

Case (HT_App): We are given

$$\frac{\Gamma \vdash V_1 : T \to C \quad \Gamma \vdash V_2 : T}{\Gamma \vdash V_1\ V_2 : C}$$

for some V_1, V_2, and T such that $M = V_1\ V_2$. By case analysis on $C.A$.

Case $C.A = \Box$: Assume that v^{k} such that $\Xi \parallel \Delta \vdash v^{\mathsf{k}} : \mathbf{1}$ is given. By the definition, it suffices to show that

$$\Xi \parallel \Delta \vdash [\![V_1]\!] \, [\![V_2]\!] \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} : [\![C.T]\!] \ ,$$

which is derived by Lemma 21 and the IHs on $\Gamma \vdash V_1 : T \to C$ and $\Gamma \vdash V_2 : T$.

Case $\exists \, C^{\mathrm{ini}}, C^{\mathrm{fin}}.\ C.A = C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$: Assume that v^{k} such that $\Xi \parallel \Delta \vdash v^{\mathsf{k}} : [\![C.T]\!] \to [\![\Box(C.\Sigma)]\!][\, [\![C^{\mathrm{ini}}]\!] \,]$ is given. By the definition, it suffices to show that

$$\Xi \parallel \Delta \vdash [\![V_1]\!] \, [\![V_2]\!] \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} : [\![C^{\mathrm{fin}}]\!] \ ,$$

which is derived by Lemma 21 and the IHs on $\Gamma \vdash V_1 : T \to C$ and $\Gamma \vdash V_2 : T$.

Case (HT_Case): We are given

$$\frac{\Gamma \vdash V : \mathsf{n} \quad \forall \, i \in [1, n].\ \Gamma \vdash M_i : C}{\Gamma \vdash \mathsf{case}(V; M_1, \cdots, M_n) : C}$$

for some V_1, $M_1, \cdots, M_n$, and n such that $M = \mathsf{case}(V; M_1, \cdots, M_n)$. By case analysis on $C.A$.

Case $C.A = \Box$: Assume that v^{k} such that $\Xi \parallel \Delta \vdash v^{\mathsf{k}} : \mathbf{1}$ is given. By the definition, it suffices to show that

$$\Xi \parallel \Delta \vdash \mathsf{case}([\![V]\!]; [\![M_1]\!]^{\mathsf{e}} @(\overline{v^{\mathsf{h}}}, \overline{v^{\mathsf{k}}}), \cdots, [\![M_n]\!]^{\mathsf{e}} @(\overline{v^{\mathsf{h}}}, \overline{v^{\mathsf{k}}})) : [\![C.T]\!] \ ,$$

which is derived by the IHs and (T_Case).

Case $\exists \, C^{\mathrm{ini}}, C^{\mathrm{fin}}.\ C.A = C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$: Assume that v^{k} such that $\Xi \parallel \Delta \vdash v^{\mathsf{k}} : [\![C.T]\!] \to [\![\Box(C.\Sigma)]\!][\, [\![C^{\mathrm{ini}}]\!] \,]$ is given. By the definition, it suffices to show that

$$\Xi \parallel \Delta \vdash \mathsf{case}([\![V]\!]; [\![M_1]\!]^{\mathsf{e}} @(\overline{v^{\mathsf{h}}}, \overline{v^{\mathsf{k}}}), \cdots, [\![M_n]\!]^{\mathsf{e}} @(\overline{v^{\mathsf{h}}}, \overline{v^{\mathsf{k}}})) : [\![C^{\mathrm{fin}}]\!] \ ,$$

which is derived by the IHs and (T_Case).

Case (HT_Op): We are given

$$\frac{\varsigma_i : U_i^{\mathrm{par}} \rightsquigarrow U_i^{\mathrm{ari}} \,/\, \Box \in C.\Sigma \quad \Gamma \vdash V' : U_i^{\mathrm{par}} \quad \Gamma, x : U_i^{\mathrm{ari}} \vdash M' : C}{\Gamma \vdash \varsigma_i(V'; x.\, M') : C}$$

for some V', x, M', and $i \in [1, n]$ such that $M = \varsigma_i(V'; x.\, M')$. Without loss of generality, we can assume that $x \notin dom(\Delta)$. Because $\Xi \parallel \Delta \vdash \overline{v^{\mathsf{h}}} : C.\Sigma$ and $\varsigma_i \in dom(\Box(C.\Sigma))$, there exists some v^{ς_i} in $\overline{v^{\mathsf{h}}}$ such that

$$\Xi \parallel \Delta \vdash v^{\varsigma_i} : [\![U_i^{\mathrm{par}}]\!] \to [\![U_i^{\mathrm{ari}}]\!] \ .$$

Thus,

$$\Xi \parallel \Delta \vdash v^{\varsigma_i} \, [\![V']\!] : [\![U_i^{\mathrm{ari}}]\!] \tag{18}$$

by

$$\dfrac{\Xi \parallel \Delta \vdash v^{\varsigma_i} : [\![U_i^{\mathrm{par}}]\!] \to [\![U_i^{\mathrm{ari}}]\!] \qquad \dfrac{}{\Xi \parallel \Delta \vdash [\![V']\!] : [\![U_i^{\mathrm{par}}]\!]}\ \textsc{By the IH}}{\Xi \parallel \Delta \vdash v^{\varsigma_i} \, [\![V']\!] : [\![U_i^{\mathrm{ari}}]\!]}\ (\text{T\_App}).$$

We proceed by case analysis on $C.A$.

Case $C.A = \Box$: Assume that v^{k} such that $\Xi \parallel \Delta \vdash v^{\mathsf{k}} : \mathbf{1}$ is given. By the definition, it suffices to show that

$$\Xi \parallel \Delta \vdash \mathsf{let}\, x = v^{\varsigma_i} \, [\![V']\!] \,\mathsf{in}\, [\![M']\!]^{\mathsf{e}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) : [\![C.T]\!] \ .$$

By (T_Let) with derivation (18), it suffices to show that

$$\Xi \parallel \Delta, x : [\![U_i^{\mathrm{ari}}]\!] \vdash [\![M']\!]^{\mathsf{e}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) : [\![C.T]\!] \ ,$$

which is derived by the IH with

– $\Gamma, x : U_i^{\mathrm{ari}} \vdash M' : C$,
– $[\![\Gamma, x : U_i^{\mathrm{ari}}]\!] \preceq \Delta, x : [\![U_i^{\mathrm{ari}}]\!]$,
– $\Xi \parallel \Delta, x : [\![U_i^{\mathrm{ari}}]\!] \vdash \overline{v^{\mathsf{h}}} : C.\Sigma$ by Lemma 15,
– $\Xi \parallel \Delta, x : [\![U_i^{\mathrm{ari}}]\!] \vdash v^{\mathsf{k}} : 1$ by Lemma 15.

Case $\exists\, C^{\mathrm{ini}}, C^{\mathrm{fin}}.\ C.A = C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$: Assume that v^{k} such that $\Xi \parallel \Delta \vdash v^{\mathsf{k}} : [\![C.T]\!] \to [\![\square(C.\Sigma)]\!][\,[\![C^{\mathrm{ini}}]\!]\,]$ is given. By the definition, it suffices to show that

$$\Xi \parallel \Delta \vdash \mathsf{let}\, x = v^{\varsigma_i}\, [\![V']\!]\, \mathsf{in}\, [\![M']\!]^{\mathsf{e}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) : [\![C^{\mathrm{fin}}]\!]\ .$$

By (T_Let) with derivation (18), it suffices to show that

$$\Xi \parallel \Delta, x : [\![U_i^{\mathrm{ari}}]\!] \vdash [\![M']\!]^{\mathsf{e}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) : [\![C^{\mathrm{fin}}]\!]\ ,$$

which is derived by the IH with
– $\Gamma, x : U_i^{\mathrm{ari}} \vdash M' : C$,
– $[\![\Gamma, x : U_i^{\mathrm{ari}}]\!] \preceq \Delta, x : [\![U_i^{\mathrm{ari}}]\!]$,
– $\Xi \parallel \Delta, x : [\![U_i^{\mathrm{ari}}]\!] \vdash \overline{v^{\mathsf{h}}} : C.\Sigma$ by Lemma 15,
– $\Xi \parallel \Delta, x : [\![U_i^{\mathrm{ari}}]\!] \vdash v^{\mathsf{k}} : [\![C.T]\!] \to [\![\square(C.\Sigma)]\!][\,[\![C^{\mathrm{ini}}]\!]\,]$ by Lemma 15.

Case (HT_OpATM):

$$\frac{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, C_i^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{fin}} \in C.\Sigma \quad \Gamma \vdash V' : T_i^{\mathrm{par}} \quad \Gamma, x : T_i^{\mathrm{ari}} \vdash M' : C.\Sigma \triangleright C.T \,/\, C^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{ini}}}{\Gamma \vdash \sigma_i(V'; x.\, M') : C.\Sigma \triangleright C.T \,/\, C^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{fin}}}$$

for some V', x, M', $i \in [1, m]$, and C^{ini} such that $M = \sigma_i(V'; x.\, M')$ and $C.A = C^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{fin}}$. Without loss of generality, we can assume that $x \notin \mathit{dom}(\Delta)$. Because $\Xi \parallel \Delta \vdash \overline{v^{\mathsf{h}}} : C.\Sigma$ and $\sigma_i \in \mathit{dom}(\not\square(C.\Sigma))$, there exists some v^{σ_i} in $\overline{v^{\mathsf{h}}}$ such that

$$\Xi \parallel \Delta \vdash v^{\sigma_i} : [\![T_i^{\mathrm{par}}]\!] \to ([\![T_i^{\mathrm{ari}}]\!] \to [\![\square(C.\Sigma)]\!][\,[\![C_i^{\mathrm{ini}}]\!]\,]) \to [\![C_i^{\mathrm{fin}}]\!]\ .$$

Thus, by (T_App) and the IH,

$$\Xi \parallel \Delta \vdash v^{\sigma_i}\, [\![V']\!] : ([\![T_i^{\mathrm{ari}}]\!] \to [\![\square(C.\Sigma)]\!][\,[\![C_i^{\mathrm{ini}}]\!]\,]) \to [\![C_i^{\mathrm{fin}}]\!]\ . \tag{19}$$

Assume that v^{k} such that $\Xi \parallel \Delta \vdash v^{\mathsf{k}} : [\![C.T]\!] \to [\![\square(C.\Sigma)]\!][\,[\![C^{\mathrm{ini}}]\!]\,]$ is given. Let $\overline{h^{\square}}$ be a sequence of fresh variables such that $|\overline{h^{\square}}| = |\square(C.\Sigma)|$. By the definition, it suffices to show that

$$\Xi \parallel \Delta \vdash v^{\sigma_i}\, [\![V']\!]\, (\lambda x, \overline{h^{\square}}.[\![M']\!]^{\mathsf{e}} @ (\overline{v^{\mathsf{h}\not\square}}, \overline{h^{\square}}, v^{\mathsf{k}})) : [\![C_i^{\mathrm{fin}}]\!]\ .$$

Let $\Delta' = \Delta, x : [\![T_i^{\mathrm{ari}}]\!], \overline{h^{\square}} : \square(C.\Sigma)$. By Lemma 21, derivation (19), (T_Abs), and (T_Return), it suffices to show that

$$\Xi \parallel \Delta' \vdash [\![M']\!]^{\mathsf{e}} @ (\overline{v^{\mathsf{h}\not\square}}, \overline{h^{\square}}, v^{\mathsf{k}}) : [\![C_i^{\mathrm{ini}}]\!]\ ,$$

which is derived by the IH with
– $\Gamma, x : T_i^{\mathrm{ari}} \vdash M' : C.\Sigma \triangleright C.T \,/\, C^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{ini}}$,
– $[\![\Gamma, x : T_i^{\mathrm{ari}}]\!] \preceq \Delta'$,
– $\Xi \parallel \Delta' \vdash \overline{v^{\mathsf{h}\not\square}}; \overline{h^{\square}} : C.\Sigma$ by Lemma 15 and (T_Var), and
– $\Xi \parallel \Delta' \vdash v^{\mathsf{k}} : [\![C.T]\!] \to [\![\square(C.\Sigma)]\!][\,[\![C^{\mathrm{ini}}]\!]\,]$ by Lemma 15.

Case (HT_Handle): We are given

$$\frac{\begin{array}{c} H' = \{\mathsf{return}\, x \mapsto L\} \uplus \{\sigma_i'(y_i; k_i) \mapsto M_i\}^{1 \le i \le m_0} \uplus \{\varsigma_i'(z_i) \mapsto N_i\}^{1 \le i \le n_0} \\ \Sigma' = \{\sigma_i' : {T'}_i^{\mathrm{par}} \rightsquigarrow {T'}_i^{\mathrm{ari}} \,/\, {C'}_i^{\mathrm{ini}} \Rightarrow {C'}_i^{\mathrm{fin}}\}^{1 \le i \le m_0} \uplus \{\varsigma_i' : {U'}_i^{\mathrm{par}} \rightsquigarrow {U'}_i^{\mathrm{ari}} \,/\, \square\}^{1 \le i \le n_0} \\ \Gamma \vdash M' : \Sigma' \triangleright T' \,/\, {C'}^{\mathrm{ini}} \Rightarrow C \quad \Gamma, x : T' \vdash L : {C'}^{\mathrm{ini}} \\ \forall\, i \in [1, m_0].\ \Gamma, y_i : {T'}_i^{\mathrm{par}}, k_i : {T'}_i^{\mathrm{ari}} \to {C'}_i^{\mathrm{ini}} \vdash M_i : {C'}_i^{\mathrm{fin}} \\ \forall\, i \in [1, n_0].\ \Gamma, z_i : {U'}_i^{\mathrm{par}} \vdash N_i : \Sigma_i' \triangleright {U'}_i^{\mathrm{ari}} \,/\, \square \\ \forall\, C' \in \{\overline{{C'}_i^{\mathrm{ini}}}^{1 \le i \le m_0}, C\}.\ \forall\, i \in [1, n_0].\ C'.\Sigma <: \Sigma_i' \end{array}}{\Gamma \vdash \mathsf{with}\, H'\, \mathsf{handle}\, M' : C}$$

for some H', M', x, L, $\sigma'_1, \cdots, \sigma'_{m_0}$, $y_1, \cdots, y_{m_0}$, $k_1, \cdots, k_{m_0}$, $M_1, \cdots, M_{m_0}$, $T'^{\mathrm{par}}_1, \cdots, T'^{\mathrm{par}}_{m_0}$, $T'^{\mathrm{ari}}_1, \cdots, T'^{\mathrm{ari}}_{m_0}$, $C'^{\mathrm{ini}}_1, \cdots, C'^{\mathrm{ini}}_{m_0}$, and $C'^{\mathrm{fin}}_1, \cdots, C'^{\mathrm{fin}}_{m_0}$, $\varsigma'_1, \cdots, \varsigma'_{n_0}$, $z_1, \cdots, z_{n_0}$, $N_1, \cdots, N_{n_0}$, $U'^{\mathrm{par}}_1, \cdots, U'^{\mathrm{par}}_{n_0}$, $U'^{\mathrm{ari}}_1, \cdots, U'^{\mathrm{ari}}_{n_0}$, Σ', T', and C'^{ini} such that $M = \mathsf{with}\, H' \,\mathsf{handle}\, M'$. Let $\overline{h^{\square}}$ be a sequence of fresh variables such that $|\overline{h^{\square}}| = n_0$. By the definition, it suffices to show that

$$\Xi \parallel \Delta \vdash \llbracket M' \rrbracket^{\mathsf{e}} @ (\overline{w_i^{\boxslash}}^{1 \le i \le m_0}, \overline{w_i^{\square}}^{1 \le i \le n_0}, \lambda x, \overline{h^{\square}}.\mathsf{return}\, \llbracket L \rrbracket) : \llbracket C \rrbracket$$

where

- $\forall\, i \in [1, m_0].\ w_i^{\boxslash} = \lambda y_i, k'_i.\mathsf{let}\, k_i = \mathsf{return}\, (\lambda y, \overline{h}, k.k'_i\, y\, \overline{v_{ij}}^{1 \le j \le n_0}\, \overline{h}\, k)\, \mathsf{in}\, \mathsf{return}\, \llbracket M_i \rrbracket$ $(|\overline{h}| = |C'^{\mathrm{ini}}_i.\Sigma|)$,
- $\forall\, i \in [1, m_0].\ \forall\, j \in [1, n_0].\ v_{ij} = \lambda z_j.\llbracket \Sigma'_j \triangleright U'^{\mathrm{ari}}_j \,/\, \square <: C'^{\mathrm{ini}}_i.\Sigma \triangleright U'^{\mathrm{ari}}_j \,/\, \square \rrbracket^{\mathsf{e}} @ (\llbracket N_j \rrbracket^{\mathsf{e}}, \overline{h}, \underline{1})$, and
- $\forall\, i \in [1, n_0].\ w_i^{\square} = \lambda z_i.\llbracket \Sigma'_i \triangleright U'^{\mathrm{ari}}_i \,/\, \square <: C.\Sigma \triangleright U'^{\mathrm{ari}}_i \,/\, \square \rrbracket^{\mathsf{e}} @ (\llbracket N_i \rrbracket^{\mathsf{e}}, \overline{v^{\mathsf{h}}}, \underline{1})$.

It is derived by the IH on $\Gamma \vdash M' : \Sigma' \triangleright T' \,/\, C'^{\mathrm{ini}} \Rightarrow C$ with the followings.

- We show that, for any $i \in [1, m_0]$,

  $$\Xi \parallel \Delta \vdash w_i^{\boxslash} : \llbracket T'^{\mathrm{par}}_i \rrbracket \to (\llbracket T'^{\mathrm{ari}}_i \rrbracket \to \llbracket \square(\Sigma') \rrbracket [\, \llbracket C'^{\mathrm{ini}}_i \rrbracket \,]) \to \llbracket C'^{\mathrm{fin}}_i \rrbracket \ .$$

  Let

  $$\Delta' = \Delta, y_i : \llbracket T'^{\mathrm{par}}_i \rrbracket, k'_i : \llbracket T'^{\mathrm{ari}}_i \rrbracket \to \llbracket \square(\Sigma') \rrbracket [\, \llbracket C'^{\mathrm{ini}}_i \rrbracket \,] \ .$$

  By (T_Abs) and (T_Return), it suffices to show that

  $$\Xi \parallel \Delta' \vdash \mathsf{let}\, k_i = \mathsf{return}\, (\lambda y, \overline{h}, k.k'_i\, y\, \overline{v_{ij}}^{1 \le j \le n_0}\, \overline{h}\, k)\, \mathsf{in}\, \mathsf{return}\, \llbracket M_i \rrbracket : \llbracket C'^{\mathrm{fin}}_i \rrbracket \ . \tag{20}$$

  First, we show that

  $$\Xi \parallel \Delta' \vdash \lambda y, \overline{h}, k.k'_i\, y\, \overline{v_{ij}}^{1 \le j \le n_0}\, \overline{h}\, k : \llbracket T'^{\mathrm{ari}}_i \rrbracket \to \llbracket C'^{\mathrm{ini}}_i \rrbracket \ . \tag{21}$$

  By (T_Abs) and (T_Return), it suffices to show that

  $$\Xi \parallel \Delta', y : \llbracket T'^{\mathrm{ari}}_i \rrbracket \vdash \lambda \overline{h}, k.k'_i\, y\, \overline{v_{ij}}^{1 \le j \le n_0}\, \overline{h}\, k : \llbracket C'^{\mathrm{ini}}_i \rrbracket \ .$$

  By (T_Abs), (T_Return), (T_Var), and Lemmas 21 and 15, it suffices to show that, for any $j \in [1, n_0]$,

  $$\Xi \parallel \Delta', \overline{h} \,:\, C'^{\mathrm{ini}}_i.\Sigma \vdash v_{ij} : \llbracket U'^{\mathrm{par}}_j \rrbracket \to \llbracket U'^{\mathrm{ari}}_j \rrbracket \ .$$

  Let $\Delta'' = \Delta', \overline{h} \,:\, C'^{\mathrm{ini}}_i.\Sigma, z_j : \llbracket U'^{\mathrm{par}}_j \rrbracket$. By (T_Abs), it suffices to show that

  $$\Xi \parallel \Delta'' \vdash \llbracket \Sigma'_j \triangleright U'^{\mathrm{ari}}_j \,/\, \square <: C'^{\mathrm{ini}}_i.\Sigma \triangleright U'^{\mathrm{ari}}_j \,/\, \square \rrbracket^{\mathsf{e}} @ (\llbracket N_j \rrbracket^{\mathsf{e}}, \overline{h}, \underline{1}) : \llbracket U'^{\mathrm{ari}}_j \rrbracket \ .$$

  By (HS_Comp) and $C'^{\mathrm{ini}}_i.\Sigma <: \Sigma'_j$ and Lemma 4, we have $\Sigma'_j \triangleright U'^{\mathrm{ari}}_j \,/\, \square <: C'^{\mathrm{ini}}_i.\Sigma \triangleright U'^{\mathrm{ari}}_j \,/\, \square$. Thus, by Lemma 22,

  $$\begin{array}{ll} & \llbracket \Sigma'_j \triangleright U'^{\mathrm{ari}}_j \,/\, \square <: C'^{\mathrm{ini}}_i.\Sigma \triangleright U'^{\mathrm{ari}}_j \,/\, \square \rrbracket^{\mathsf{e}} \\ : & \mathsf{comp}[\, \Xi \parallel \Delta'' \vdash \Sigma'_j \triangleright U'^{\mathrm{ari}}_j \,/\, \square \,] \times \mathsf{vals}[\, \Xi \parallel \Delta'' \vdash C'^{\mathrm{ini}}_i.\Sigma \,] \times \mathsf{val}[\, \Xi \parallel \Delta'' \vdash 1 \,] \to \mathsf{term}[\, \Xi \parallel \Delta'' \vdash \llbracket U'^{\mathrm{ari}}_j \rrbracket \,] \ . \end{array}$$

  Because

  * $\llbracket N_j \rrbracket^{\mathsf{e}} : \mathsf{comp}[\, \Xi \parallel \Delta'' \vdash \Sigma'_j \triangleright U'^{\mathrm{ari}}_j \,/\, \square \,]$ by the IH with $\Gamma, z_j \,:\, U'^{\mathrm{par}}_j \vdash N_j : \Sigma'_j \triangleright U'^{\mathrm{ari}}_j \,/\, \square$ and $\llbracket \Gamma, z_j : U'^{\mathrm{par}}_j \rrbracket \preceq \Delta''$,
  * $\Xi \parallel \Delta'' \vdash \overline{h} : C'^{\mathrm{ini}}_i.\Sigma$ by (T_Var),
  * $\Xi \parallel \Delta'' \vdash \underline{1} : 1$ by (T_EConst),

  we have the conclusion.

– We show that, for any $i \in [1, n_0]$,

$$\Xi \parallel \Delta \vdash w_i^{\square} : [\![{U'}_i^{\mathrm{par}}]\!] \rightarrow [\![{U'}_i^{\mathrm{ari}}]\!] \ .$$

Let $\Delta' = \Delta, z_i : [\![{U'}_i^{\mathrm{par}}]\!]$. By (T_Abs), it suffices to show that

$$\Xi \parallel \Delta' \vdash [\![\Sigma'_i \triangleright {U'}_i^{\mathrm{ari}} \,/\, \square <: C.\Sigma \triangleright {U'}_i^{\mathrm{ari}} \,/\, \square]\!]^{\mathsf{e}} @([\![N_i]\!]^{\mathsf{e}}, \overline{v^{\mathsf{h}}}, \underline{1}) : [\![{U'}_i^{\mathrm{ari}}]\!] \ .$$

By (HS_Comp) and $C.\Sigma <: \Sigma'_i$ and Lemma 4, we have $\Sigma'_i \triangleright {U'}_i^{\mathrm{ari}} \,/\, \square <: C.\Sigma \triangleright {U'}_i^{\mathrm{ari}} \,/\, \square$. Thus, by Lemma 22,

$$\begin{array}{ll} & [\![\Sigma'_i \triangleright {U'}_i^{\mathrm{ari}} \,/\, \square <: C.\Sigma \triangleright {U'}_i^{\mathrm{ari}} \,/\, \square]\!]^{\mathsf{e}} \\ : & \mathsf{comp}[\, \Xi \parallel \Delta' \vdash \Sigma'_i \triangleright {U'}_i^{\mathrm{ari}} \,/\, \square \,] \times \mathsf{vals}[\, \Xi \parallel \Delta' \vdash C.\Sigma \,] \times \mathsf{val}[\, \Xi \parallel \Delta' \vdash \mathsf{1} \,] \rightarrow \mathsf{term}[\, \Xi \parallel \Delta' \vdash [\![{U'}_i^{\mathrm{ari}}]\!] \,] \ . \end{array}$$

Because

* $[\![N_i]\!]^{\mathsf{e}} : \mathsf{comp}[\, \Xi \parallel \Delta' \vdash \Sigma'_i \triangleright {U'}_i^{\mathrm{ari}} \,/\, \square \,]$ by the IH with $\Gamma, z_i : {U'}_i^{\mathrm{par}} \vdash N_i : \Sigma'_i \triangleright {U'}_i^{\mathrm{ari}} \,/\, \square$ and $[\![\Gamma, z_i : {U'}_i^{\mathrm{par}}]\!] \preceq \Delta'$,
* $\Xi \parallel \Delta' \vdash \overline{v^{\mathsf{h}}} : C.\Sigma$ by (T_Var),
* $\Xi \parallel \Delta' \vdash \underline{1} : \mathsf{1}$ by (T_EConst),

we have the conclusion.

– We show that

$$\Xi \parallel \Delta \vdash \lambda x, \overline{h^{\square}}.\mathsf{return}\, [\![L]\!] : [\![T']\!] \rightarrow [\![\square(\Sigma')]\!] [\, [\![{C'}^{\mathrm{ini}}]\!] \,] \ .$$

By (T_Abs) and (T_Return) it suffices to show that

$$\Xi \parallel \Delta, x : [\![T']\!], \overline{h^{\square} \ : \square(\Sigma')} \vdash [\![L]\!] : [\![{C'}^{\mathrm{ini}}]\!] \ ,$$

which is derived by the IH with

* $\Gamma, x : T' \vdash L : {C'}^{\mathrm{ini}}$ and
* $[\![\Gamma, x : T']\!] \preceq \Delta, x : [\![T']\!], \overline{h^{\square} \ : \square(\Sigma')}$.

Case (HT_SubC): We are given

$$\frac{\Gamma \vdash_{\mathcal{D}'} M : C' \qquad C' <: C}{\Gamma \vdash_{\mathcal{D}} M : C}$$

for some C' and $\mathcal{D}'$. By case analysis on $C.A$.

Case $C.A = \square$: Assume that v^{k} such that $\Xi \parallel \Delta \vdash v^{\mathsf{k}} : \mathsf{1}$ is given. By the definition, it suffices to show that

$$\Xi \parallel \Delta \vdash [\![C' <: C]\!]^{\mathsf{e}} @([\![M]\!]^{\mathsf{e}}_{\mathcal{D}'}, \overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) : [\![C.T]\!] \ ,$$

which is derived by Lemma 22 and the IH.

Case $\exists\, C^{\mathrm{ini}}, C^{\mathrm{fin}}.\ C.A = C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$: Assume that v^{k} such that $\Xi \parallel \Delta \vdash v^{\mathsf{k}} : [\![C.T]\!] \rightarrow [\![\square(C.\Sigma)]\!] [\, [\![C^{\mathrm{ini}}]\!] \,]$ is given. By the definition, it suffices to show that

$$\Xi \parallel \Delta \vdash [\![C' <: C]\!]^{\mathsf{e}} @([\![M]\!]^{\mathsf{e}}_{\mathcal{D}'}, \overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) : [\![C^{\mathrm{fin}}]\!] \ ,$$

which is derived by Lemma 22 and the IH.

□

3.4 Semantics Preservation

Lemma 24 (Well-Definedness and Closedness of CPS Transformation on Subtyping Derivation)**.**

1. If $T_1 <: T_2$, then $[\![T_1 <: T_2]\!]$ is well defined and closed.
2. If $C_1 <: C_2$, then $[\![C_1 <: C_2]\!]$ is well defined and closed.
3. If $C_1 <: C_2$, then $[\![C_1 <: C_2]\!]^{\mathsf{e}}$ is well defined and closed.
4. If $\Sigma_1 <: \Sigma_2$, then $[\![\Sigma_1 <: \Sigma_2]\!]$ is well defined and closed.

Proof. Straightforward by induction on the subtyping derivations. □

Lemma 25 (Well-Definedness of CPS Transformation on Typing Derivation)**.**

1. If $\Gamma \vdash M : C$ and $|\overline{v^{\mathsf{h}}}| = |C.\Sigma|$, then, for any v^{k}, $[\![M]\!]^{\mathsf{e}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})$ is well defined.
2. If $\Gamma \vdash M : C$, then $[\![M]\!]$ is well defined.
3. If $\Gamma \vdash V : T$, then $[\![V]\!]$ is well defined.

Proof. Straightforward by induction on the typing derivations with Lemma 24. □

Lemma 26 (CPS Transformation of Reflexive Subtyping)**.**

1. $\forall\, T, v.\ [\![T <: T]\!] @ (v) \hookrightarrow v$.
2. $\forall\, C, v.\ [\![C <: C]\!] @ (v) \hookrightarrow v$.
3. $\forall\, \Sigma, \overline{v^{\mathsf{h}}}.\ |\overline{v^{\mathsf{h}}}| = |\Sigma| \Longrightarrow [\![\Sigma <: \Sigma]\!] @ (\overline{v^{\mathsf{h}}}) \hookrightarrow \overline{v^{\mathsf{h}}}$.

Proof. By mutual structural induction on T, C, and Σ.

1. By case analysis on T.

   Case $\exists\, B.\ T = B$: Obvious.
   Case $\exists\, n.\ T = \mathsf{n}$: Obvious.
   Case $\exists\, T', C'.\ T = T' \to C'$: The conclusion is proven by

$$\begin{array}{rcl} [\![T' \to C' <: T' \to C']\!]\,(v) & = & \lambda y_2.\mathsf{let}\, z_1 = v\,([\![T' <: T']\!] @ (y_2))\,\mathsf{in\ return}\,[\![C' <: C']\!] @ (z_1) \\ & \hookrightarrow & \lambda y_2.\mathsf{let}\, z_1 = v\, y_2\,\mathsf{in\ return}\, z_1 \quad \text{(by the IHs)} \\ & \hookrightarrow & \lambda y_2. v\, y_2 \\ & \hookrightarrow & v \end{array}$$

2. By the definition of the CPS transformation, we have

$$[\![C <: C]\!] @ (v) \quad = \quad \lambda \overline{h_2}, k_2.[\![C <: C]\!]^{\mathsf{e}} @ ((\boldsymbol{\lambda}(\overline{\mathbf{h}_1}, \mathbf{k}_1).\, v\, \overline{\mathbf{h}_1}\, \mathbf{k}_1), \overline{h_2}, k_2)\ .$$

   By case analysis on $C.A$.

   Case $C.A = \square$: By the IHs and the definition of the CPS transformation, the conclusion is proven as follows:

$$\begin{array}{rcl} [\![C <: C]\!] @ (v) & = & \lambda \overline{h_2}, k_2.\mathsf{let}\, x_1 = v\, [\![C.\Sigma <: C.\Sigma]\!] @ (\overline{h_2})\, k_2\, \mathsf{in\ return}\, [\![C.T <: C.T]\!] @ (x_1) \\ & \hookrightarrow & \lambda \overline{h_2}, k_2.\mathsf{let}\, x_1 = v\, \overline{h_2}\, k_2\, \mathsf{in\ return}\, x_1 \quad \text{(by the IHs)} \\ & \hookrightarrow & \lambda \overline{h_2}, k_2. v\, \overline{h_2}\, k_2 \\ & \hookrightarrow & v\ . \end{array}$$

Case $\exists\, C^{\mathrm{ini}}, C^{\mathrm{fin}}.\ C.A = C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$: By the IH and the definition of the CPS transformation, the conclusion is proven as follows:

$$\begin{array}{rcl} [\![C <: C]\!]@(v) & = & \lambda \overline{h_2}, k_2.\mathsf{let}\, x_1 = v\, \overline{h_2}\, k_2\, \mathsf{in\ return}\, [\![C^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!]@(x_1) \\ & \hookrightarrow & \lambda \overline{h_2}, k_2.\mathsf{let}\, x_1 = v\, \overline{h_2}\, k_2\, \mathsf{in\ return}\, x_1 \quad \text{(by the IH)} \\ & \hookrightarrow & \lambda \overline{h_2}, k_2.v\, \overline{h_2}\, k_2 \\ & \hookrightarrow & v\ . \end{array}$$

3. Assume
   - $\Sigma = \{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, C_i^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{fin}}\}^{1 \le i \le m} \uplus \{\varsigma_i : U_i^{\mathrm{par}} \rightsquigarrow U_i^{\mathrm{ari}} \,/\, \square\}^{1 \le i \le n}$ and
   - $\overline{v^{\mathsf{h}}} = \overline{v^{\mathsf{h}\sigma_i}}^{1 \le i \le m}, \overline{v^{\mathsf{h}\varsigma_i}}^{1 \le i \le n}$.

   Then, $[\![\Sigma <: \Sigma]\!]@(\overline{v^{\mathsf{h}}})$ returns a value sequence $\overline{v^{\sigma_i}}^{1 \le i \le m}, \overline{v^{\varsigma_i}}^{1 \le i \le n}$ such that
   - $\forall\, i \in [1, m].\ v^{\sigma_i} = \lambda x_2, k_2.\mathsf{let}\, x_1 = v^{\mathsf{h}\sigma_i}\, ([\![T_i^{\mathrm{par}} <: T_i^{\mathrm{par}}]\!]@(x_2))\, v_i\, \mathsf{in\ return}\, [\![C_i^{\mathrm{fin}} <: C_i^{\mathrm{fin}}]\!]@(x_1)$,
   - $\forall\, i \in [1, m].\ v_i = \lambda y_1, \overline{h^{\varsigma_j}}^{1 \le j \le n}.\mathsf{let}\, y_2 = k_2\, ([\![T_i^{\mathrm{ari}} <: T_i^{\mathrm{ari}}]\!]@(y_1))\, \overline{w^{\varsigma_j}}^{1 \le j \le n}\, \mathsf{in\ return}\, [\![C_i^{\mathrm{ini}} <: C_i^{\mathrm{ini}}]\!]@(y_2)$, and
   - $\forall\, i \in [1, n].\ w^{\varsigma_i} = \lambda z_2.\mathsf{let}\, z_1 = h^{\varsigma_i}\, ([\![U_i^{\mathrm{par}} <: U_i^{\mathrm{par}}]\!]@(z_2))\, \mathsf{in\ return}\, [\![U_i^{\mathrm{ari}} <: U_i^{\mathrm{ari}}]\!]@(z_1)$, and
   - $\forall\, i \in [1, n].\ v^{\varsigma_i} = \lambda x_2.\mathsf{let}\, x_1 = v^{\mathsf{h}\varsigma_i}\, ([\![U_i^{\mathrm{par}} <: U_i^{\mathrm{par}}]\!]@(x_2))\, \mathsf{in\ return}\, [\![U_i^{\mathrm{ari}} <: U_i^{\mathrm{ari}}]\!]@(x_1)$.

   By the IHs, the conclusion is proven as follows:

$$\begin{array}{rcl} \forall i \in [1,n].\ w^{\varsigma_i} & = & \lambda z_2.\mathsf{let}\, z_1 = h^{\varsigma_i}\, ([\![U_i^{\mathrm{par}} <: U_i^{\mathrm{par}}]\!]@(z_2))\, \mathsf{in\ return}\, [\![U_i^{\mathrm{ari}} <: U_i^{\mathrm{ari}}]\!]@(z_1) \\ & \hookrightarrow & \lambda z_2.\mathsf{let}\, z_1 = h^{\varsigma_i}\, z_2\, \mathsf{in\ return}\, z_1 \quad \text{(by the IHs)} \\ & \hookrightarrow & \lambda z_2.h^{\varsigma_i}\, z_2 \\ & \hookrightarrow & h^{\varsigma_i} \\ \forall i \in [1,m].\ \ v_i & = & \lambda y_1, \overline{h^{\varsigma_j}}^{1 \le j \le n}.\mathsf{let}\, y_2 = k_2\, ([\![T_i^{\mathrm{ari}} <: T_i^{\mathrm{ari}}]\!]@(y_1))\, \overline{w^{\varsigma_j}}^{1 \le j \le n}\, \mathsf{in\ return}\, [\![C_i^{\mathrm{ini}} <: C_i^{\mathrm{ini}}]\!]@(y_2) \\ & \hookrightarrow & \lambda y_1, \overline{h^{\varsigma_j}}^{1 \le j \le n}.\mathsf{let}\, y_2 = k_2\, y_1\, \overline{w^{\varsigma_j}}^{1 \le j \le n}\, \mathsf{in\ return}\, y_2 \quad \text{(by the IHs)} \\ & \hookrightarrow & \lambda y_1, \overline{h^{\varsigma_j}}^{1 \le j \le n}.k_2\, y_1\, \overline{w^{\varsigma_j}}^{1 \le j \le n} \\ & = & \lambda y_1, \overline{h^{\varsigma_j}}^{1 \le j \le n}.k_2\, y_1\, \overline{h^{\varsigma_j}}^{1 \le j \le n} \quad \text{(by the above reasoning)} \\ & \hookrightarrow & k_2 \\ \forall i \in [1,m].\ v^{\sigma_i} & = & \lambda x_2, k_2.\mathsf{let}\, x_1 = v^{\mathsf{h}\sigma_i}\, ([\![T_i^{\mathrm{par}} <: T_i^{\mathrm{par}}]\!]@(x_2))\, v_i\, \mathsf{in\ return}\, [\![C_i^{\mathrm{fin}} <: C_i^{\mathrm{fin}}]\!]@(x_1) \\ & \hookrightarrow & \lambda x_2, k_2.\mathsf{let}\, x_1 = v^{\mathsf{h}\sigma_i}\, x_2\, k_2\, \mathsf{in\ return}\, x_1 \quad \text{(by the IHs and the above reasoning)} \\ & \hookrightarrow & \lambda x_2, k_2.v^{\mathsf{h}\sigma_i}\, x_2\, k_2 \\ & \hookrightarrow & v^{\mathsf{h}\sigma_i} \\ \forall i \in [1,m].\ v^{\varsigma_i} & = & \lambda x_2.\mathsf{let}\, x_1 = v^{\mathsf{h}\varsigma_i}\, ([\![U_i^{\mathrm{par}} <: U_i^{\mathrm{par}}]\!]@(x_2))\, \mathsf{in\ return}\, [\![U_i^{\mathrm{ari}} <: U_i^{\mathrm{ari}}]\!]@(x_1) \\ & \hookrightarrow & \lambda x_2.\mathsf{let}\, x_1 = v^{\mathsf{h}\varsigma_i}\, x_2\, \mathsf{in\ return}\, x_1 \quad \text{(by the IHs)} \\ & \hookrightarrow & \lambda x_2.v^{\mathsf{h}\varsigma_i}\, x_2 \\ & \hookrightarrow & v^{\mathsf{h}\varsigma_i}\ . \end{array}$$

$\square$

Lemma 27 (CPS Transformation of Subtyping on Final Answer Types)**.** If $\overline{C} = C_1, \cdots, C_n$ and $E^{\overline{C}}$ is well defined and $\Gamma \vdash_{\mathcal{D}} M : \Sigma \triangleright T \,/\, C \Rightarrow C_1$ and $|\overline{v^{\mathsf{h}}}| = |\Sigma|$, then, for any v^{k},

$$E^{\overline{C}}[[\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})] = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}'}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})$$

for some $\mathcal{D}'$ and $\Gamma \vdash_{\mathcal{D}'} M : \Sigma \triangleright T \,/\, C \Rightarrow C_n$.

Proof. By induction on n, it suffices to show that: if $\Gamma \vdash_{\mathcal{D}} M : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C_1^{\mathrm{fin}}$ and $C_1^{\mathrm{fin}} <: C_2^{\mathrm{fin}}$ and $|\overline{v^{\mathsf{h}}}| = |\Sigma|$, then, for any v^{k} and fresh variable x,

$$\mathsf{let}\, x = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})\, \mathsf{in\ return}\, [\![C_1^{\mathrm{fin}} <: C_2^{\mathrm{fin}}]\!]@(x) = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}'}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})$$

for some $\mathcal{D}'$ such that $\Gamma \vdash_{\mathcal{D}'} M : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}$

Assume that $\Gamma \vdash_{\mathcal{D}} M : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C_1^{\mathrm{fin}}$ and $C_1^{\mathrm{fin}} <: C_2^{\mathrm{fin}}$ and $|\overline{v^{\mathsf{h}}}| = |\Sigma|$ and v^{k} are given.

By Lemma 4 and (HS_Comp), we have $\Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C_1^{\mathrm{fin}} <: \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}$. Let

$$\mathcal{D}' \;=\; \frac{\Gamma \vdash_{\mathcal{D}} M : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C_1^{\mathrm{fin}} \quad \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C_1^{\mathrm{fin}} <: \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}}{\Gamma \vdash M : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}} \;(\text{HS\_SubC}).$$

Let x is fresh variable. Then, the conclusion is proven as follows:

$$\begin{array}{rcl} [\![M]\!]^{\mathsf{e}}_{\mathcal{D}'} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) & = & [\![\Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C_1^{\mathrm{fin}} <: \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}]\!]^{\mathsf{e}} @([\![M]\!]^{\mathsf{e}}_{\mathcal{D}}, \overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\ & = & \mathsf{let}\, x = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \,\mathsf{in}\, \mathsf{return}\, [\![C_1^{\mathrm{fin}} <: C_2^{\mathrm{fin}}]\!] @(x) \;. \end{array}$$

□

Lemma 28 (CPS Transformation of Operation Signature Subtyping)**.** If $\Sigma_1 <: \Sigma_2$ and $|\Sigma_1| = |\overline{v^{\mathsf{h}}}|$, then $|\Sigma_2| = |[\![\Sigma_1 <: \Sigma_2]\!] @(\overline{v^{\mathsf{h}}})|$.

Proof. By the definition of the CPS transformation. □

Lemma 29 (Evaluation Under Evaluation Contexts)**.** If $e_1 \longrightarrow^n e_2$, then $E[e_1] \longrightarrow^n E[e_2]$ for any E.

Proof. First, it is easy to show that

$$\forall\, e_1, e_2, E.\; e_1 \longrightarrow e_2 \Longrightarrow E[e_1] \longrightarrow E[e_2]$$

by induction on E. Then, the conclusion is proven by induction on n. □

Lemma 30 (Weakening of CPS Transformation)**.** Assume that $dom(\Gamma_2) \cap dom(\Gamma_1, \Gamma_3)$ is empty.

- If $\Gamma_1, \Gamma_3 \vdash_{\mathcal{D}} V : T$, then $[\![V]\!]_{\mathcal{D}} = [\![V]\!]_{\mathcal{D}'}$ for some $\mathcal{D}'$ such that $\Gamma_1, \Gamma_2, \Gamma_3 \vdash_{\mathcal{D}'} V : T$.
- If $\Gamma_1, \Gamma_3 \vdash_{\mathcal{D}} M : C$, then $[\![M]\!]_{\mathcal{D}} = [\![M]\!]_{\mathcal{D}'}$ for some $\mathcal{D}'$ such that $\Gamma_1, \Gamma_2, \Gamma_3 \vdash_{\mathcal{D}'} M : C$.
- If $\Gamma_1, \Gamma_3 \vdash M : C$ and $|\overline{v^{\mathsf{h}}}| = |C.\Sigma|$, then, for any v^{k}, $[\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}'} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})$ for some $\mathcal{D}'$ such that $\Gamma_1, \Gamma_2, \Gamma_3 \vdash_{\mathcal{D}'} M : C$.

Proof. Straightforward by mutual induction on the typing derivations. □

Lemma 31 (Substitution is a Homomorphism)**.** Assume that $\Gamma_1 \vdash V' : T'$.

1. If $\Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}} M : C$, then $[\![M]\!]^{\mathsf{e}}_{\mathcal{D}}[[\![V']\!]/x] = [\![M[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'}$ for some $\mathcal{D}'$ such that $\Gamma_1, \Gamma_2 \vdash M[V'/x] : C$.
2. If $\Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}} M : C$, then $[\![M]\!]_{\mathcal{D}}[[\![V']\!]/x] = [\![M[V'/x]]\!]_{\mathcal{D}'}$ for some $\mathcal{D}'$ such that $\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'} M[V'/x] : C$.
3. If $\Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}} V : T$, then $[\![V]\!]_{\mathcal{D}}[[\![V']\!]/x] = [\![V[V'/x]]\!]_{\mathcal{D}'}$ for some $\mathcal{D}'$ such that $\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'} V[V'/x] : T$.

Proof. By mutual induction on the typing derivations.

1. By case analysis on the typing rule applied last to derive $\Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}} M : C$. We may omit typing derivations in the CPS transformation but they are clear from the context.

   Case (HT_Return): We are given

   $$\frac{\Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}''} V : C.T}{\Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}} \mathsf{return}\; V : C.\Sigma \triangleright C.T \,/\, \square}$$

   and $C.A = \square$ for some V and $\mathcal{D}''$ such that $M = \mathsf{return}\; V$. The conclusion is shown as follows:

   $$\begin{array}{rll} & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}[[\![V']\!]/x] & \\ = & (\boldsymbol{\lambda}(\overline{\mathbf{h}}, \mathbf{k}).\, \mathsf{return}\, [\![V]\!]_{\mathcal{D}''})[[\![V']\!]/x] & \\ = & \boldsymbol{\lambda}(\overline{\mathbf{h}}, \mathbf{k}).\, \mathsf{return}\, [\![V[V'/x]]\!]_{\mathcal{D}'''} & \text{(by the IH)} \\ = & [\![\mathsf{return}\; V[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'} & \\ = & [\![M[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'} & \end{array}$$

for some $\mathcal{D}'''$ and $\mathcal{D}'$ such that $\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'''} V[V'/x] : C.T$ and

$$\frac{\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'''} V[V'/x] : C.T}{\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'} \mathsf{return}\ V[V'/x] : C.\Sigma \triangleright C.T \,/\, \square}\ (\text{T\_Return}).$$

Case (HT_SubC): We are given

$$\frac{\Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}''} M : C' \quad C' <: C}{\Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}} M : C}$$

for some C' and $\mathcal{D}''$. Then,

$$\begin{array}{rl} & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}[[\![V']\!]/x] \\ = & (\boldsymbol{\lambda}(\overline{\mathbf{h}}, \mathbf{k}).\, [\![C' <: C]\!]^{\mathsf{e}} @([\![M]\!]^{\mathsf{e}}_{\mathcal{D}''}, \overline{\mathbf{h}}, \mathbf{k}))[[\![V']\!]/x] \\ = & (\boldsymbol{\lambda}(\overline{\mathbf{h}}, \mathbf{k}).\, [\![C' <: C]\!]^{\mathsf{e}} @([\![M[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'''}, \overline{\mathbf{h}}, \mathbf{k})) \qquad \text{(by the IH and Lemma 24)} \\ = & [\![M[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'} \end{array}$$

for some $\mathcal{D}'''$ and $\mathcal{D}'$ such that $\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'''} M[V'/x] : C'$ and

$$\frac{\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'''} M[V'/x] : C' \quad C' <: C}{\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'} M[V'/x] : C}\ (\text{HT\_SubC}).$$

(HT_Let): We are given

$$\frac{\Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}_1} M_1 : C.\Sigma \triangleright T_1 \,/\, \square \quad \Gamma_1, x : T', \Gamma_2, y : T_1 \vdash_{\mathcal{D}_2} M_2 : C}{\Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}} \mathsf{let}\, y = M_1 \,\mathsf{in}\, M_2 : C}$$

for some y, M_1, M_2, T_1, $\mathcal{D}_1$, and $\mathcal{D}_2$ such that $M = (\mathsf{let}\, y = M_1 \,\mathsf{in}\, M_2)$. Without loss of generality, we can assume that $y \notin \mathit{fv}(V') \cup \{x\}$. Then, the conclusion is shown as follows:

$$\begin{array}{rl} & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}[[\![V']\!]/x] \\ = & (\boldsymbol{\lambda}(\overline{\mathbf{h}}, \mathbf{k}).\, \mathsf{let}\, y = [\![M_1]\!]^{\mathsf{e}}_{\mathcal{D}_1} @(\overline{\mathbf{h}}, \underline{1}) \,\mathsf{in}\, [\![M_2]\!]^{\mathsf{e}}_{\mathcal{D}_2} @(\overline{\mathbf{h}}, \mathbf{k}))[[\![V']\!]/x] \\ = & \boldsymbol{\lambda}(\overline{\mathbf{h}}, \mathbf{k}).\, \mathsf{let}\, y = [\![M_1[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'_1} @(\overline{\mathbf{h}}, \underline{1}) \,\mathsf{in}\, [\![M_2[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'_2} @(\overline{\mathbf{h}}, \mathbf{k}) \quad \text{(by the IHs)} \\ = & [\![\mathsf{let}\, y = M_1[V'/x] \,\mathsf{in}\, M_2[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'} \\ = & [\![M[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'} \end{array}$$

for some $\mathcal{D}'_1$, $\mathcal{D}'_2$, and $\mathcal{D}'$ such that $\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'_1} M_1[V'/x] : C.\Sigma \triangleright T_1 \,/\, \square$ and $\Gamma_1, \Gamma_2, y : T_1 \vdash_{\mathcal{D}'_2} M_2[V'/x] : C$ and

$$\frac{\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'_1} M_1[V'/x] : C.\Sigma \triangleright T_1 \,/\, \square \qquad \Gamma_1, \Gamma_2, y : T_1 \vdash_{\mathcal{D}'_2} M_2[V'/x] : C}{\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'} \mathsf{let}\, y = M_1[V'/x] \,\mathsf{in}\, M_2[V'/x] : C}\ (\text{HT\_Let}).$$

(HT_LetATM): We are given

$$\frac{\Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}_1} M_1 : C.\Sigma \triangleright T_1 \,/\, C' \Rightarrow C^{\mathrm{fin}} \quad \Gamma_1, x : T', \Gamma_2, y : T_1 \vdash_{\mathcal{D}_2} M_2 : C.\Sigma \triangleright C.T \,/\, C^{\mathrm{ini}} \Rightarrow C'}{\Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}} \mathsf{let}\, y = M_1 \,\mathsf{in}\, M_2 : C.\Sigma \triangleright C.T \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}}$$

for some y, M_1, M_2, T_1, C', C^{ini}, C^{fin}, $\mathcal{D}_1$, and $\mathcal{D}_2$ such that $M = (\mathsf{let}\, y = M_1 \,\mathsf{in}\, M_2)$ and $C.A = C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$. Without loss of generality, we can assume that $y \notin \mathit{fv}(V') \cup \{x\}$. Let $\overline{h^{\square}}$ be a sequence of fresh variables such that $|\overline{h^{\square}}| = |\square(C.\Sigma)|$. Then, the conclusion is shown as follows:

$$\begin{array}{rl} & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}[[\![V']\!]/x] \\ = & (\boldsymbol{\lambda}(\overline{\mathbf{h}^{\not\square}}, \overline{\mathbf{h}^{\square}}, \mathbf{k}).\, [\![M_1]\!]^{\mathsf{e}}_{\mathcal{D}_1} @(\overline{\mathbf{h}^{\not\square}}, \overline{\mathbf{h}^{\square}}, \lambda y, \overline{h^{\square}}.[\![M_2]\!]^{\mathsf{e}}_{\mathcal{D}_2} @(\overline{\mathbf{h}^{\not\square}}, \overline{h^{\square}}, \mathbf{k})))[[\![V']\!]/x] \\ & \quad \text{(where } |\overline{\mathbf{h}^{\not\square}}| = |\not\square(C.\Sigma)| \text{ and } |\overline{\mathbf{h}^{\square}}| = |\square(C.\Sigma)|) \\ = & \boldsymbol{\lambda}(\overline{\mathbf{h}^{\not\square}}, \overline{\mathbf{h}^{\square}}, \mathbf{k}).\, [\![M_1[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'_1} @(\overline{\mathbf{h}^{\not\square}}, \overline{\mathbf{h}^{\square}}, \lambda y, \overline{h^{\square}}.[\![M_2[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'_2} @(\overline{\mathbf{h}^{\not\square}}, \overline{h^{\square}}, \mathbf{k})) \quad \text{(by the IHs)} \\ = & [\![\mathsf{let}\, y = M_1[V'/x] \,\mathsf{in}\, M_2[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'} \\ = & [\![M[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'} \end{array}$$

for some $\mathcal{D}'_1$, $\mathcal{D}'_2$, and $\mathcal{D}'$ such that $\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'_1} M_1[V'/x] : C.\Sigma \triangleright T_1 \,/\, C' \Rightarrow C^{\mathrm{fin}}$ and $\Gamma_1, \Gamma_2, y : T_1 \vdash_{\mathcal{D}'_2} M_2[V'/x] : C.\Sigma \triangleright C.T \,/\, C^{\mathrm{ini}} \Rightarrow C'$ and

$$\frac{\begin{array}{c}\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'_1} M_1[V'/x] : C.\Sigma \triangleright T_1 \,/\, C' \Rightarrow C^{\mathrm{fin}} \\ \Gamma_1, \Gamma_2, y : T_1 \vdash_{\mathcal{D}'_2} M_2[V'/x] : C.\Sigma \triangleright C.T \,/\, C^{\mathrm{ini}} \Rightarrow C'\end{array}}{\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'} \mathsf{let}\, y = M_1[V'/x] \,\mathsf{in}\, M_2[V'/x] : C.\Sigma \triangleright C.T \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}} \text{ (HT\_LetATM)}.$$

Case (HT_App): We are given

$$\frac{\Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}_1} V_1 : T' \to C \quad \Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}_2} V_2 : T'}{\Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}} V_1\ V_2 : C}$$

for some V_1, V_2, T', $\mathcal{D}_1$, and $\mathcal{D}_2$ such that $M = V_1\ V_2$. The conclusion is shown as follows:

$$\begin{array}{cll} & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}[[\![V']\!]/x] & \\ = & (\boldsymbol{\lambda}(\overline{\mathbf{h}}, \mathbf{k}).\, [\![V_1]\!]_{\mathcal{D}_1}\, [\![V_2]\!]_{\mathcal{D}_2}\, \overline{\mathbf{h}}\, \mathbf{k})[[\![V']\!]/x] & \\ = & \boldsymbol{\lambda}(\overline{\mathbf{h}}, \mathbf{k}).\, [\![V_1[V'/x]]\!]_{\mathcal{D}'_1}\, [\![V_2[V'/x]]\!]_{\mathcal{D}'_2}\, \overline{\mathbf{h}}\, \mathbf{k} & \text{(by the IHs)} \\ = & [\![V_1[V'/x]\ V_2[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'} & \\ = & [\![M[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'} & \end{array}$$

for some $\mathcal{D}'_1$, $\mathcal{D}'_2$, and $\mathcal{D}'$ such that $\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'_1} V_1[V'/x] : T' \to C$ and $\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'_2} V_2[V'/x] : T'$ and

$$\frac{\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'_1} V_1[V'/x] : T' \to C \qquad \Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'_2} V_2[V'/x] : T'}{\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'} V_1[V'/x]\ V_2[V'/x] : C} \text{ (HT\_App)}.$$

Case (HT_Case): We are given

$$\frac{\Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}_0} V : \mathsf{n} \quad \forall\, i \in [1, n].\ \Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}_i} M_i : C}{\Gamma_1, x : T', \Gamma_2 \vdash \mathsf{case}(V; M_1, \cdots, M_n) : C}$$

for some V_1, $M_1, \cdots, M_n$, n, $\mathcal{D}_0$, and $\mathcal{D}_1, \cdots, \mathcal{D}_n$ such that $M = \mathsf{case}(V; M_1, \cdots, M_n)$. The conclusion is shown as follows:

$$\begin{array}{cll} & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}[[\![V']\!]/x] & \\ = & (\boldsymbol{\lambda}(\overline{\mathbf{h}}, \mathbf{k}).\, \mathsf{case}([\![V]\!]_{\mathcal{D}_0}; [\![M_1]\!]^{\mathsf{e}}_{\mathcal{D}_1} @(\overline{\mathbf{h}}, \mathbf{k}), \cdots, [\![M_n]\!]^{\mathsf{e}}_{\mathcal{D}_n} @(\overline{\mathbf{h}}, \mathbf{k})))[[\![V']\!]/x] & \\ = & \boldsymbol{\lambda}(\overline{\mathbf{h}}, \mathbf{k}).\, \mathsf{case}([\![V[V'/x]]\!]_{\mathcal{D}'_0}; [\![M_1[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'_1} @(\overline{\mathbf{h}}, \mathbf{k}), \cdots, [\![M_n[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'_n} @(\overline{\mathbf{h}}, \mathbf{k})) & \text{(by the IHs)} \\ = & [\![\mathsf{case}(V[V'/x]; M_1[V'/x], \cdots, M_n[V'/x])]\!]^{\mathsf{e}}_{\mathcal{D}'} & \\ = & [\![M[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'}\ . & \end{array}$$

for some $\mathcal{D}'_0$, $\mathcal{D}'_1, \cdots, \mathcal{D}'_n$, and $\mathcal{D}'$ such that $\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'_0} V[V'/x] : \mathsf{n}$ and $\forall\, i \in [1, n].\ \Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'_i} M_i[V'/x] : C$ and

$$\frac{\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'_0} V[V'/x] : \mathsf{n} \qquad \forall\, i \in [1, n].\ \Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'_i} M_i[V'/x] : C}{\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'} \mathsf{case}(V[V'/x]; M_1[V'/x], \cdots, M_n[V'/x]) : C} \text{ (HT\_Case)}.$$

Case (HT_Op): We are given

$$\frac{\varsigma : U^{\mathrm{par}} \rightsquigarrow U^{\mathrm{ari}} \,/\, \square \in C.\Sigma \quad \Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}_1} V : U^{\mathrm{par}} \quad \Gamma_1, x : T', \Gamma_2, y : U^{\mathrm{ari}} \vdash_{\mathcal{D}_2} M' : C}{\Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}} \varsigma(V; y.\, M') : C}$$

for some ς, V, y, M', U^{par}, U^{ari}, $\mathcal{D}_1$, and $\mathcal{D}_2$ such that $M = \varsigma(V; y.\, M')$. Without loss of generality, we can assume that $y \notin fv(V') \cup \{x\}$. The conclusion is shown as follows:

$$\begin{array}{cll} & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}[[\![V']\!]/x] & \\ = & (\boldsymbol{\lambda}(\overline{\mathbf{h}}, \mathbf{k}).\, \mathsf{let}\, y = \mathbf{h}^{\varsigma}\, [\![V]\!]_{\mathcal{D}_1} \,\mathsf{in}\, [\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2} @(\overline{\mathbf{h}}, \mathbf{k}))[[\![V']\!]/x] & \text{(where } \mathbf{h}^{\varsigma} \text{ corresponds to } \varsigma \text{ in } \overline{\mathbf{h}}\text{)} \\ = & \boldsymbol{\lambda}(\overline{\mathbf{h}}, \mathbf{k}).\, \mathsf{let}\, y = \mathbf{h}^{\varsigma}\, [\![V[V'/x]]\!]_{\mathcal{D}'_1} \,\mathsf{in}\, [\![M'[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'_2} @(\overline{\mathbf{h}}, \mathbf{k}) & \text{(by the IHs)} \\ = & [\![\varsigma(V[V'/x]; y.\, M'[V'/x])]\!]^{\mathsf{e}}_{\mathcal{D}'} & \\ = & [\![M[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'} & \end{array}$$

for some $\mathcal{D}'_1$, $\mathcal{D}'_2$, and $\mathcal{D}'$ such that $\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'_1} V[V'/x] : U^{\mathrm{par}}$ and $\Gamma_1, \Gamma_2, y : U^{\mathrm{ari}} \vdash_{\mathcal{D}'_2} M'[V'/x] : C$ and

$$\frac{\varsigma : U^{\mathrm{par}} \rightsquigarrow U^{\mathrm{ari}} \,/\, \square \in C.\Sigma \qquad \Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'_1} V[V'/x] : U^{\mathrm{par}} \qquad \Gamma_1, \Gamma_2, y : U^{\mathrm{ari}} \vdash_{\mathcal{D}'_2} M'[V'/x] : C}{\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'} \varsigma(V[V'/x]; y.\, M'[V'/x]) : C} \text{ (HT\_Op)}.$$

Case (HT_OpATM): We are given

$$\frac{\begin{array}{c}\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}} \in C.\Sigma \\ \Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}_1} V : T^{\mathrm{par}} \qquad \Gamma_1, x : T', \Gamma_2, y : T^{\mathrm{ari}} \vdash_{\mathcal{D}_2} M' : C.\Sigma \triangleright C.T \,/\, C' \Rightarrow C^{\mathrm{ini}}\end{array}}{\Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}} \sigma(V; y.\, M') : C.\Sigma \triangleright C.T \,/\, C' \Rightarrow C^{\mathrm{fin}}}$$

for some σ, V, y, M', T^{par}, T^{ari}, C^{ini}, C^{fin}, C', $\mathcal{D}_1$, and $\mathcal{D}_2$ such that $M = \sigma(V; y.\, M')$. Without loss of generality, we can assume that $y \notin fv(V') \cup \{x\}$. Let $\overline{h^{\square}}$ be a sequence of fresh variables such that $|\overline{h^{\square}}| = |\square(C.\Sigma)|$. The conclusion is shown as follows:

$$\begin{array}{cl}
& [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}[[\![V']\!]/x] \\
= & (\boldsymbol{\lambda}(\overline{\mathbf{h}^{\not\square}}, \overline{\mathbf{h}^{\square}}, \mathbf{k}).\, \mathbf{h}^{\sigma}\ [\![V]\!]_{\mathcal{D}_1}\ (\lambda x, \overline{h^{\square}}.[\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2} @(\overline{\mathbf{h}^{\not\square}}, \overline{h^{\square}}, \mathbf{k})))[[\![V']\!]/x] \\
& \quad (\text{where } |\overline{\mathbf{h}^{\not\square}}| = |\not\square(C.\Sigma)| \text{ and } |\overline{\mathbf{h}^{\square}}| = |\square(C.\Sigma)| \text{ and } \mathbf{h}^{\sigma} \text{ corresponds to } \sigma \text{ in } \overline{\mathbf{h}^{\not\square}}) \\
= & \boldsymbol{\lambda}(\overline{\mathbf{h}^{\not\square}}, \overline{\mathbf{h}^{\square}}, \mathbf{k}).\, \mathbf{h}^{\sigma}\ [\![V[V'/x]]\!]_{\mathcal{D}'_1}\ (\lambda x, \overline{h^{\square}}.[\![M'[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'_2} @(\overline{\mathbf{h}^{\not\square}}, \overline{h^{\square}}, \mathbf{k})) \qquad \text{(by the IHs)} \\
= & [\![\sigma(V[V'/x]; y.\, M'[V'/x])]\!]^{\mathsf{e}}_{\mathcal{D}'} \\
= & [\![M[V'/x]]\!]^{\mathsf{e}}_{\mathcal{D}'}
\end{array}$$

for some $\mathcal{D}'_1$, $\mathcal{D}'_2$, and $\mathcal{D}'$ such that $\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'_1} V[V'/x] : T^{\mathrm{par}}$ and $\Gamma_1, \Gamma_2, y : T^{\mathrm{ari}} \vdash_{\mathcal{D}'_2} M'[V'/x] : C.\Sigma \triangleright C.T \,/\, C' \Rightarrow C^{\mathrm{ini}}$ and

$$\frac{\begin{array}{c}\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}} \in C.\Sigma \\ \Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'_1} V[V'/x] : T^{\mathrm{par}} \qquad \Gamma_1, \Gamma_2, y : T^{\mathrm{ari}} \vdash_{\mathcal{D}'_2} M'[V'/x] : C.\Sigma \triangleright C.T \,/\, C' \Rightarrow C^{\mathrm{ini}}\end{array}}{\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}'} \sigma(V; y.\, M') : C.\Sigma \triangleright C.T \,/\, C' \Rightarrow C^{\mathrm{fin}}} \text{ (HT\_OpATM)}.$$

Case (HT_Handle): We are given

$$\frac{\begin{array}{c}H' = \{\mathsf{return}\, y \mapsto L\} \uplus \{\sigma_i(y_i; k_i) \mapsto M_i\}^{1 \le i \le m} \uplus \{\varsigma_i(z_i) \mapsto N_i\}^{1 \le i \le n} \\ \Sigma_0 = \{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, C_i^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{fin}}\}^{1 \le i \le m} \uplus \{\varsigma_i : U_i^{\mathrm{par}} \rightsquigarrow U_i^{\mathrm{ari}} \,/\, \square\}^{1 \le i \le n} \\ \Gamma_1, x : T', \Gamma_2 \vdash M_0 : \Sigma_0 \triangleright T_0 \,/\, C_0^{\mathrm{ini}} \Rightarrow C \quad \Gamma_1, x : T', \Gamma_2, y : T_0 \vdash L : C_0^{\mathrm{ini}} \\ \forall\, i \in [1, m].\ \Gamma_1, x : T', \Gamma_2, y_i : T_i^{\mathrm{par}}, k_i : T_i^{\mathrm{ari}} \to C_i^{\mathrm{ini}} \vdash M_i : C_i^{\mathrm{fin}} \\ a\forall\, i \in [1, n].\ \Gamma_1, x : T', \Gamma_2, z_i : U_i^{\mathrm{par}} \vdash N_i : \Sigma_i \triangleright U_i^{\mathrm{ari}} \,/\, \square \\ \forall\, C' \in \{\overline{C_i^{\mathrm{ini}}}^{1 \le i \le m}, C\}.\ \forall\, i \in [1, n].\ C'.\Sigma <: \Sigma_i\end{array}}{\Gamma_1, x : T', \Gamma_2 \vdash \mathsf{with}\, H'\, \mathsf{handle}\, M_0 : C}$$

for some H', M_0, y, L, $\sigma_1, \cdots, \sigma_m$, $y_1, \cdots, y_m$, $k_1, \cdots, k_m$, $M_1, \cdots, M_m$, $T_1^{\mathrm{par}}, \cdots, T_m^{\mathrm{par}}$, $T_1^{\mathrm{ari}}, \cdots, T_m^{\mathrm{ari}}$, $C_1^{\mathrm{ini}}, \cdots, C_m^{\mathrm{ini}}$, and $C_1^{\mathrm{fin}}, \cdots, C_m^{\mathrm{fin}}$, $\varsigma_1, \cdots, \varsigma_n$, $z_1, \cdots, z_n$, $N_1, \cdots, N_n$, $U_1^{\mathrm{par}}, \cdots, U_n^{\mathrm{par}}$, $U_1^{\mathrm{ari}}, \cdots, U_n^{\mathrm{ari}}$, Σ_0, T_0, and C_0^{ini} such that $M = \mathsf{with}\, H'\, \mathsf{handle}\, M_0$. Without loss of generality, we can assume that the variables $y, y_1, \cdots, y_m, k_1, \ldots, k_n, z_1, \cdots, z_n$ are distinct from the variables in $fv(V') \cup \{x\}$. By the definition of the CPS transformation, we have

$$[\![M]\!]^{\mathsf{e}}_{\mathcal{D}} = \boldsymbol{\lambda}(\overline{\mathbf{h}}, \mathbf{k}).\, [\![M_0]\!]^{\mathsf{e}} @(\overline{w_i^{\not\square}}^{1 \le i \le m}, \overline{w_i^{\square}}^{1 \le i \le n}, \lambda y, \overline{h^{\square}}.\mathsf{return}\, [\![L]\!])\, \overline{\mathbf{h}}\, \mathbf{k}$$

where

- $\forall\, i \in [1, m].\ w_i^{\not\square} = \lambda y_i, k'_i.\mathsf{let}\, k_i = \mathsf{return}\, (\lambda y, \overline{h}, k.k'_i\, y\, \overline{v_{ij}}^{1 \le j \le n}\, \overline{h}\, k)\, \mathsf{in}\, \mathsf{return}\, [\![M_i]\!]\ (|\overline{h}| = |C_i^{\mathrm{ini}}.\Sigma|)$,
- $\forall\, i \in [1, m].\ \forall\, j \in [1, n].\ v_{ij} = \lambda z_j.[\![\Sigma_j \triangleright U_j^{\mathrm{ari}} \,/\, \square <: C_i^{\mathrm{ini}}.\Sigma \triangleright U_j^{\mathrm{ari}} \,/\, \square]\!]^{\mathsf{e}} @([\![N_j]\!]^{\mathsf{e}}, \overline{h}, \underline{1})$,
- $\forall\, i \in [1, n].\ w_i^{\square} = \lambda z_i.[\![\Sigma_i \triangleright U_i^{\mathrm{ari}} \,/\, \square <: C.\Sigma \triangleright U_i^{\mathrm{ari}} \,/\, \square]\!]^{\mathsf{e}} @([\![N_i]\!]^{\mathsf{e}}, \overline{\mathbf{h}}, \underline{1})$, and
- $\overline{h^{\square}}$ is a sequence of fresh variables such that $|\overline{h^{\square}}| = n$.

By the IHs and Lemma 24,

- $\forall\, i \in [1,m].\ w_i^{\not\square}[\llbracket V' \rrbracket/x] = \lambda y_i, k'_i.\mathsf{let}\ k_i = \mathsf{return}\ (\lambda y, \overline{h}, k.k'_i\ y\ \overline{v'_{ij}}^{1\le j\le n}\ \overline{h}\ k)\ \mathsf{in\ return}\ \llbracket M_i[V'/x] \rrbracket$,
- $\forall\, i \in [1,m].\ \forall\, j \in [1,n].\ v'_{ij} = \lambda z_j.\llbracket \Sigma_j \triangleright U_j^{\mathrm{ari}} \,/\, \square <: C_i^{\mathrm{ini}}.\Sigma \triangleright U_j^{\mathrm{ari}} \,/\, \square \rrbracket^{\mathsf{e}} @(\llbracket N_j[V'/x] \rrbracket^{\mathsf{e}}, \overline{h}, \underline{1})$, and
- $\forall\, i \in [1,n].\ w_i^{\square}[\llbracket V' \rrbracket/x] = \lambda z_i.\llbracket \Sigma_i \triangleright U_i^{\mathrm{ari}} \,/\, \square <: C.\Sigma \triangleright U_i^{\mathrm{ari}} \,/\, \square \rrbracket^{\mathsf{e}} @(\llbracket N_i[V'/x] \rrbracket^{\mathsf{e}}, \overline{\mathbf{h}}, \underline{1})$.

Then, the conclusion is shown as follows:

$$\begin{array}{cl}
 & \llbracket M \rrbracket^{\mathsf{e}}_{\mathcal{D}}[\llbracket V' \rrbracket/x] \\
= & (\boldsymbol{\lambda}(\overline{\mathbf{h}}, \mathbf{k}).\, \llbracket M_0 \rrbracket^{\mathsf{e}} @(\overline{w_i^{\not\square}}^{1\le i\le m}, \overline{w_i^{\square}}^{1\le i\le n}, \lambda y, \overline{h^{\square}}.\mathsf{return}\ \llbracket L \rrbracket))[\llbracket V' \rrbracket/x] \\
= & \boldsymbol{\lambda}(\overline{\mathbf{h}}, \mathbf{k}).\, \llbracket M_0[V'/x] \rrbracket^{\mathsf{e}} @(\overline{w_i^{\not\square}[\llbracket V' \rrbracket/x]}^{1\le i\le m}, \overline{w_i^{\square}[\llbracket V' \rrbracket/x]}^{1\le i\le n}, \lambda y, \overline{h^{\square}}.\mathsf{return}\ \llbracket L[V'/x] \rrbracket) \\
 & \quad \text{(by the IHs on } M_0 \text{ and } L) \\
= & \llbracket (\mathsf{with}\ H'\ \mathsf{handle}\ M_0)[V'/x] \rrbracket^{\mathsf{e}}_{\mathcal{D}'} \\
= & \llbracket M[V'/x] \rrbracket^{\mathsf{e}}_{\mathcal{D}'}
\end{array}$$

for some $\mathcal{D}'$ such that $\Gamma_1, \Gamma_2 \vdash (\mathsf{with}\ H'\ \mathsf{handle}\ M_0)[V'/x] : C$.

2. By case (1).
3. Straightforward by case analysis on the typing rule applied last to derive $\Gamma_1, x : T', \Gamma_2 \vdash_{\mathcal{D}} V : T$. The case for (HT_Var) rests on Lemma 30, and the case for (HT_SubV) rests on Lemma 24.

□

Lemma 32 (Handler and Continuation Substitution)**.**

1. If $\Gamma \vdash_{\mathcal{D}} M : C$ and $x \notin \mathit{dom}(\Gamma)$, then $\llbracket M \rrbracket^{\mathsf{e}}_{\mathcal{D}}[v/x] = \llbracket M \rrbracket^{\mathsf{e}}_{\mathcal{D}}$ for any v.
2. If $\Gamma \vdash_{\mathcal{D}} M : C$ and $x \notin \mathit{dom}(\Gamma)$, then $\llbracket M \rrbracket_{\mathcal{D}}[v/x] = \llbracket M \rrbracket_{\mathcal{D}}$ for any v.
3. If $\Gamma \vdash_{\mathcal{D}} V : T$ and $x \notin \mathit{dom}(\Gamma)$, then $\llbracket V \rrbracket_{\mathcal{D}}[v/x] = \llbracket V \rrbracket_{\mathcal{D}}$ for any v.

Proof. Straightforward by mutual induction on the typing derivations with Lemma 24.

□

Lemma 33 (CPS Transformation of Lambda Function Applications)**.** If $\Gamma \vdash_{\mathcal{D}_1} \lambda x.M : T \to C$ and $\Gamma \vdash_{\mathcal{D}_2} V : T$, then, for any $\overline{v^{\mathsf{h}}}$ and v^{k} such that $|\overline{v^{\mathsf{h}}}| = |C.\Sigma|$,

- $\llbracket \lambda x.M \rrbracket_{\mathcal{D}_1}\, \llbracket V \rrbracket_{\mathcal{D}_2} \longrightarrow^{+} \mathsf{return}\ v$,
- $v\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} \longrightarrow^{+} \llbracket M[V/x] \rrbracket^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})$, and
- $\Gamma \vdash_{\mathcal{D}} M[V/x] : C$

for some v and $\mathcal{D}$.

Proof. By induction on the typing derivation of $\Gamma \vdash_{\mathcal{D}_1} \lambda x.M : T \to C$.

Case (HT_Abs): We are given

$$\frac{\Gamma, x : T \vdash_{\mathcal{D}'_1} M : C}{\Gamma \vdash_{\mathcal{D}_1} \lambda x.M : T \to C}$$

for some $\mathcal{D}'_1$. Let $\overline{h}$ be a sequence of fresh variables such that $|\overline{h}| = |C.\Sigma|$, and k be a fresh variable. Then, the conclusion is proven as follows:

$$\begin{array}{cl}
 & \llbracket \lambda x.M \rrbracket_{\mathcal{D}_1}\, \llbracket V \rrbracket_{\mathcal{D}_2}\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \\
= & (\lambda x, \overline{h}, k.\llbracket M \rrbracket^{\mathsf{e}}_{\mathcal{D}'_1} @(\overline{h}, k))\, \llbracket V \rrbracket_{\mathcal{D}_2}\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \\
\longrightarrow^{+} & (\llbracket M \rrbracket^{\mathsf{e}}_{\mathcal{D}'_1} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}))[\llbracket V \rrbracket_{\mathcal{D}_2}/x] \\
= & (\llbracket M[V/x] \rrbracket^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})) \quad \text{(by Lemmas 31 and 32)}
\end{array}$$

for some $\mathcal{D}$ such that $\Gamma \vdash_{\mathcal{D}} M[V/x] : C$.

Case (HT_SUBV): We are given

$$\frac{\Gamma \vdash_{\mathcal{D}'_1} \lambda x.M : T_0 \qquad T_0 <: T \rightarrow C}{\Gamma \vdash_{\mathcal{D}_1} \lambda x.M : T \rightarrow C}$$

for some T_0 and $\mathcal{D}'_1$. By Lemma 3,

- $T_0 = T' \rightarrow C'$,
- $T <: T'$,
- $C' <: C$ (that is, $C.\Sigma <: C'.\Sigma$ and $C'.T <: C.T$ and $C'.A <: C.A$)

for some T' and C'. Let y_2, z_1, and z_2 be fresh variables. Then,

$$\begin{array}{cl}
 & \llbracket \lambda x.M \rrbracket_{\mathcal{D}_1} \, \llbracket V \rrbracket_{\mathcal{D}_2} \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} \\
= & \llbracket T_0 <: T \rightarrow C \rrbracket @ (\llbracket \lambda x.M \rrbracket_{\mathcal{D}'_1}) \, \llbracket V \rrbracket_{\mathcal{D}_2} \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} \\
= & (\lambda y_2.\mathsf{let}\, z_1 = \llbracket \lambda x.M \rrbracket_{\mathcal{D}'_1} \, (\llbracket T <: T' \rrbracket @ (y_2)) \,\mathsf{in}\,\mathsf{return}\, \llbracket C' <: C \rrbracket @ (z_1)) \, \llbracket V \rrbracket_{\mathcal{D}_2} \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} \\
\longrightarrow & (\mathsf{let}\, z_1 = \llbracket \lambda x.M \rrbracket_{\mathcal{D}'_1} \, (\llbracket T <: T' \rrbracket @ (\llbracket V \rrbracket_{\mathcal{D}_2})) \,\mathsf{in}\,\mathsf{return}\, \llbracket C' <: C \rrbracket @ (z_1)) \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} \\
 & \quad \text{(by (E_BETA) and (E_LETE))} \\
= & (\mathsf{let}\, z_1 = \llbracket \lambda x.M \rrbracket_{\mathcal{D}'_1} \, \llbracket V \rrbracket_{\mathcal{D}'_2} \,\mathsf{in}\,\mathsf{return}\, \llbracket C' <: C \rrbracket @ (z_1)) \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} \\
= & \mathsf{let}\, z_2 = (\mathsf{let}\, z_1 = \llbracket \lambda x.M \rrbracket_{\mathcal{D}'_1} \, \llbracket V \rrbracket_{\mathcal{D}'_2} \,\mathsf{in}\,\mathsf{return}\, \llbracket C' <: C \rrbracket @ (z_1)) \,\mathsf{in}\, z_2 \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} \\
\longrightarrow^{+} & \mathsf{let}\, z_2 = (\mathsf{let}\, z_1 = \mathsf{return}\, v' \,\mathsf{in}\,\mathsf{return}\, \llbracket C' <: C \rrbracket @ (z_1)) \,\mathsf{in}\, z_2 \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} \\
 & \quad \text{(for some } v' \text{ by the IH, and (E_LETE))} \\
\longrightarrow & \mathsf{let}\, z_2 = \mathsf{return}\, \llbracket C' <: C \rrbracket @ (v') \,\mathsf{in}\, z_2 \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} \\
 & \quad \text{(by (E_LETE)/(E_LETV))}
\end{array}$$

for some $\mathcal{D}'_2$ such that

$$\frac{\Gamma \vdash_{\mathcal{D}_2} V : T \quad T <: T'}{\Gamma \vdash_{\mathcal{D}'_2} V : T'} \text{ (HT\_SUBV)}.$$

We proceed by case analysis on the subtyping rule applied last to derive $C'.A <: C.A$.

Case (HS_ANSBOX): We are given $C'.A = C.A = \square$. Let k_2 and x_1 be fresh variables, and $\overline{h_2}$ be a sequence of fresh variables such that $|\overline{h_2}| = |C.\Sigma|$. Then, we derive the conclusion as follows:

$$\begin{array}{cl}
 & \llbracket \lambda x.M \rrbracket_{\mathcal{D}_1} \, \llbracket V \rrbracket_{\mathcal{D}_2} \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} \\
\longrightarrow^{+} & \mathsf{let}\, z_2 = \mathsf{return}\, \llbracket C' <: C \rrbracket @ (v') \,\mathsf{in}\, z_2 \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} \\
 & \quad \text{(by the above reasoning)} \\
= & \mathsf{let}\, z_2 = \mathsf{return}\, \lambda \overline{h_2}, k_2.\mathsf{let}\, x_1 = v' \, \llbracket C.\Sigma <: C'.\Sigma \rrbracket @ (\overline{h_2}) \, k_2 \,\mathsf{in}\,\mathsf{return}\, \llbracket C'.T <: C.T \rrbracket @ (x_1) \,\mathsf{in}\, z_2 \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} \\
\longrightarrow^{+} & \mathsf{let}\, x_1 = v' \, \llbracket C.\Sigma <: C'.\Sigma \rrbracket @ (\overline{v^{\mathsf{h}}}) \, v^{\mathsf{k}} \,\mathsf{in}\,\mathsf{return}\, \llbracket C'.T <: C.T \rrbracket @ (x_1) \\
 & \quad \text{(by (E_LETV), (E_BETA), and (E_LETE))} \\
\longrightarrow^{+} & \mathsf{let}\, x_1 = \llbracket M[V/x] \rrbracket^{\mathsf{e}}_{\mathcal{D}'} @ (\llbracket C.\Sigma <: C'.\Sigma \rrbracket @ (\overline{v^{\mathsf{h}}}), v^{\mathsf{k}}) \,\mathsf{in}\,\mathsf{return}\, \llbracket C'.T <: C.T \rrbracket @ (x_1) \\
 & \quad \text{(by (E_LETE), Lemma 28, and the assumption on } v' \text{ by the IH)} \\
= & \llbracket C' <: C \rrbracket^{\mathsf{e}} @ (\llbracket M[V/x] \rrbracket^{\mathsf{e}}_{\mathcal{D}'}, \overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & \llbracket M[V/x] \rrbracket^{\mathsf{e}}_{\mathcal{D}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})
\end{array}$$

for some $\mathcal{D}'$ and $\mathcal{D}$ such that

$$\frac{\Gamma \vdash_{\mathcal{D}'} M[V/x] : C' \quad C' <: C}{\Gamma \vdash_{\mathcal{D}} M[V/x] : C} \text{ (HT\_SUBC)}.$$

Case (HS_ANSEMB): We are given $C'.A = \square$ and $C.A = C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$ for some C^{ini} and C^{fin}. Let $\overline{h_2^{\not\square}}$ and $\overline{h_2^{\square}}$ be sequences of fresh variables such that $|\overline{h_2^{\not\square}}| = |\not\square(C.\Sigma)|$ and $|\overline{h_2^{\square}}| = |\square(C.\Sigma)|$. Furthermore, let $(\overline{v^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}) =$

$\mathit{split}(\overline{v^{\mathsf{h}}}, C.\Sigma)$ and k_2, x_1, and y_2 be fresh variables. Then, the conclusion is proven as follows:

$$
\begin{array}{ll}
 & \llbracket \lambda x.M \rrbracket_{\mathcal{D}_1} \, \llbracket V \rrbracket_{\mathcal{D}_2} \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} \\
\longrightarrow^{+} & \mathsf{let}\, z_2 = \mathsf{return}\, \llbracket C' <: C \rrbracket @ (v') \,\mathsf{in}\, z_2 \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} \\
 & \quad \text{(by the above reasoning)} \\
= & \mathsf{let}\, z_2 = \mathsf{return}\, (\lambda \overline{h_2^{\not\square}}, \overline{h_2^{\square}}, k_2 . \mathsf{let}\, x_1 = v' \, \llbracket C.\Sigma <: C'.\Sigma \rrbracket @ (\overline{h_2^{\not\square}}, \overline{h_2^{\square}}) \, \underline{1} \,\mathsf{in}\, e) \,\mathsf{in}\, z_2 \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} \\
 & \quad \text{(where } e = (\mathsf{let}\, y_2 = k_2 \, (\llbracket C'.T <: C.T \rrbracket @ (x_1)) \, \overline{h_2^{\square}} \,\mathsf{in}\, \mathsf{return}\, \llbracket C^{\mathrm{ini}} <: C^{\mathrm{fin}} \rrbracket @ (y_2))) \\
\longrightarrow^{+} & \mathsf{let}\, x_1 = v' \, \llbracket C.\Sigma <: C'.\Sigma \rrbracket @ (\overline{v^{\mathsf{h}}}) \, \underline{1} \,\mathsf{in}\, \mathsf{let}\, y_2 = v^{\mathsf{k}} \, (\llbracket C'.T <: C.T \rrbracket @ (x_1)) \, \overline{v^{\mathsf{h}\square}} \,\mathsf{in}\, \mathsf{return}\, \llbracket C^{\mathrm{ini}} <: C^{\mathrm{fin}} \rrbracket @ (y_2) \\
 & \quad \text{(by (E\_LetV), (E\_Beta), and (E\_LetE))} \\
\longrightarrow^{+} & \mathsf{let}\, x_1 = \llbracket M[V/x] \rrbracket^{\mathsf{e}}_{\mathcal{D}'} @ (\llbracket C.\Sigma <: C'.\Sigma \rrbracket @ (\overline{v^{\mathsf{h}}}), \underline{1}) \,\mathsf{in}\, \mathsf{let}\, y_2 = v^{\mathsf{k}} \, (\llbracket C'.T <: C.T \rrbracket @ (x_1)) \, \overline{v^{\mathsf{h}\square}} \,\mathsf{in}\, \mathsf{return}\, \llbracket C^{\mathrm{ini}} <: C^{\mathrm{fin}} \rrbracket @ (y_2) \\
 & \quad \text{(by (E\_LetE), Lemma 28, and the assumption on } v' \text{ by the IH))} \\
= & \llbracket C' <: C \rrbracket^{\mathsf{e}} @ (\llbracket M[V/x] \rrbracket^{\mathsf{e}}_{\mathcal{D}'}, \overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & \llbracket M[V/x] \rrbracket^{\mathsf{e}}_{\mathcal{D}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})
\end{array}
$$

for some $\mathcal{D}'$ and $\mathcal{D}$ such that

$$
\frac{\Gamma \vdash_{\mathcal{D}'} M[V/x] : C' \qquad C' <: C}{\Gamma \vdash_{\mathcal{D}} M[V/x] : C} \text{ (HT\_SubC).}
$$

Case (HS_AnsMod): We are given $C'.A = C'^{\mathrm{ini}} \Rightarrow C'^{\mathrm{fin}}$ and $C.A = C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$ for some C'^{ini}, C'^{fin}, C^{ini}, and C^{fin}.

Assume that $C'.\Sigma = C.\Sigma$ and $C'.T = C.T$ and $C'^{\mathrm{ini}} = C^{\mathrm{ini}}$. Let $\overline{h_2}$ be a sequence of fresh variables such that $|\overline{h_2}| = |C.\Sigma|$. Then, the conclusion is proven as follows:

$$
\begin{array}{ll}
 & \llbracket \lambda x.M \rrbracket_{\mathcal{D}_1} \, \llbracket V \rrbracket_{\mathcal{D}_2} \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} \\
\longrightarrow^{+} & \mathsf{let}\, z_2 = \mathsf{return}\, \llbracket C' <: C \rrbracket @ (v') \,\mathsf{in}\, z_2 \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} \\
 & \quad \text{(by the above reasoning)} \\
= & \mathsf{let}\, z_2 = \mathsf{return}\, (\lambda \overline{h_2}, k_2 . \mathsf{let}\, x_1 = v' \, \overline{h_2} \, k_2 \,\mathsf{in}\, \mathsf{return}\, \llbracket C'^{\mathrm{fin}} <: C^{\mathrm{fin}} \rrbracket @ (x_1)) \,\mathsf{in}\, z_2 \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} \\
\longrightarrow^{+} & \mathsf{let}\, x_1 = v' \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} \,\mathsf{in}\, \mathsf{return}\, \llbracket C'^{\mathrm{fin}} <: C^{\mathrm{fin}} \rrbracket @ (x_1) \\
 & \quad \text{(by (E\_LetV), (E\_Beta), and (E\_LetE))} \\
\longrightarrow^{+} & \mathsf{let}\, x_1 = \llbracket M[V/x] \rrbracket^{\mathsf{e}}_{\mathcal{D}'} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \,\mathsf{in}\, \mathsf{return}\, \llbracket C'^{\mathrm{fin}} <: C^{\mathrm{fin}} \rrbracket @ (x_1) \\
 & \quad \text{(by (E\_LetE) and the assumption on } v' \text{ by the IH)} \\
= & \llbracket C' <: C \rrbracket^{\mathsf{e}} @ (\llbracket M[V/x] \rrbracket^{\mathsf{e}}_{\mathcal{D}'}, \overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & \llbracket M[V/x] \rrbracket^{\mathsf{e}}_{\mathcal{D}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})
\end{array}
$$

for some $\mathcal{D}'$ and $\mathcal{D}$ such that

$$
\frac{\Gamma \vdash_{\mathcal{D}'} M[V/x] : C' \qquad C' \Rightarrow C}{\Gamma \vdash_{\mathcal{D}} M[V/x] : C} \text{ (HT\_SubC).}
$$

Otherwise, assume that $C'.\Sigma \neq C.\Sigma$ or $C'.T \neq C.T$ or $C'^{\mathrm{ini}} \neq C^{\mathrm{ini}}$. By Lemmas 3 and 20, $\square(C.\Sigma) = \square(C'.\Sigma)$. Let $\overline{h_2^{\not\square}}$, $\overline{h_2^{\square}}$, and $\overline{h_1^{\square}}$ be sequences of fresh variables such that $|\overline{h_2^{\not\square}}| = |\not\square(C.\Sigma)|$ and $|\overline{h_2^{\square}}| = |\overline{h_1^{\square}}| = |\square(C.\Sigma)|$. Furthermore, let $(\overline{v^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}) = \mathit{split}(\overline{v^{\mathsf{h}}}, C.\Sigma)$ and k_2, x_1, y_1, and z_2' be fresh variables. Then, the conclusion is proven as follows:

$$
\begin{array}{ll}
 & \llbracket \lambda x.M \rrbracket_{\mathcal{D}_1} \, \llbracket V \rrbracket_{\mathcal{D}_2} \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} \\
\longrightarrow^{+} & \mathsf{let}\, z_2 = \mathsf{return}\, \llbracket C' <: C \rrbracket @ (v') \,\mathsf{in}\, z_2 \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} \\
 & \quad \text{(by the above reasoning)} \\
= & \mathsf{let}\, z_2 = \mathsf{return}\, (\lambda \overline{h_2^{\not\square}}, \overline{h_2^{\square}}, k_2 . \mathsf{let}\, x_1 = v' \, \llbracket \not\square(C.\Sigma) <: \not\square(C'.\Sigma) \rrbracket @ (\overline{h_2^{\not\square}}) \, \overline{h_2^{\square}} \, v_1^{\mathsf{k}} \,\mathsf{in}\, \mathsf{return}\, \llbracket C'^{\mathrm{fin}} <: C^{\mathrm{fin}} \rrbracket @ (x_1)) \,\mathsf{in}\, z_2 \, \overline{v^{\mathsf{h}}} \, v^{\mathsf{k}} \\
 & \quad \text{(where } v_1^{\mathsf{k}} = \lambda y_1, \overline{h_1^{\square}} . \mathsf{let}\, z_2' = k_2 \, (\llbracket C'.T <: C.T \rrbracket @ (y_1)) \, \overline{h_1^{\square}} \,\mathsf{in}\, \mathsf{return}\, \llbracket C^{\mathrm{ini}} <: C'^{\mathrm{ini}} \rrbracket @ (z_2')) \\
\longrightarrow^{+} & \mathsf{let}\, x_1 = v' \, \llbracket \not\square(C.\Sigma) <: \not\square(C'.\Sigma) \rrbracket @ (\overline{v^{\mathsf{h}\not\square}}) \, \overline{v^{\mathsf{h}\square}} \, v_1^{\mathsf{k}}[v^{\mathsf{k}}/k_2] \,\mathsf{in}\, \mathsf{return}\, \llbracket C'^{\mathrm{fin}} <: C^{\mathrm{fin}} \rrbracket @ (x_1) \\
 & \quad \text{(by (E\_LetV), (E\_Beta), and (E\_LetE))} \\
\longrightarrow^{+} & \mathsf{let}\, x_1 = \llbracket M[V/x] \rrbracket^{\mathsf{e}}_{\mathcal{D}'} @ (\llbracket \not\square(C.\Sigma) <: \not\square(C'.\Sigma) \rrbracket @ (\overline{v^{\mathsf{h}\not\square}}), \overline{v^{\mathsf{h}\square}}, v_1^{\mathsf{k}}[v^{\mathsf{k}}/k_2]) \,\mathsf{in}\, \mathsf{return}\, \llbracket C'^{\mathrm{fin}} <: C^{\mathrm{fin}} \rrbracket @ (x_1) \\
 & \quad \text{(by (E\_LetE), Lemma 28, and the assumption on } v' \text{ by the IH)} \\
= & \llbracket C' <: C \rrbracket^{\mathsf{e}} @ (\llbracket M[V/x] \rrbracket^{\mathsf{e}}_{\mathcal{D}'}, \overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & \llbracket M[V/x] \rrbracket^{\mathsf{e}}_{\mathcal{D}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})
\end{array}
$$

for some $\mathcal{D}'$ and $\mathcal{D}$ such that

$$\frac{\Gamma \vdash_{\mathcal{D}'} M[V/x] : C' \qquad C' \Rightarrow C}{\Gamma \vdash_{\mathcal{D}} M[V/x] : C} \text{(HT\_SubC)}.$$

Otherwise: Contradictory.

$\square$

Lemma 34 (CPS Transformation of Recursive Function Application)**.** If $\Gamma \vdash_{\mathcal{D}_1} \mathsf{fix}\, x. V_1 : T \to C$ and $\Gamma \vdash_{\mathcal{D}_2} V_2 : T$, then, for any $\overline{v^{\mathsf{h}}}$ and v^{k} such that $|\overline{v^{\mathsf{h}}}| = |C.\Sigma|$,

- $[\![\mathsf{fix}\, x. V_1]\!]_{\mathcal{D}_1} [\![V]\!]_{\mathcal{D}_2} \longrightarrow^{+} \mathsf{return}\, v$,
- $v\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \longrightarrow^{+} [\![V_1[\mathsf{fix}\, x. V_1/x]\ V_2]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})$, and
- $\Gamma \vdash_{\mathcal{D}} V_1[\mathsf{fix}\, x. V_1/x]\ V_2 : C$

for some $\mathcal{D}$.

Proof.
By induction on the typing derivation of $\Gamma \vdash_{\mathcal{D}_1} \mathsf{fix}\, x. V_1 : T \to C$.

Case (HT_Fix): We are given

$$\frac{\Gamma, x : T \to C \vdash_{\mathcal{D}'_1} V_1 : T \to C}{\Gamma \vdash_{\mathcal{D}_1} \mathsf{fix}\, x. V_1 : T \to C}$$

for some $\mathcal{D}'_1$. Then, the conclusion is proven as follows:

$$\begin{array}{ll}
 & [\![\mathsf{fix}\, x. V_1]\!]_{\mathcal{D}_1} [\![V_2]\!]_{\mathcal{D}_2}\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \\
= & \mathsf{fix}\, x. [\![V_1]\!]_{\mathcal{D}'_1} [\![V_2]\!]_{\mathcal{D}_2}\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \\
\longrightarrow & [\![V_1]\!]_{\mathcal{D}'_1} [\mathsf{fix}\, x. [\![V_1]\!]_{\mathcal{D}'_1}/x]\, [\![V_2]\!]_{\mathcal{D}_2}\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \quad \text{by (E_Fix) and (E_LetE)} \\
= & [\![V_1]\!]_{\mathcal{D}'_1} [[\![\mathsf{fix}\, x. V_1]\!]_{\mathcal{D}_1}/x]\, [\![V_2]\!]_{\mathcal{D}_2}\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \\
= & [\![V_1[\mathsf{fix}\, x. V_1/x]]\!]_{\mathcal{D}''_1} [\![V_2]\!]_{\mathcal{D}_2}\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \quad \text{(by Lemmas 31 and 32)} \\
= & [\![V_1[\mathsf{fix}\, x. V_1/x]\ V_2]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})
\end{array}$$

for some $\mathcal{D}''_1$ and $\mathcal{D}$ such that $\Gamma \vdash_{\mathcal{D}''_1} V_1[\mathsf{fix}\, x. V_1/x] : T \to C$ and

$$\frac{\Gamma \vdash_{\mathcal{D}''_1} V_1[\mathsf{fix}\, x. V_1/x] : T \to C \qquad \Gamma \vdash_{\mathcal{D}_2} V_2 : T}{\Gamma \vdash_{\mathcal{D}} V_1[\mathsf{fix}\, x. V_1/x]\ V_2 : C} \text{(HT\_App)}.$$

Case (HT_SubV): We are given

$$\frac{\Gamma \vdash_{\mathcal{D}'_1} \mathsf{fix}\, x. V_1 : T_0 \qquad T_0 <: T \to C}{\Gamma \vdash_{\mathcal{D}_1} \mathsf{fix}\, x. V_1 : T \to C}$$

for some T_0 and $\mathcal{D}'_1$. By Lemma 3,

- $T_0 = T' \to C'$,
- $T <: T'$,
- $C' <: C$ (that is, $C.\Sigma <: C'.\Sigma$ and $C'.T <: C.T$ and $C'.A <: C.A$)

for some T' and C'. Let y_2, z_1, and z_2 be fresh variables. Then,

$$\begin{array}{rl}
 & \llbracket \mathsf{fix}\, x.\, V_1 \rrbracket_{\mathcal{D}_1}\, \llbracket V \rrbracket_{\mathcal{D}_2}\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \\
= & \llbracket T_0 <: T \to C \rrbracket @ (\llbracket \mathsf{fix}\, x.\, V_1 \rrbracket_{\mathcal{D}'_1})\, \llbracket V \rrbracket_{\mathcal{D}_2}\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \\
= & (\lambda y_2.\mathsf{let}\, z_1 = \llbracket \mathsf{fix}\, x.\, V_1 \rrbracket_{\mathcal{D}'_1}\, (\llbracket T <: T' \rrbracket @ (y_2))\, \mathsf{in}\, \mathsf{return}\, \llbracket C' <: C \rrbracket @ (z_1))\, \llbracket V \rrbracket_{\mathcal{D}_2}\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \\
\longrightarrow & (\mathsf{let}\, z_1 = \llbracket \mathsf{fix}\, x.\, V_1 \rrbracket_{\mathcal{D}'_1}\, (\llbracket T <: T' \rrbracket @ (\llbracket V \rrbracket_{\mathcal{D}_2}))\, \mathsf{in}\, \mathsf{return}\, \llbracket C' <: C \rrbracket @ (z_1))\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \\
 & \quad \text{(by (E_Beta) and (E_LetE))} \\
= & (\mathsf{let}\, z_1 = \llbracket \mathsf{fix}\, x.\, V_1 \rrbracket_{\mathcal{D}'_1}\, \llbracket V \rrbracket_{\mathcal{D}'_2}\, \mathsf{in}\, \mathsf{return}\, \llbracket C' <: C \rrbracket @ (z_1))\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \\
= & \mathsf{let}\, z_2 = (\mathsf{let}\, z_1 = \llbracket \mathsf{fix}\, x.\, V_1 \rrbracket_{\mathcal{D}'_1}\, \llbracket V \rrbracket_{\mathcal{D}'_2}\, \mathsf{in}\, \mathsf{return}\, \llbracket C' <: C \rrbracket @ (z_1))\, \mathsf{in}\, z_2\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \\
\longrightarrow^{+} & \mathsf{let}\, z_2 = (\mathsf{let}\, z_1 = \mathsf{return}\, v'\, \mathsf{in}\, \mathsf{return}\, \llbracket C' <: C \rrbracket @ (z_1))\, \mathsf{in}\, z_2\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \\
 & \quad \text{(for some } v' \text{ by the IH, and (E_LetE))} \\
\longrightarrow & \mathsf{let}\, z_2 = \mathsf{return}\, \llbracket C' <: C \rrbracket @ (v')\, \mathsf{in}\, z_2\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \\
 & \quad \text{(by (E_LetE)/(E_LetV))}
\end{array}$$

for some $\mathcal{D}'_2$ such that

$$\frac{\Gamma \vdash_{\mathcal{D}_2} V : T \quad T <: T'}{\Gamma \vdash_{\mathcal{D}'_2} V : T'}\ \text{(HT\_SubV)}.$$

We proceed by case analysis on the subtyping rule applied last to derive $C'.A <: C.A$.

Case (HS_AnsBox): We are given $C'.A = C.A = \Box$. Let k_2 and x_1 be fresh variables, and $\overline{h_2}$ be a sequence of fresh variables such that $|\overline{h_2}| = |C.\Sigma|$. Then, we derive the conclusion as follows:

$$\begin{array}{rl}
 & \llbracket \mathsf{fix}\, x.\, V_1 \rrbracket_{\mathcal{D}_1}\, \llbracket V \rrbracket_{\mathcal{D}_2}\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \\
= & \mathsf{let}\, z_2 = \mathsf{return}\, \llbracket C' <: C \rrbracket @ (v')\, \mathsf{in}\, z_2\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \\
 & \quad \text{(by the above reasoning)} \\
= & \mathsf{let}\, z_2 = \mathsf{return}\, \lambda \overline{h_2}, k_2.\mathsf{let}\, x_1 = v'\, \llbracket C.\Sigma <: C'.\Sigma \rrbracket @ (\overline{h_2})\, k_2\, \mathsf{in}\, \llbracket C'.T <: C.T \rrbracket @ (x_1)\, \mathsf{in}\, z_2\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \\
\longrightarrow^{+} & \mathsf{let}\, x_1 = v'\, \llbracket C.\Sigma <: C'.\Sigma \rrbracket @ (\overline{v^{\mathsf{h}}})\, v^{\mathsf{k}}\, \mathsf{in}\, \llbracket C'.T <: C.T \rrbracket @ (x_1) \\
 & \quad \text{(by (E_LetV), (E_Beta), and (E_LetE))} \\
\longrightarrow^{+} & \mathsf{let}\, x_1 = \llbracket V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2 \rrbracket^{\mathsf{e}}_{\mathcal{D}'} @ (\llbracket C.\Sigma <: C'.\Sigma \rrbracket @ (\overline{v^{\mathsf{h}}}), v^{\mathsf{k}})\, \mathsf{in}\, \llbracket C'.T <: C.T \rrbracket @ (x_1) \\
 & \quad \text{(by (E_LetE), Lemma 28, and the assumption on } v' \text{ by the IH)} \\
= & \llbracket C' <: C \rrbracket^{\mathsf{e}} @ (\llbracket V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2 \rrbracket^{\mathsf{e}}_{\mathcal{D}'}, \overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & \llbracket V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2 \rrbracket^{\mathsf{e}}_{\mathcal{D}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})
\end{array}$$

for some $\mathcal{D}'$ and $\mathcal{D}$ such that

$$\frac{\Gamma \vdash_{\mathcal{D}'} V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2 : C' \quad C' <: C}{\Gamma \vdash_{\mathcal{D}} V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2 : C}\ \text{(HT\_SubC)}.$$

Case (HS_AnsEmb): We are given $C'.A = \Box$ and $C.A = C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$ for some C^{ini} and C^{fin}. Let $\overline{h_2^{\not\Box}}$ and $\overline{h_2^{\Box}}$ be sequences of fresh variables such that $|\overline{h_2^{\not\Box}}| = |\not\Box(\Sigma)|$ and $|\overline{h_2^{\Box}}| = |\Box(\Sigma)|$. Furthermore, let $(\overline{v^{\mathsf{h}\not\Box}}, \overline{v^{\mathsf{h}\Box}}) = \mathit{split}(\overline{v^{\mathsf{h}}}, \Sigma)$ and k_2, x_1, y_2 be fresh variables. Then, the conclusion is proven as follows:

$$\begin{array}{rl}
 & \llbracket \mathsf{fix}\, x.\, V_1 \rrbracket_{\mathcal{D}_1}\, \llbracket V \rrbracket_{\mathcal{D}_2}\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \\
\longrightarrow^{+} & \mathsf{let}\, z_2 = \mathsf{return}\, \llbracket C' <: C \rrbracket @ (v')\, \mathsf{in}\, z_2\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \\
 & \quad \text{(by the above reasoning)} \\
= & \mathsf{let}\, z_2 = \mathsf{return}\, (\lambda \overline{h_2^{\not\Box}}, \overline{h_2^{\Box}}, k_2.\mathsf{let}\, x_1 = v'\, \llbracket C.\Sigma <: C'.\Sigma \rrbracket @ (\overline{h_2^{\not\Box}}, \overline{h_2^{\Box}})\, \underline{1}\, \mathsf{in}\, e)\, \mathsf{in}\, z_2\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \\
 & \quad \text{(where } e = (\mathsf{let}\, y_2 = k_2\, (\llbracket C'.T <: C.T \rrbracket @ (x_1))\, \overline{h_2^{\Box}}\, \mathsf{in}\, \mathsf{return}\, \llbracket C^{\mathrm{ini}} <: C^{\mathrm{fin}} \rrbracket @ (y_2))) \\
\longrightarrow^{+} & \mathsf{let}\, x_1 = v'\, \llbracket C.\Sigma <: C'.\Sigma \rrbracket @ (\overline{v^{\mathsf{h}}})\, \underline{1}\, \mathsf{in}\, \mathsf{let}\, y_2 = v^{\mathsf{k}}\, (\llbracket C'.T <: C.T \rrbracket @ (x_1))\, \overline{v^{\mathsf{h}\Box}}\, \mathsf{in}\, \mathsf{return}\, \llbracket C^{\mathrm{ini}} <: C^{\mathrm{fin}} \rrbracket @ (y_2) \\
 & \quad \text{(by (E_LetV), (E_Beta), and (E_LetE))} \\
\longrightarrow^{+} & \mathsf{let}\, x_1 = \llbracket V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2 \rrbracket^{\mathsf{e}}_{\mathcal{D}'} @ (\llbracket C.\Sigma <: C'.\Sigma \rrbracket @ (\overline{v^{\mathsf{h}}}), \underline{1})\, \mathsf{in}\, \mathsf{let}\, y_2 = v^{\mathsf{k}}\, (\llbracket C'.T <: C.T \rrbracket @ (x_1))\, \overline{v^{\mathsf{h}\Box}}\, \mathsf{in}\, \mathsf{return}\, \llbracket C^{\mathrm{ini}} <: C^{\mathrm{fin}} \rrbracket @ (y_2) \\
 & \quad \text{(by (E_LetE), Lemma 28, and the assumption on } v' \text{ by the IH)} \\
= & \llbracket C' <: C \rrbracket^{\mathsf{e}} @ (\llbracket V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2 \rrbracket^{\mathsf{e}}_{\mathcal{D}'}, \overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & \llbracket V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2 \rrbracket^{\mathsf{e}}_{\mathcal{D}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})
\end{array}$$

for some $\mathcal{D}'$ and $\mathcal{D}$ such that

$$\dfrac{\Gamma \vdash_{\mathcal{D}'} V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2 : C' \qquad C' <: C}{\Gamma \vdash_{\mathcal{D}} V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2 : C}\ (\text{HT\_SubC}).$$

Case (HS_AnsMod): We are given $C'.A = C'^{\mathrm{ini}} \Rightarrow C'^{\mathrm{fin}}$ and $C.A = C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$ for some C'^{ini}, C'^{fin}, C^{ini}, and C^{fin}.

Assume that $C'.\Sigma = C.\Sigma$ and $C'.T = C.T$ and $C'^{\mathrm{ini}} = C^{\mathrm{ini}}$. Let $\overline{h_2}$ be a sequence of fresh variables such that $|\overline{h_2}| = |C.\Sigma|$. Then, the conclusion is proven as follows:

$$\begin{array}{rl}
& [\![\mathsf{fix}\, x.\, V_1]\!]_{\mathcal{D}_1}\ [\![V]\!]_{\mathcal{D}_2}\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} \\
\longrightarrow^{+} & \mathsf{let}\ z_2 = \mathsf{return}\ [\![C' <: C]\!]@(v')\ \mathsf{in}\ z_2\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} \\
& \quad \text{(by the above reasoning)} \\
= & \mathsf{let}\ z_2 = \mathsf{return}\ (\lambda \overline{h_2}, k_2.\mathsf{let}\ x_1 = v'\ \overline{h_2}\ k_2\ \mathsf{in}\ \mathsf{return}\ [\![C'^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!]@(x_1))\ \mathsf{in}\ z_2\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} \\
\longrightarrow^{+} & \mathsf{let}\ x_1 = v'\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}}\ \mathsf{in}\ \mathsf{return}\ [\![C'^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!]@(x_1) \\
& \quad \text{(by (E_LetV), (E_Beta), and (E_LetE))} \\
\longrightarrow^{+} & \mathsf{let}\ x_1 = [\![V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2]\!]^{\mathsf{e}}_{\mathcal{D}'}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})\ \mathsf{in}\ \mathsf{return}\ [\![C'^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!]@(x_1) \\
& \quad \text{(by (E_LetE) and the assumption on } v' \text{ by the IH)} \\
= & [\![C' <: C]\!]^{\mathsf{e}}@([\![V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2]\!]^{\mathsf{e}}_{\mathcal{D}'}, \overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & [\![V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})
\end{array}$$

for some $\mathcal{D}'$ and $\mathcal{D}$ such that

$$\dfrac{\Gamma \vdash_{\mathcal{D}'} V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2 : C' \qquad C' \Rightarrow C}{\Gamma \vdash_{\mathcal{D}} V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2 : C}\ (\text{HT\_SubC}).$$

Otherwise, assume that $C'.\Sigma \neq C.\Sigma$ or $C'.T \neq C.T$ or $C'^{\mathrm{ini}} \neq C^{\mathrm{ini}}$. By Lemmas 3 and 20, $\Box(C.\Sigma) = \Box(C'.\Sigma)$. Let $\overline{h_2^{\not\Box}}$, $\overline{h_2^{\Box}}$, and $\overline{h_1^{\Box}}$ be sequences of fresh variables such that $|\overline{h_2^{\not\Box}}| = |\not\Box(C.\Sigma)|$ and $|\overline{h_2^{\Box}}| = |\overline{h_1^{\Box}}| = |\Box(C.\Sigma)|$. Furthermore, let $(\overline{v^{\mathsf{h}\not\Box}}, \overline{v^{\mathsf{h}\Box}}) = \mathit{split}(\overline{v^{\mathsf{h}}}, C.\Sigma)$ and k_2, x_1, y_1, and z_2' are fresh variables. Then, the conclusion is proven as follows:

$$\begin{array}{rl}
& [\![\mathsf{fix}\, x.\, V_1]\!]_{\mathcal{D}_1}\ [\![V]\!]_{\mathcal{D}_2}\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} \\
\longrightarrow^{+} & \mathsf{let}\ z_2 = \mathsf{return}\ [\![C' <: C]\!]@(v')\ \mathsf{in}\ z_2\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} \\
& \quad \text{(by the above reasoning)} \\
= & \mathsf{let}\ z_2 = \mathsf{return}\ (\lambda \overline{h_2^{\not\Box}}, \overline{h_2^{\Box}}, k_2.\mathsf{let}\ x_1 = v'\ [\![\not\Box(C.\Sigma) <: \not\Box(C'.\Sigma)]\!]@(\overline{h_2^{\not\Box}})\ \overline{h_2^{\Box}}\ v_1^{\mathsf{k}}\ \mathsf{in}\ \mathsf{return}\ [\![C'^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!]@(x_1))\ \mathsf{in}\ z_2\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} \\
& \quad \text{(where } v_1^{\mathsf{k}} = \lambda y_1, \overline{h_1^{\Box}}.\mathsf{let}\ z_2' = k_2\ ([\![C'.T <: C.T]\!]@(y_1))\ \overline{h_1^{\Box}}\ \mathsf{in}\ \mathsf{return}\ [\![C^{\mathrm{ini}} <: C'^{\mathrm{ini}}]\!]@(z_2')) \\
\longrightarrow^{+} & \mathsf{let}\ x_1 = v'\ [\![\not\Box(C.\Sigma) <: \not\Box(C'.\Sigma)]\!]@(\overline{v^{\mathsf{h}\not\Box}})\ \overline{v^{\mathsf{h}\Box}}\ v_1^{\mathsf{k}}[v^{\mathsf{k}}/k_2]\ \mathsf{in}\ \mathsf{return}\ [\![C'^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!]@(x_1) \\
& \quad \text{(by (E_LetV), (E_Beta), and (E_LetE))} \\
\longrightarrow^{+} & \mathsf{let}\ x_1 = [\![V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2]\!]^{\mathsf{e}}_{\mathcal{D}'}@([\![\not\Box(C.\Sigma) <: \not\Box(C'.\Sigma)]\!]@(\overline{v^{\mathsf{h}\not\Box}}), \overline{v^{\mathsf{h}\Box}}, v_1^{\mathsf{k}}[v^{\mathsf{k}}/k_2])\ \mathsf{in}\ \mathsf{return}\ [\![C'^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!]@(x_1) \\
& \quad \text{(by (E_LetE), Lemma 28, and the assumption on } v' \text{ by the IH)} \\
= & [\![C' <: C]\!]^{\mathsf{e}}@([\![V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2]\!]^{\mathsf{e}}_{\mathcal{D}'}, \overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & [\![V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})
\end{array}$$

for some $\mathcal{D}'$ and $\mathcal{D}$ such that

$$\dfrac{\Gamma \vdash_{\mathcal{D}'} V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2 : C' \qquad C' <: C}{\Gamma \vdash_{\mathcal{D}} V_1[\mathsf{fix}\, x.\, V_1/x]\ V_2 : C}\ (\text{HT\_SubC}).$$

Otherwise: Contradictory.

$\Box$

Lemma 35 (CPS Transformation of Case Matches)**.** If $\Gamma \vdash_{\mathcal{D}} \underline{i} : \mathsf{n}$ and $\forall j \in [1, n].\ \Gamma \vdash_{\mathcal{D}_j} M_j : C$, then, for any $\overline{v^{\mathsf{h}}}$ and v^{k} such that $|\overline{v^{\mathsf{h}}}| = |C.\Sigma|$,

$$\mathsf{case}([\![\underline{i}]\!]_{\mathcal{D}}; [\![M_1]\!]^{\mathsf{e}}_{\mathcal{D}_1}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}), \cdots, [\![M_n]\!]^{\mathsf{e}}_{\mathcal{D}_n}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})) \longrightarrow [\![M_i]\!]^{\mathsf{e}}_{\mathcal{D}_i}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})\ .$$

Proof. By (E_CASE), it suffices to show that $[\![\underline{\mathsf{i}}]\!]_{\mathcal{D}} = \underline{\mathsf{i}}$. We show it by induction on the typing derivation of $\Gamma \vdash_{\mathcal{D}} \underline{\mathsf{i}} : \mathsf{n}$.

Case (HT_ECONST): Obvious by the definition of the CPS transformation.

Case (HT_SUBV): We are given

$$\frac{\Gamma \vdash_{\mathcal{D}'} \underline{\mathsf{i}} : T \qquad T <: \mathsf{n}}{\Gamma \vdash_{\mathcal{D}} \underline{\mathsf{i}} : \mathsf{n}}$$

for some T. By Lemma 3, $T = \mathsf{n}$. The conclusion is proven as follows:

$$\begin{array}{rcll} [\![\underline{\mathsf{i}}]\!]_{\mathcal{D}} & = & [\![\mathsf{n} <: \mathsf{n}]\!] @ ([\![\underline{\mathsf{i}}]\!]_{\mathcal{D}'}) & \\ & = & [\![\mathsf{n} <: \mathsf{n}]\!] @ (\underline{\mathsf{i}}) & \text{(by the IH)} \\ & = & \underline{\mathsf{i}} \ . & \end{array}$$

Otherwise: Contradictory.

□

Lemma 36 (Rolling Up Final Answer Types)**.** If $\overline{C} = C_1, \cdots, C_n$ and $E^{\overline{C}}$ is well defined and $\Gamma \vdash_{\mathcal{D}} M : C_1$ and $|\overline{v^{\mathsf{h}}}| = |C_n.\Sigma|$, then

$$E^{\overline{C}}[\mathsf{return}\ [\![M]\!]_{\mathcal{D}}]\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} \hookrightarrow [\![M]\!]^{\mathsf{e}}_{\mathcal{D}'} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})$$

for some $\mathcal{D}'$ and $\Gamma \vdash_{\mathcal{D}'} M : C_n$.

Proof. By induction on n.

Case $n = 1$: The conclusion is proven as follows:

$$\begin{array}{rcl} E^{\overline{C}}[\mathsf{return}\ [\![M]\!]_{\mathcal{D}}]\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} & \hookrightarrow & [\![M]\!]_{\mathcal{D}}\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} \\ & \hookrightarrow & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \ . \end{array}$$

Case $n > 1$: Let $\overline{C'} = C_2, \cdots, C_n$ and x be a fresh variable. We have $C_1 <: C_2$. By the definition of the CPS transformation,

$$\begin{array}{rcl} E^{\overline{C}}[\mathsf{return}\ [\![M]\!]_{\mathcal{D}}]\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} & = & E^{\overline{C'}}[\mathsf{let}\, x = \mathsf{return}\ [\![M]\!]_{\mathcal{D}}\ \mathsf{in\ return}\ [\![C_1 <: C_2]\!] @ (x)]\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} \\ & \hookrightarrow & E^{\overline{C'}}[\mathsf{return}\ [\![C_1 <: C_2]\!] @ ([\![M]\!]_{\mathcal{D}})]\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} \ . \end{array}$$

By Lemma 3 with $C_1 <: C_2$, we have $C_2.\Sigma <: C_1.\Sigma$ and $C_1.T <: C_2.T$ and $C_1.A <: C_2.A$. By case analysis on the subtyping rule applied to derive $C_1.A <: C_2.A$.

Case (HS_ANSBOX): We are given $C_1.A = C_2.A = \square$. Let k_2 and x_1 be fresh variables and $\overline{h_2}$ be a sequence of fresh variables such that $|\overline{h_2}| = |C_2.\Sigma|$. By the definition of the CPS transformation,

$$\begin{array}{rll} & E^{\overline{C}}[\mathsf{return}\ [\![M]\!]_{\mathcal{D}}]\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} & \\ \hookrightarrow & E^{\overline{C'}}[\mathsf{return}\ [\![C_1 <: C_2]\!] @ ([\![M]\!]_{\mathcal{D}})]\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} & \text{(by the above reasoning)} \\ = & E^{\overline{C'}}[\mathsf{return}\ \lambda \overline{h_2}, k_2.\mathsf{let}\, x_1 = [\![M]\!]_{\mathcal{D}}\ [\![C_2.\Sigma <: C_1.\Sigma]\!] @ (\overline{h_2})\ k_2\ \mathsf{in\ return}\ [\![C_1.T <: C_2.T]\!] @ (x_1)]\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} & \\ \hookrightarrow & E^{\overline{C'}}[\mathsf{return}\ \lambda \overline{h_2}, k_2.\mathsf{let}\, x_1 = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @ ([\![C_2.\Sigma <: C_1.\Sigma]\!] @ (\overline{h_2}), k_2)\ \mathsf{in\ return}\ [\![C_1.T <: C_2.T]\!] @ (x_1)]\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} & \\ = & E^{\overline{C'}}[\mathsf{return}\ \lambda \overline{h_2}, k_2.[\![M]\!]^{\mathsf{e}}_{\mathcal{D}''} @ (\overline{h_2}, k_2)]\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} & \\ = & E^{\overline{C'}}[\mathsf{return}\ [\![M]\!]_{\mathcal{D}''}]\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} & \\ \hookrightarrow & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}'} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \quad \text{(by the IH)} & \end{array}$$

for some $\mathcal{D}''$ and $\mathcal{D}'$ such that

$$\frac{\Gamma \vdash_{\mathcal{D}} M : C_1 \qquad C_1 <: C_2}{\Gamma \vdash_{\mathcal{D}''} M : C_2}\ \text{(HT\_SUBC)}$$

and $\Gamma \vdash_{\mathcal{D}'} M : C_n$.

Case (HS_AnsEmb): We are given $C_1.A = \square$ and $C_2.A = C_2^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}$ for some C_2^{ini} and C_2^{fin} such that $C_2^{\mathrm{ini}} <: C_2^{\mathrm{fin}}$. Let k_2, x_1, and y_2 be fresh variables and $\overline{h_2}$ be a sequence of fresh variables such that $|\overline{h_2}| = |C_2.\Sigma|$, and $(\overline{h_2^{\not\square}}, \overline{h_2^{\square}}) = \mathit{split}(\overline{h_2}, C_2.\Sigma)$. Then, the conclusion is proven as follows:

$$\begin{array}{rl} & E^{\overline{C}}[\mathsf{return}\,[\![M]\!]_{\mathcal{D}}]\;\overline{v^{\mathsf{h}}}\;v^{\mathsf{k}} \\ \hookrightarrow & E^{\overline{C'}}[\mathsf{return}\,[\![C_1 <: C_2]\!]@([\![M]\!]_{\mathcal{D}})]\;\overline{v^{\mathsf{h}}}\;v^{\mathsf{k}} \quad \text{(by the above reasoning)} \\ = & E^{\overline{C'}}[\mathsf{return}\,\lambda\overline{h_2}, k_2.\mathsf{let}\,x_1 = [\![M]\!]_{\mathcal{D}}\;[\![C_2.\Sigma <: C_1.\Sigma]\!]@(\overline{h_2})\;\underline{1}\;\mathsf{in}\;e]\;\overline{v^{\mathsf{h}}}\;v^{\mathsf{k}} \\ & \quad \text{(where } e = (\mathsf{let}\,y_2 = k_2\,([\![C_1.T <: C_2.T]\!]@(x_1))\;\overline{h_2^{\square}}\;\mathsf{in\;return}\,[\![C_2^{\mathrm{ini}} <: C_2^{\mathrm{fin}}]\!]@(y_2))) \\ \hookrightarrow & E^{\overline{C'}}[\mathsf{return}\,\lambda\overline{h_2}, k_2.\mathsf{let}\,x_1 = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@([\![C_2.\Sigma <: C_1.\Sigma]\!]@(\overline{h_2}), \underline{1})\;\mathsf{in}\;e]\;\overline{v^{\mathsf{h}}}\;v^{\mathsf{k}} \\ = & E^{\overline{C'}}[\mathsf{return}\,\lambda\overline{h_2}, k_2.[\![M]\!]^{\mathsf{e}}_{\mathcal{D}''}@(\overline{h_2}, k_2)]\;\overline{v^{\mathsf{h}}}\;v^{\mathsf{k}} \\ = & E^{\overline{C'}}[\mathsf{return}\,[\![M]\!]_{\mathcal{D}''}]\;\overline{v^{\mathsf{h}}}\;v^{\mathsf{k}} \\ \hookrightarrow & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}'}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \quad \text{(by the IH)} \end{array}$$

for some $\mathcal{D}''$ and $\mathcal{D}'$ such that

$$\frac{\Gamma \vdash_{\mathcal{D}} M : C_1 \qquad C_1 <: C_2}{\Gamma \vdash_{\mathcal{D}''} M : C_2}\ \text{(HT\_SubC)}$$

and $\Gamma \vdash_{\mathcal{D}'} M : C_n$.

Case (HS_AnsMod): We are given

$$\frac{C_2^{\mathrm{ini}} <: C_1^{\mathrm{ini}} \qquad C_1^{\mathrm{fin}} <: C_2^{\mathrm{fin}}}{C_1^{\mathrm{ini}} \Rightarrow C_1^{\mathrm{fin}} <: C_2^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}}$$

for some C_1^{ini}, C_1^{fin}, C_2^{ini}, and C_2^{fin} such that $C_1.A = C_1^{\mathrm{ini}} \Rightarrow C_1^{\mathrm{fin}}$ and $C_2.A = C_2^{\mathrm{ini}} \Rightarrow C_2^{\mathrm{fin}}$.
Assume that $C_1.\Sigma = C_2.\Sigma$ and $C_1.T = C_2.T$ and $C_1^{\mathrm{ini}} = C_2^{\mathrm{ini}}$. Let x_1 be a fresh variable. Then, the conclusion is proven as follows:

$$\begin{array}{rl} & E^{\overline{C}}[\mathsf{return}\,[\![M]\!]_{\mathcal{D}}]\;\overline{v^{\mathsf{h}}}\;v^{\mathsf{k}} \\ \hookrightarrow & E^{\overline{C'}}[\mathsf{return}\,[\![C_1 <: C_2]\!]@([\![M]\!]_{\mathcal{D}})]\;\overline{v^{\mathsf{h}}}\;v^{\mathsf{k}} \quad \text{(by the above reasoning)} \\ = & E^{\overline{C'}}[\mathsf{return}\,\lambda\overline{h_2}, k_2.\mathsf{let}\,x_1 = [\![M]\!]_{\mathcal{D}}\;\overline{h_2}\;k_2\;\mathsf{in\;return}\,[\![C_1^{\mathrm{fin}} <: C_2^{\mathrm{fin}}]\!]@(x_1)]\;\overline{v^{\mathsf{h}}}\;v^{\mathsf{k}} \\ \hookrightarrow & E^{\overline{C'}}[\mathsf{return}\,\lambda\overline{h_2}, k_2.\mathsf{let}\,x_1 = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{h_2}, k_2)\;\mathsf{in\;return}\,[\![C_1^{\mathrm{fin}} <: C_2^{\mathrm{fin}}]\!]@(x_1)]\;\overline{v^{\mathsf{h}}}\;v^{\mathsf{k}} \\ = & E^{\overline{C'}}[\mathsf{return}\,\lambda\overline{h_2}, k_2.[\![M]\!]^{\mathsf{e}}_{\mathcal{D}''}@(\overline{h_2}, k_2)]\;\overline{v^{\mathsf{h}}}\;v^{\mathsf{k}} \\ = & E^{\overline{C'}}[\mathsf{return}\,[\![M]\!]_{\mathcal{D}''}]\;\overline{v^{\mathsf{h}}}\;v^{\mathsf{k}} \\ \hookrightarrow & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}'}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \quad \text{(by the IH)} \end{array}$$

for some $\mathcal{D}''$ and $\mathcal{D}'$ such that

$$\frac{\Gamma \vdash_{\mathcal{D}} M : C_1 \qquad C_1 <: C_2}{\Gamma \vdash_{\mathcal{D}''} M : C_2}\ \text{(HT\_SubC)}$$

and $\Gamma \vdash_{\mathcal{D}'} M : C_n$.
Otherwise, assume that $C_1.\Sigma \neq C_2.\Sigma$ or $C_1.T \neq C_2.T$ or $C_1^{\mathrm{ini}} \neq C_2^{\mathrm{ini}}$. Let k_2, x_1, and y_1 be fresh variables, and $\overline{h_1^{\square}}$ and $\overline{h_2}$ be a sequence of fresh variables such that $|\overline{h_1^{\square}}| = |\square(C_2.\Sigma)|$ and $|\overline{h_2}| = |C_2.\Sigma|$. Then, the conclusion is proven as follows:

$$\begin{array}{rl} & E^{\overline{C}}[\mathsf{return}\,[\![M]\!]_{\mathcal{D}}]\;\overline{v^{\mathsf{h}}}\;v^{\mathsf{k}} \\ \hookrightarrow & E^{\overline{C'}}[\mathsf{return}\,[\![C_1 <: C_2]\!]@([\![M]\!]_{\mathcal{D}})]\;\overline{v^{\mathsf{h}}}\;v^{\mathsf{k}} \quad \text{(by the above reasoning)} \\ = & E^{\overline{C'}}[\mathsf{return}\,\lambda\overline{h_2}, k_2.\mathsf{let}\,x_1 = [\![M]\!]_{\mathcal{D}}\;[\![C_2.\Sigma <: C_1.\Sigma]\!]@(\overline{h_2})\;v\;\mathsf{in\;return}\,[\![C_1^{\mathrm{fin}} <: C_2^{\mathrm{fin}}]\!]@(x_1)]\;\overline{v^{\mathsf{h}}}\;v^{\mathsf{k}} \\ & \quad \text{(where } v = \lambda y_1, \overline{h_1^{\square}}.\mathsf{let}\,z_2 = k_2\,([\![C_1.T <: C_2.T]\!]\,(y_1))\;\overline{h_1^{\square}}\;\mathsf{in\;return}\,[\![C_2^{\mathrm{ini}} <: C_1^{\mathrm{ini}}]\!]@(z_2)) \\ \hookrightarrow & E^{\overline{C'}}[\mathsf{return}\,\lambda\overline{h_2}, k_2.\mathsf{let}\,x_1 = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@([\![C_2.\Sigma <: C_1.\Sigma]\!]@(\overline{h_2}), v)\;\mathsf{in\;return}\,[\![C_1^{\mathrm{fin}} <: C_2^{\mathrm{fin}}]\!]@(x_1)]\;\overline{v^{\mathsf{h}}}\;v^{\mathsf{k}} \\ = & E^{\overline{C'}}[\mathsf{return}\,\lambda\overline{h_2}, k_2.[\![M]\!]^{\mathsf{e}}_{\mathcal{D}''}@(\overline{h_2}, k_2)]\;\overline{v^{\mathsf{h}}}\;v^{\mathsf{k}} \\ = & E^{\overline{C'}}[\mathsf{return}\,[\![M]\!]_{\mathcal{D}''}]\;\overline{v^{\mathsf{h}}}\;v^{\mathsf{k}} \\ \hookrightarrow & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}'}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \quad \text{(by the IH)} \end{array}$$

for some $\mathcal{D}''$ and $\mathcal{D}'$ such that

$$\frac{\Gamma \vdash_{\mathcal{D}} M : C_1 \qquad C_1 <: C_2}{\Gamma \vdash_{\mathcal{D}''} M : C_2}\ (\text{HT\_SubC})$$

and $\Gamma \vdash_{\mathcal{D}'} M : C_n$.

$\square$

Lemma 37 (Simulation up to Reduction)**.** If $\emptyset \vdash_{\mathcal{D}} M : \Sigma \triangleright T \,/\, A$ and $|\overline{v^{\mathsf{h}}}| = |\Sigma|$, then, for any v^{k}, one of the following holds:

1. there exist some V and $\mathcal{D}'$ such that
   - $A = \square$,
   - $M = \mathsf{return}\ V$,
   - $\llbracket \mathsf{return}\ V \rrbracket^{\mathsf{e}}_{\mathcal{D}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \longrightarrow^{*} \mathsf{return}\ \llbracket V \rrbracket_{\mathcal{D}'}$, and
   - $\emptyset \vdash_{\mathcal{D}'} V : T$;
2. there exist some C^{ini}, C^{fin}, V, $\mathcal{D}'$, $\overline{C}$, $\overline{v^{\mathsf{h}\not\square}}$, and $\overline{v^{\mathsf{h}\square}}$ such that
   - $A = C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$,
   - $M = \mathsf{return}\ V$,
   - $(\overline{v^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}) = \mathit{split}(\overline{v^{\mathsf{h}}}, \Sigma)$,
   - $\llbracket \mathsf{return}\ V \rrbracket^{\mathsf{e}}_{\mathcal{D}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \longrightarrow^{*} E^{\overline{C}}[v^{\mathsf{k}}\ \llbracket V \rrbracket_{\mathcal{D}'}\ \overline{v^{\mathsf{h}\square}}]$,
   - $\emptyset \vdash_{\mathcal{D}'} V : T$, and
   - the first and last computation types of $\overline{C}$ are C^{ini} and C^{fin}, respectively;
3. there exist σ, V', x, M', T^{par}, T^{ari}, v, $\mathcal{D}_1$, and $\mathcal{D}_2$ such that
   - $M \longrightarrow^{*} \sigma(V'; x.\, M')$,
   - $\llbracket M \rrbracket^{\mathsf{e}}_{\mathcal{D}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \hookrightarrow \mathsf{let}\, x = v\ \llbracket V' \rrbracket_{\mathcal{D}_1}\ \mathsf{in}\ \llbracket M' \rrbracket^{\mathsf{e}}_{\mathcal{D}_2} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})$,
   - $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, \square \in \Sigma$,
   - v is a value in the sequence $\overline{v^{\mathsf{h}}}$ that corresponds to σ in Σ,
   - $\emptyset \vdash_{\mathcal{D}_1} V' : T^{\mathrm{par}}$, and
   - $\emptyset, x : T^{\mathrm{ari}} \vdash_{\mathcal{D}_2} M' : \Sigma \triangleright T \,/\, A$;
4. there exist some C^{ini}, C^{fin}, σ, V', x, M', T^{par}, T^{ari}, $C^{\sigma\mathrm{ini}}$, $C^{\sigma\mathrm{fin}}$, $\overline{C}$, v, $\overline{v^{\mathsf{h}\not\square}}$, $\overline{v^{\mathsf{h}\square}}$, $\overline{h^{\square}}$, $\mathcal{D}_1$, and $\mathcal{D}_2$ such that
   - $A = C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$,
   - $M \longrightarrow^{*} \sigma(V'; x.\, M')$,
   - $\overline{h^{\square}}$ is a sequence of fresh variables such that $|\overline{h^{\square}}| = |\square(\Sigma)|$,
   - $(\overline{v^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}) = \mathit{split}(\overline{v^{\mathsf{h}}}, \Sigma)$,
   - $\llbracket M \rrbracket^{\mathsf{e}}_{\mathcal{D}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \hookrightarrow E^{\overline{C}}[v\ \llbracket V' \rrbracket_{\mathcal{D}_1}\ (\lambda x, \overline{h^{\square}}. \llbracket M' \rrbracket^{\mathsf{e}}_{\mathcal{D}_2} @ (\overline{v^{\mathsf{h}\not\square}}, \overline{h^{\square}}, v^{\mathsf{k}}))]$,
   - $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, C^{\sigma\mathrm{ini}} \Rightarrow C^{\sigma\mathrm{fin}} \in \Sigma$,
   - v is a value in the sequence $\overline{v^{\mathsf{h}}}$ that corresponds to σ in Σ,
   - the first and last computation types of $\overline{C}$ are $C^{\sigma\mathrm{fin}}$ and C^{fin}, respectively,
   - $\emptyset \vdash_{\mathcal{D}_1} V' : T^{\mathrm{par}}$,
   - $\emptyset, x : T^{\mathrm{ari}} \vdash_{\mathcal{D}_2} M' : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C^{\sigma\mathrm{ini}}$;

or

5. there exist some M' and $\mathcal{D}'$ such that
   - $M \longrightarrow^{+} M'$,
   - $[\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \hookrightarrow\longrightarrow^{+}\hookrightarrow [\![M']\!]^{\mathsf{e}}_{\mathcal{D}'} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})$, and
   - $\emptyset \vdash_{\mathcal{D}'} M' : \Sigma \triangleright T \,/\, A$.

Proof. By induction on the derivation of $\emptyset \vdash_{\mathcal{D}} M : \Sigma \triangleright T \,/\, A$.

Case (HT_Return): The conclusion (case (1)) holds obviously.

Case (HT_SubC): We are given

$$\frac{\emptyset \vdash_{\mathcal{D}''} M : C \qquad C <: \Sigma \triangleright T \,/\, A}{\emptyset \vdash_{\mathcal{D}} M : \Sigma \triangleright T \,/\, A}$$

for some $\mathcal{D}''$ and C. By Lemma 3, $\Sigma <: C.\Sigma$ and $C.T <: T$ and $C.A <: A$. By case analysis on the subtyping rule applied last to derive $C.A <: A$.

Case (HS_AnsBox): We are given $C.A = A = \square$. Let x_1 be a fresh variable. By the definition of the CPS transformation,

$$\begin{array}{rcl} [\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) & = & [\![C <: \Sigma \triangleright T \,/\, \square]\!]^{\mathsf{e}} @([\![M]\!]^{\mathsf{e}}_{\mathcal{D}''}, \overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\ & = & \mathsf{let}\, x_1 = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}''} @([\![\Sigma <: C.\Sigma]\!] @(\overline{v^{\mathsf{h}}}), v^{\mathsf{k}})\ \mathsf{in\ return}\ [\![C.T <: T]\!] @(x_1)\ . \end{array}$$

Let $\overline{v'^{\mathsf{h}}} = [\![\Sigma <: C.\Sigma]\!] @(\overline{v^{\mathsf{h}}})$. By Lemma 28, $|\overline{v'^{\mathsf{h}}}| = |C.\Sigma|$. Thus, we can apply the IH on $\emptyset \vdash_{\mathcal{D}''} M : C$. We proceed by case analysis on the result.

Case 1: We are given some V and $\mathcal{D}'''$ such that
- $M = \mathsf{return}\ V$,
- $[\![\mathsf{return}\ V]\!]^{\mathsf{e}}_{\mathcal{D}''} @(\overline{v'^{\mathsf{h}}}, v^{\mathsf{k}}) \longrightarrow^{*} \mathsf{return}\ [\![V]\!]_{\mathcal{D}'''}$, and
- $\emptyset \vdash_{\mathcal{D}'''} V : C.T$.

Then, the conclusion (case (1)) is proven as follows:

$$\begin{array}{rll} & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) & \\ = & \mathsf{let}\, x_1 = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}''} @(\overline{v'^{\mathsf{h}}}, v^{\mathsf{k}})\ \mathsf{in\ return}\ [\![C.T <: T]\!] @(x_1) & \text{(by the above reasoning)} \\ \longrightarrow^{*} & \mathsf{let}\, x_1 = \mathsf{return}\ [\![V]\!]_{\mathcal{D}'''}\ \mathsf{in\ return}\ [\![C.T <: T]\!] @(x_1) & \text{(by case (1) and (E\_LetE))} \\ \longrightarrow & [\![C.T <: T]\!] @([\![V]\!]_{\mathcal{D}'''}) & \\ = & [\![V]\!]_{\mathcal{D}'} & \end{array}$$

for some $\mathcal{D}'$ such that

$$\frac{\emptyset \vdash_{\mathcal{D}'''} V : C.T \qquad C.T <: T}{\emptyset \vdash_{\mathcal{D}'} V : T}\ \text{(HT\_SubV)}.$$

Case 2: Contradictory with $C.A = \square$.

Case 3: We are given some σ, V', x, M', T'^{par}, T'^{ari}, v', $\mathcal{D}'_1$, and $\mathcal{D}'_2$ such that
- $M \longrightarrow^{*} \sigma(V'; x.\, M')$,
- $[\![M]\!]^{\mathsf{e}}_{\mathcal{D}''} @(\overline{v'^{\mathsf{h}}}, v^{\mathsf{k}}) \hookrightarrow \mathsf{let}\, x = v'\ [\![V']\!]_{\mathcal{D}'_1}\ \mathsf{in}\ [\![M']\!]^{\mathsf{e}}_{\mathcal{D}'_2} @(\overline{v'^{\mathsf{h}}}, v^{\mathsf{k}})$,
- $\sigma : T'^{\mathrm{par}} \rightsquigarrow T'^{\mathrm{ari}} \,/\, \square \in C.\Sigma$,
- v' is a value in the sequence $\overline{v'^{\mathsf{h}}}$ that corresponds to σ in $C.\Sigma$,
- $\emptyset \vdash_{\mathcal{D}'_1} V' : T'^{\mathrm{par}}$, and
- $\emptyset, x : T'^{\mathrm{ari}} \vdash_{\mathcal{D}'_2} M' : C$.

Because $\Sigma <: C.\Sigma$ and $\sigma : T'^{\mathrm{par}} \rightsquigarrow T'^{\mathrm{ari}} \,/\, \square \in C.\Sigma$, there exist some T^{par} and T^{ari} such that

- $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, \square \in \Sigma$,
- $T'^{\mathrm{par}} <: T^{\mathrm{par}}$, and
- $T^{\mathrm{ari}} <: T'^{\mathrm{ari}}$

by Lemma 3. Let v be a value in the sequence $\overline{v^{\mathsf{h}}}$ that corresponds to σ in Σ (there exists such a value because $|\overline{v^{\mathsf{h}}}| = |\Sigma|$), and let y_2 and y_1 be fresh variables. By the definition of the CPS transformation,

$$v' = \lambda y_2.\mathsf{let}\, y_1 = v\,(\llbracket T'^{\mathrm{par}} <: T^{\mathrm{par}} \rrbracket @(y_2))\,\mathsf{in\ return}\,\llbracket T^{\mathrm{ari}} <: T'^{\mathrm{ari}} \rrbracket @(y_1)\ .$$

Then, the conclusion (case (3)) is proven as follows:

$$\begin{array}{cl}
 & \llbracket M \rrbracket^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & \mathsf{let}\, x_1 = \llbracket M \rrbracket^{\mathsf{e}}_{\mathcal{D}''} @(\overline{v'^{\mathsf{h}}}, v^{\mathsf{k}})\,\mathsf{in\ return}\,\llbracket C.T <: T \rrbracket @(x_1) \quad \text{(by the above reasoning)} \\
\hookrightarrow & \mathsf{let}\, x_1 = (\mathsf{let}\, x = v'\,\llbracket V' \rrbracket_{\mathcal{D}'_1}\,\mathsf{in}\,\llbracket M' \rrbracket^{\mathsf{e}}_{\mathcal{D}'_2} @(\overline{v'^{\mathsf{h}}}, v^{\mathsf{k}}))\,\mathsf{in\ return}\,\llbracket C.T <: T \rrbracket @(x_1) \quad \text{(by case (3))} \\
\hookrightarrow & \mathsf{let}\, x = v'\,\llbracket V' \rrbracket_{\mathcal{D}'_1}\,\mathsf{in\ let}\, x_1 = \llbracket M' \rrbracket^{\mathsf{e}}_{\mathcal{D}'_2} @(\overline{v'^{\mathsf{h}}}, v^{\mathsf{k}})\,\mathsf{in\ return}\,\llbracket C.T <: T \rrbracket @(x_1) \\
= & \mathsf{let}\, x = v'\,\llbracket V' \rrbracket_{\mathcal{D}'_1}\,\mathsf{in}\,\llbracket M' \rrbracket^{\mathsf{e}}_{\mathcal{D}''_2} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
\hookrightarrow & \mathsf{let}\, x = (\mathsf{let}\, y_1 = v\,(\llbracket T'^{\mathrm{par}} <: T^{\mathrm{par}} \rrbracket @(\llbracket V' \rrbracket_{\mathcal{D}'_1}))\,\mathsf{in\ return}\,\llbracket T^{\mathrm{ari}} <: T'^{\mathrm{ari}} \rrbracket @(y_1))\,\mathsf{in}\,\llbracket M' \rrbracket^{\mathsf{e}}_{\mathcal{D}''_2} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & \mathsf{let}\, x = (\mathsf{let}\, y_1 = v\,\llbracket V' \rrbracket_{\mathcal{D}_1}\,\mathsf{in\ return}\,\llbracket T^{\mathrm{ari}} <: T'^{\mathrm{ari}} \rrbracket @(y_1))\,\mathsf{in}\,\llbracket M' \rrbracket^{\mathsf{e}}_{\mathcal{D}''_2} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
\hookrightarrow & \mathsf{let}\, y_1 = v\,\llbracket V' \rrbracket_{\mathcal{D}_1}\,\mathsf{in\ let}\, x = \mathsf{return}\,\llbracket T^{\mathrm{ari}} <: T'^{\mathrm{ari}} \rrbracket @(y_1)\,\mathsf{in}\,\llbracket M' \rrbracket^{\mathsf{e}}_{\mathcal{D}''_2} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
\hookrightarrow & \mathsf{let}\, x = v\,\llbracket V' \rrbracket_{\mathcal{D}_1}\,\mathsf{in}\,\llbracket M' \rrbracket^{\mathsf{e}}_{\mathcal{D}_2} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \quad \text{(by Lemmas 30, 31, 32)}
\end{array}$$

for some $\mathcal{D}''_2$, $\mathcal{D}_1$, and $\mathcal{D}_2$ such that

$$\frac{\emptyset, x : T'^{\mathrm{ari}} \vdash_{\mathcal{D}'_2} M' : C \qquad C <: \Sigma \triangleright T \,/\, A}{\emptyset, x : T'^{\mathrm{ari}} \vdash_{\mathcal{D}''_2} M' : \Sigma \triangleright T \,/\, A}\ (\text{HT\_SubC}),$$

$$\frac{\emptyset \vdash_{\mathcal{D}'_1} V' : T'^{\mathrm{par}} \qquad T'^{\mathrm{par}} <: T^{\mathrm{par}}}{\emptyset \vdash_{\mathcal{D}_1} V' : T^{\mathrm{par}}}\ (\text{HT\_SubV}),$$

and

$$\emptyset, x : T^{\mathrm{ari}} \vdash_{\mathcal{D}_2} M' : \Sigma \triangleright T \,/\, A\ .$$

Case 4: Contradictory with $C.A = \square$.

Case 5: We are given some M' and $\mathcal{D}'''$ such that

- $M \longrightarrow^{+} M'$,
- $\llbracket M \rrbracket^{\mathsf{e}}_{\mathcal{D}''} @(\overline{v'^{\mathsf{h}}}, v^{\mathsf{k}}) \hookrightarrow\longrightarrow^{+}\hookrightarrow \llbracket M' \rrbracket^{\mathsf{e}}_{\mathcal{D}'''} @(\overline{v'^{\mathsf{h}}}, v^{\mathsf{k}})$, and
- $\emptyset \vdash_{\mathcal{D}'''} M' : C$.

Then, the conclusion (case (5)) is proven as follows:

$$\begin{array}{cl}
 & \llbracket M \rrbracket^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & \mathsf{let}\, x_1 = \llbracket M \rrbracket^{\mathsf{e}}_{\mathcal{D}''} @(\overline{v'^{\mathsf{h}}}, v^{\mathsf{k}})\,\mathsf{in\ return}\,\llbracket C.T <: T \rrbracket @(x_1) \quad \text{(by the above reasoning)} \\
\hookrightarrow\longrightarrow^{+}\hookrightarrow & \mathsf{let}\, x_1 = \llbracket M' \rrbracket^{\mathsf{e}}_{\mathcal{D}'''} @(\overline{v'^{\mathsf{h}}}, v^{\mathsf{k}})\,\mathsf{in\ return}\,\llbracket C.T <: T \rrbracket @(x_1) \quad \text{(by case (5) and (E_LetE))} \\
= & \llbracket M' \rrbracket^{\mathsf{e}}_{\mathcal{D}'} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})
\end{array}$$

for some $\mathcal{D}'$ such that

$$\frac{\emptyset \vdash_{\mathcal{D}'''} M' : C \qquad C <: \Sigma \triangleright T \,/\, A}{\emptyset \vdash_{\mathcal{D}'} M' : \Sigma \triangleright T \,/\, A}\ (\text{HT\_SubC}).$$

Case (HS_AnsEmb): We are given $C.A = \square$ and $A = C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$ for some C^{ini} and C^{fin} such that $C^{\mathrm{ini}} <: C^{\mathrm{fin}}$. Let $(\overline{v^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}) = \mathit{split}(\overline{v^{\mathsf{h}}}, \Sigma)$. Let x_1 and y_2 be fresh variables. By the definition of the CPS transformation,

$$\begin{array}{cl}
 & \llbracket M \rrbracket^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & \llbracket C <: \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}} \rrbracket^{\mathsf{e}} @(\llbracket M \rrbracket^{\mathsf{e}}_{\mathcal{D}''}, \overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & \mathsf{let}\, x_1 = \llbracket M \rrbracket^{\mathsf{e}}_{\mathcal{D}''} @(\llbracket \Sigma <: C.\Sigma \rrbracket @(\overline{v^{\mathsf{h}}}), \underline{1})\,\mathsf{in\ let}\, y_2 = v^{\mathsf{k}}\,(\llbracket C.T <: T \rrbracket @(x_1))\,\overline{v^{\mathsf{h}\square}}\,\mathsf{in\ return}\,\llbracket C^{\mathrm{ini}} <: C^{\mathrm{fin}} \rrbracket @(y_2)\ .
\end{array}$$

Because $|[\![\Sigma <: C.\Sigma]\!]@(\overline{v^{\mathsf{h}}})| = |C.\Sigma|$ by Lemma 28, we can apply the IH on $\emptyset \vdash_{\mathcal{D}''} M : C$. We proceed by case analysis on the result.

Case 1: We are given some V and $\mathcal{D}'''$ such that

- $M = \mathsf{return}\ V$,
- $[\![\mathsf{return}\ V]\!]^{\mathsf{e}}_{\mathcal{D}''}@([\![\Sigma <: C.\Sigma]\!]@(\overline{v^{\mathsf{h}}}), \underline{1}) \longrightarrow^* \mathsf{return}\,[\![V]\!]_{\mathcal{D}'''}$, and
- $\emptyset \vdash_{\mathcal{D}'''} V : C.T$.

Then:

$$\begin{array}{rl} & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\ = & \mathsf{let}\, x_1 = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}''}@([\![\Sigma <: C.\Sigma]\!]@(\overline{v^{\mathsf{h}}}), \underline{1})\,\mathsf{in}\,\mathsf{let}\, y_2 = v^{\mathsf{k}}\,([\![C.T <: T]\!]@(x_1))\,\overline{v^{\mathsf{h}\square}}\,\mathsf{in}\,\mathsf{return}\,[\![C^{\mathrm{ini}} <: C^{\mathrm{fin}}]\!]@(y_2) \\ & \quad \text{(by the above reasoning)} \\ \longrightarrow^* & \mathsf{let}\, x_1 = \mathsf{return}\,[\![V]\!]_{\mathcal{D}'''}\,\mathsf{in}\,\mathsf{let}\, y_2 = v^{\mathsf{k}}\,([\![C.T <: T]\!]@(x_1))\,\overline{v^{\mathsf{h}\square}}\,\mathsf{in}\,\mathsf{return}\,[\![C^{\mathrm{ini}} <: C^{\mathrm{fin}}]\!]@(y_2) \\ & \quad \text{(by case (1) and (E\_LetE))} \\ \longrightarrow & \mathsf{let}\, y_2 = v^{\mathsf{k}}\,([\![C.T <: T]\!]@([\![V]\!]_{\mathcal{D}'''}))\,\overline{v^{\mathsf{h}\square}}\,\mathsf{in}\,\mathsf{return}\,[\![C^{\mathrm{ini}} <: C^{\mathrm{fin}}]\!]@(y_2) \\ = & \mathsf{let}\, y_2 = v^{\mathsf{k}}\,[\![V]\!]_{\mathcal{D}'}\,\overline{v^{\mathsf{h}\square}}\,\mathsf{in}\,\mathsf{return}\,[\![C^{\mathrm{ini}} <: C^{\mathrm{fin}}]\!]@(y_2) \end{array}$$

for some $\mathcal{D}'$ such that

$$\frac{\emptyset \vdash_{\mathcal{D}'''} V : C.T \qquad C.T <: T}{\emptyset \vdash_{\mathcal{D}'} V : T}\ (\text{HT\_SubV}).$$

We have the conclusion (case (2)) by letting $\overline{C} = C^{\mathrm{ini}}, C^{\mathrm{fin}}$.

Case 2: Contradictory with $C.A = \square$.

Case 3: We are given some σ, V', x, M', T'^{par}, T'^{ari}, v', $\mathcal{D}'_1$, and $\mathcal{D}'_2$ such that

- $M \longrightarrow^* \sigma(V'; x.\, M')$,
- $[\![M]\!]^{\mathsf{e}}_{\mathcal{D}''}@([\![\Sigma <: C.\Sigma]\!]@(\overline{v^{\mathsf{h}}}), \underline{1}) \hookrightarrow \mathsf{let}\, x = v'\,[\![V']\!]_{\mathcal{D}'_1}\,\mathsf{in}\,[\![M']\!]^{\mathsf{e}}_{\mathcal{D}'_2}@([\![\Sigma <: C.\Sigma]\!]@(\overline{v^{\mathsf{h}}}), \underline{1})$,
- $\sigma : T'^{\mathrm{par}} \rightsquigarrow T'^{\mathrm{ari}} / \square \in C.\Sigma$,
- v' is a value in the sequence $[\![\Sigma <: C.\Sigma]\!]@(\overline{v^{\mathsf{h}}})$ that corresponds to σ in $C.\Sigma$,
- $\emptyset \vdash_{\mathcal{D}'_1} V' : T'^{\mathrm{par}}$, and
- $\emptyset, x : T'^{\mathrm{ari}} \vdash_{\mathcal{D}'_2} M' : C$.

Because $\Sigma <: C.\Sigma$ and $\sigma : T'^{\mathrm{par}} \rightsquigarrow T'^{\mathrm{ari}} / \square \in C.\Sigma$, there exist some T^{par} and T^{ari} such that

- $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} / \square \in \Sigma$,
- $T'^{\mathrm{par}} <: T^{\mathrm{par}}$, and
- $T^{\mathrm{ari}} <: T'^{\mathrm{ari}}$

by Lemma 3. Let v be a value in the sequence $\overline{v^{\mathsf{h}}}$ that corresponds to σ in Σ (there exists such a value because $|\overline{v^{\mathsf{h}}}| = |\Sigma|$), and let y_2 and y_1 be fresh variables. By the definition of the CPS transformation,

$$v' = \lambda y_2.\mathsf{let}\, y_1 = v\,([\![T'^{\mathrm{par}} <: T^{\mathrm{par}}]\!]@(y_2))\,\mathsf{in}\,\mathsf{return}\,[\![T^{\mathrm{ari}} <: T'^{\mathrm{ari}}]\!]@(y_1)\ .$$

Therefore,

$$\begin{array}{rl} & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}''}@([\![\Sigma <: C.\Sigma]\!]@(\overline{v^{\mathsf{h}}}), \underline{1}) \\ \hookrightarrow & \mathsf{let}\, x = v'\,[\![V']\!]_{\mathcal{D}'_1}\,\mathsf{in}\,[\![M']\!]^{\mathsf{e}}_{\mathcal{D}'_2}@([\![\Sigma <: C.\Sigma]\!]@(\overline{v^{\mathsf{h}}}), \underline{1}) \quad \text{(by case (3))} \\ \hookrightarrow & \mathsf{let}\, x = (\mathsf{let}\, y_1 = v\,([\![T'^{\mathrm{par}} <: T^{\mathrm{par}}]\!]@([\![V']\!]_{\mathcal{D}'_1}))\,\mathsf{in}\,\mathsf{return}\,[\![T^{\mathrm{ari}} <: T'^{\mathrm{ari}}]\!]@(y_1))\,\mathsf{in}\,[\![M']\!]^{\mathsf{e}}_{\mathcal{D}'_2}@([\![\Sigma <: C.\Sigma]\!]@(\overline{v^{\mathsf{h}}}), \underline{1}) \\ = & \mathsf{let}\, x = (\mathsf{let}\, y_1 = v\,[\![V']\!]_{\mathcal{D}_1}\,\mathsf{in}\,\mathsf{return}\,[\![y_1]\!]_{\mathcal{D}^{y_1}})\,\mathsf{in}\,[\![M']\!]^{\mathsf{e}}_{\mathcal{D}'_2}@([\![\Sigma <: C.\Sigma]\!]@(\overline{v^{\mathsf{h}}}), \underline{1}) \\ \hookrightarrow & \mathsf{let}\, y_1 = v\,[\![V']\!]_{\mathcal{D}_1}\,\mathsf{in}\,\mathsf{let}\, x = \mathsf{return}\,[\![y_1]\!]_{\mathcal{D}^{y_1}}\,\mathsf{in}\,[\![M']\!]^{\mathsf{e}}_{\mathcal{D}'_2}@([\![\Sigma <: C.\Sigma]\!]@(\overline{v^{\mathsf{h}}}), \underline{1}) \\ \hookrightarrow & \mathsf{let}\, y_1 = v\,[\![V']\!]_{\mathcal{D}_1}\,\mathsf{in}\,[\![M']\!]^{\mathsf{e}}_{\mathcal{D}'_2}@([\![\Sigma <: C.\Sigma]\!]@(\overline{v^{\mathsf{h}}}), \underline{1})[[\![y_1]\!]_{\mathcal{D}^{y_1}}/x] \\ = & \mathsf{let}\, x = v\,[\![V']\!]_{\mathcal{D}_1}\,\mathsf{in}\,[\![M']\!]^{\mathsf{e}}_{\mathcal{D}''_2}@([\![\Sigma <: C.\Sigma]\!]@(\overline{v^{\mathsf{h}}}), \underline{1}) \quad \text{(by Lemmas 30, 31, and 32)} \end{array}$$

for some $\mathcal{D}_1$, $\mathcal{D}^{y_1}$, and $\mathcal{D}''_2$ such that

$$\frac{\emptyset \vdash_{\mathcal{D}'_1} V' : T'^{\mathrm{par}} \qquad T'^{\mathrm{par}} <: T^{\mathrm{par}}}{\emptyset \vdash_{\mathcal{D}_1} V' : T^{\mathrm{par}}} \ (\text{HT\_SubV}),$$

and

$$\frac{\dfrac{}{\emptyset, y_1 : T^{\mathrm{ari}} \vdash y_1 : T^{\mathrm{ari}}} \ (\text{HT\_Var}) \qquad T^{\mathrm{ari}} <: T'^{\mathrm{ari}}}{\emptyset, y_1 : T^{\mathrm{ari}} \vdash_{\mathcal{D}^{y_1}} y_1 : T'^{\mathrm{ari}}} \ (\text{HT\_SubV}),$$

and

$$\emptyset, x : T^{\mathrm{ari}} \vdash_{\mathcal{D}''_2} M' : C \ .$$

Then, the conclusion (case (3)) is proven as follows:

$$\begin{array}{ll}
 & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
\hookrightarrow & \mathsf{let}\, x_1 = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}''} @ ([\![\Sigma <: C.\Sigma]\!] @ (\overline{v^{\mathsf{h}}}), \underline{1}) \,\mathsf{in}\, \mathsf{let}\, y_2 = v^{\mathsf{k}} \, ([\![C.T <: T]\!] @ (x_1)) \, \overline{v^{\mathsf{h}\square}} \,\mathsf{in}\, \mathsf{return}\, [\![C^{\mathrm{ini}} <: C^{\mathrm{fin}}]\!] @ (y_2) \\
 & \quad \text{(by the above reasoning)} \\
\hookrightarrow & \mathsf{let}\, x_1 = (\mathsf{let}\, x = v \, [\![V']\!]_{\mathcal{D}_1} \,\mathsf{in}\, [\![M']\!]^{\mathsf{e}}_{\mathcal{D}''_2} @ ([\![\Sigma <: C.\Sigma]\!] @ (\overline{v^{\mathsf{h}}}), \underline{1})) \,\mathsf{in}\, \mathsf{let}\, y_2 = v^{\mathsf{k}} \, ([\![C.T <: T]\!] @ (x_1)) \, \overline{v^{\mathsf{h}\square}} \,\mathsf{in}\, \mathsf{return}\, [\![C^{\mathrm{ini}} <: C^{\mathrm{fin}}]\!] @ (y_2) \\
 & \quad \text{(by the above reasoning)} \\
\hookrightarrow & \mathsf{let}\, x = v \, [\![V']\!]_{\mathcal{D}_1} \,\mathsf{in}\, \mathsf{let}\, x_1 = [\![M']\!]^{\mathsf{e}}_{\mathcal{D}''_2} @ ([\![\Sigma <: C.\Sigma]\!] @ (\overline{v^{\mathsf{h}}}), \underline{1}) \,\mathsf{in}\, \mathsf{let}\, y_2 = v^{\mathsf{k}} \, ([\![C.T <: T]\!] @ (x_1)) \, \overline{v^{\mathsf{h}\square}} \,\mathsf{in}\, \mathsf{return}\, [\![C^{\mathrm{ini}} <: C^{\mathrm{fin}}]\!] @ (y_2) \\
= & \mathsf{let}\, x = v \, [\![V']\!]_{\mathcal{D}_1} \,\mathsf{in}\, [\![C <: \Sigma \triangleright T \,/\, A]\!]^{\mathsf{e}} @ ([\![M']\!]^{\mathsf{e}}_{\mathcal{D}''_2}, \overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & \mathsf{let}\, x = v \, [\![V']\!]_{\mathcal{D}_1} \,\mathsf{in}\, [\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})
\end{array}$$

for some $\mathcal{D}_2$ such that

$$\frac{\emptyset, x : T^{\mathrm{ari}} \vdash_{\mathcal{D}''_2} M' : C \qquad C <: \Sigma \triangleright T \,/\, A}{\emptyset, x : T^{\mathrm{ari}} \vdash_{\mathcal{D}_2} M' : \Sigma \triangleright T \,/\, A} \ (\text{HT\_SubC}).$$

Case 4: Contradictory with $C.A = \square$.

Case 5: We are given some M' and $\mathcal{D}'''$ such that

- $M \longrightarrow^{+} M'$,
- $[\![M]\!]^{\mathsf{e}}_{\mathcal{D}''} @ ([\![\Sigma <: C.\Sigma]\!] @ (\overline{v^{\mathsf{h}}}), \underline{1}) \hookrightarrow\!\longrightarrow^{+}\!\hookrightarrow [\![M']\!]^{\mathsf{e}}_{\mathcal{D}'''} @ ([\![\Sigma <: C.\Sigma]\!] @ (\overline{v^{\mathsf{h}}}), \underline{1})$, and
- $\emptyset \vdash_{\mathcal{D}'''} M' : C$

for some M' and $\mathcal{D}'''$. Then, we have the conclusion (case (5)) because

$$\begin{array}{ll}
 & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & \mathsf{let}\, x_1 = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}''} @ ([\![\Sigma <: C.\Sigma]\!] @ (\overline{v^{\mathsf{h}}}), \underline{1}) \,\mathsf{in}\, \mathsf{let}\, y_2 = v^{\mathsf{k}} \, ([\![C.T <: T]\!] @ (x_1)) \, \overline{v^{\mathsf{h}\square}} \,\mathsf{in}\, \mathsf{return}\, [\![C^{\mathrm{ini}} <: C^{\mathrm{fin}}]\!] @ (y_2) \\
 & \quad \text{(by the above reasoning)} \\
\hookrightarrow\!\longrightarrow^{+}\!\hookrightarrow & \mathsf{let}\, x_1 = [\![M']\!]^{\mathsf{e}}_{\mathcal{D}'''} @ ([\![\Sigma <: C.\Sigma]\!] @ (\overline{v^{\mathsf{h}}}), \underline{1}) \,\mathsf{in}\, \mathsf{let}\, y_2 = v^{\mathsf{k}} \, ([\![C.T <: T]\!] @ (x_1)) \, \overline{v^{\mathsf{h}\square}} \,\mathsf{in}\, \mathsf{return}\, [\![C^{\mathrm{ini}} <: C^{\mathrm{fin}}]\!] @ (y_2) \\
 & \quad \text{(by case (5) and (E_LetE))} \\
= & [\![C <: \Sigma \triangleright T \,/\, A]\!]^{\mathsf{e}} @ ([\![M']\!]^{\mathsf{e}}_{\mathcal{D}'''}, \overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & [\![M']\!]^{\mathsf{e}}_{\mathcal{D}'} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})
\end{array}$$

for some $\mathcal{D}'$ such that

$$\frac{\emptyset \vdash_{\mathcal{D}'''} M' : C \quad C <: \Sigma \triangleright T \,/\, A}{\emptyset \vdash_{\mathcal{D}'} M' : \Sigma \triangleright T \,/\, A} \ (\text{HT\_SubC}).$$

Case (HS_AnsMod): We are given

$$\frac{C^{\mathrm{ini}} <: C'^{\mathrm{ini}} \quad C'^{\mathrm{fin}} <: C^{\mathrm{fin}}}{C'^{\mathrm{ini}} \Rightarrow C'^{\mathrm{fin}} <: C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}}$$

for some C'^{ini}, C'^{fin}, C^{ini}, and C^{fin} such that $C.A = C'^{\mathrm{ini}} \Rightarrow C'^{\mathrm{fin}}$ and $A = C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$. We consider two cases as follows.

Case $C.\Sigma = \Sigma$ and $C.T = T$ and ${C'}^{\mathrm{ini}} = C^{\mathrm{ini}}$: Let x_1 be a fresh variable. Also, let $(\overline{v^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}) = \mathit{split}(\overline{v^{\mathsf{h}}}, \Sigma)$. By the definition of the CPS transformation,

$$\begin{array}{rl} & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\ = & [\![C <: \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}]\!]^{\mathsf{e}} @([\![M]\!]^{\mathsf{e}}_{\mathcal{D}''}, \overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\ = & \mathsf{let}\, x_1 = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}''} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \,\mathsf{in}\, \mathsf{return}\, [\![{C'}^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!] @(x_1) \ . \end{array}$$

We proceed by case analysis on the result of applying the IH on $\emptyset \vdash_{\mathcal{D}''} M : C$.

Case 1: Contradictory with $C.A \neq \square$.

Case 2: We are given some V, $\mathcal{D}'$, and $\overline{C'}$ such that

- $M = \mathsf{return}\ V$,
- $[\![\mathsf{return}\ V]\!]^{\mathsf{e}}_{\mathcal{D}''} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \ \longrightarrow^* \ E^{\overline{C'}}[v^{\mathsf{k}}\, [\![V]\!]_{\mathcal{D}'}\, \overline{v^{\mathsf{h}\square}}]$,
- $\emptyset \vdash_{\mathcal{D}'} V : C.T$, and
- the first and last computation types of $\overline{C'}$ are ${C'}^{\mathrm{ini}}$ and ${C'}^{\mathrm{fin}}$, respectively.

Then:

$$\begin{array}{rl} & [\![\mathsf{return}\ V]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\ = & \mathsf{let}\, x_1 = [\![\mathsf{return}\ V]\!]^{\mathsf{e}}_{\mathcal{D}''} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \,\mathsf{in}\, \mathsf{return}\, [\![{C'}^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!] @(x_1) \\ & \quad \text{(by the above reasoning)} \\ \longrightarrow^* & \mathsf{let}\, x_1 = E^{\overline{C'}}[v^{\mathsf{k}}\, [\![V]\!]_{\mathcal{D}'}\, \overline{v^{\mathsf{h}\square}}] \,\mathsf{in}\, \mathsf{return}\, [\![{C'}^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!] @(x_1) \\ & \quad \text{(by case (2) and (E\_LetE))} \\ = & E^{\overline{C'}, C^{\mathrm{fin}}}[v^{\mathsf{k}}\, [\![V]\!]_{\mathcal{D}'}\, \overline{v^{\mathsf{h}\square}}] \ . \end{array}$$

Because $C.T = T$ and ${C'}^{\mathrm{ini}} = C^{\mathrm{ini}}$, We have the conclusion (case (2)) by letting $\overline{C} = \overline{C'}, C^{\mathrm{fin}}$.

Case 3: We are given some σ, V', x, M', T^{par}, T^{ari}, v, $\mathcal{D}_1$, and $\mathcal{D}'_2$ such that

- $M \ \longrightarrow^* \ \sigma(V'; x.\, M')$,
- $[\![M]\!]^{\mathsf{e}}_{\mathcal{D}''} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \hookrightarrow \mathsf{let}\, x = v\, [\![V']\!]_{\mathcal{D}_1} \,\mathsf{in}\, [\![M']\!]^{\mathsf{e}}_{\mathcal{D}'_2} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})$,
- $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, \square \in C.\Sigma$,
- v is a value in the sequence $\overline{v^{\mathsf{h}}}$ that corresponds to σ in $C.\Sigma$,
- $\emptyset \vdash_{\mathcal{D}_1} V' : T^{\mathrm{par}}$, and
- $\emptyset, x : T^{\mathrm{ari}} \vdash_{\mathcal{D}'_2} M' : C$.

Then, the conclusion (case (3)) is proven as follows:

$$\begin{array}{rl} & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\ = & \mathsf{let}\, x_1 = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}''} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \,\mathsf{in}\, \mathsf{return}\, [\![{C'}^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!] @(x_1) \\ & \quad \text{(by the above reasoning)} \\ \hookrightarrow & \mathsf{let}\, x_1 = (\mathsf{let}\, x = v\, [\![V']\!]_{\mathcal{D}_1} \,\mathsf{in}\, [\![M']\!]^{\mathsf{e}}_{\mathcal{D}'_2} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})) \,\mathsf{in}\, \mathsf{return}\, [\![{C'}^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!] @(x_1) \\ & \quad \text{(by case (3))} \\ \hookrightarrow & \mathsf{let}\, x = v\, [\![V']\!]_{\mathcal{D}_1} \,\mathsf{in}\, \mathsf{let}\, x_1 = [\![M']\!]^{\mathsf{e}}_{\mathcal{D}'_2} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \,\mathsf{in}\, \mathsf{return}\, [\![{C'}^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!] @(x_1) \\ = & \mathsf{let}\, x = v\, [\![V']\!]_{\mathcal{D}_1} \,\mathsf{in}\, [\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \end{array}$$

for some $\mathcal{D}_2$ such that

$$\frac{\emptyset, x : T^{\mathrm{ari}} \vdash_{\mathcal{D}'_2} M' : C \qquad C <: \Sigma \triangleright T \,/\, A}{\emptyset, x : T^{\mathrm{ari}} \vdash_{\mathcal{D}_2} M' : \Sigma \triangleright T \,/\, A} \ (\text{HT\_SubC}).$$

Note that because $\Sigma = C.\Sigma$, we have

- $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, \square \in \Sigma$ and
- v is a value in the sequence $\overline{v^{\mathsf{h}}}$ that corresponds to σ in Σ.

Case 4: We are given some σ, V', x, M', T^{par}, T^{ari}, $C^{\sigma\mathrm{ini}}$, $C^{\sigma\mathrm{fin}}$, $\overline{C'}$, v, $\overline{h^{\square}}$, $\mathcal{D}_1$, and $\mathcal{D}_2$ such that

- $M \ \longrightarrow^* \ \sigma(V'; x.\, M')$,

- $\overline{h^{\Box}}$ is a sequence of fresh variables such that $|\overline{h^{\Box}}| = |\Box(C.\Sigma)|$,
- $[\![M]\!]^{\mathsf{e}}_{\mathcal{D}''} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \hookrightarrow E^{\overline{C'}}[v\,[\![V']\!]_{\mathcal{D}_1}\,(\lambda x, \overline{h^{\Box}}.[\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2} @ (\overline{v^{\mathsf{h}\not\Box}}, \overline{h^{\Box}}, v^{\mathsf{k}}))]$,
- $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, C^{\sigma\mathrm{ini}} \Rightarrow C^{\sigma\mathrm{fin}} \in C.\Sigma$,
- v is a value in the sequence $\overline{v^{\mathsf{h}}}$ that corresponds to σ in $C.\Sigma$,
- the first and last computation types of $\overline{C'}$ are $C^{\sigma\mathrm{fin}}$ and ${C'}^{\mathrm{fin}}$, respectively,
- $\emptyset \vdash_{\mathcal{D}_1} V' : T^{\mathrm{par}}$, and
- $\emptyset, x : T^{\mathrm{ari}} \vdash_{\mathcal{D}_2} M' : C.\Sigma \triangleright C.T \,/\, {C'}^{\mathrm{ini}} \Rightarrow C^{\sigma\mathrm{ini}}$.

Then, the conclusion (case (4)) is proven as follows:

$$\begin{array}{cl}
 & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & \mathsf{let}\, x_1 = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}''} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})\ \mathsf{in}\ \mathsf{return}\ [\![{C'}^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!] @ (x_1) \\
 & \quad \text{(by the above reasoning)} \\
\hookrightarrow & \mathsf{let}\, x_1 = E^{\overline{C'}}[v\,[\![V']\!]_{\mathcal{D}_1}\,(\lambda x, \overline{h^{\Box}}.[\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2} @ (\overline{v^{\mathsf{h}\not\Box}}, \overline{h^{\Box}}, v^{\mathsf{k}}))]\ \mathsf{in}\ \mathsf{return}\ [\![{C'}^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!] @ (x_1) \\
 & \quad \text{(by case (4))} \\
= & E^{\overline{C'}, C^{\mathrm{fin}}}[v\,[\![V']\!]_{\mathcal{D}_1}\,(\lambda x, \overline{h^{\Box}}.[\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2} @ (\overline{v^{\mathsf{h}\not\Box}}, \overline{h^{\Box}}, v^{\mathsf{k}}))]\ .
\end{array}$$

Note that because $C.\Sigma = \Sigma$ and $C.T = T$ and ${C'}^{\mathrm{ini}} = C^{\mathrm{ini}}$, we have
- $|\overline{h^{\Box}}| = |\Box(\Sigma)|$,
- $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, C^{\sigma\mathrm{ini}} \Rightarrow C^{\sigma\mathrm{fin}} \in \Sigma$,
- v is a value in the sequence $\overline{v^{\mathsf{h}}}$ that corresponds to σ in Σ, and
- $\emptyset, x : T^{\mathrm{ari}} \vdash_{\mathcal{D}_2} M' : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C^{\sigma\mathrm{ini}}$.

Case 5: We are given some M' and $\mathcal{D}'''$ such that
- $M \longrightarrow^{+} M'$,
- $[\![M]\!]^{\mathsf{e}}_{\mathcal{D}''} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \hookrightarrow\longrightarrow^{+}\hookrightarrow [\![M']\!]^{\mathsf{e}}_{\mathcal{D}'''} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})$, and
- $\emptyset \vdash_{\mathcal{D}'''} M' : C$.

Then, the conclusion (case (5)) is proven as follows:

$$\begin{array}{cll}
 & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) & \\
= & \mathsf{let}\, x_1 = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}''} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})\ \mathsf{in}\ \mathsf{return}\ [\![{C'}^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!] @ (x_1) & \text{(by the above reasoning)} \\
\hookrightarrow\longrightarrow^{+}\hookrightarrow & \mathsf{let}\, x_1 = [\![M']\!]^{\mathsf{e}}_{\mathcal{D}'''} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})\ \mathsf{in}\ \mathsf{return}\ [\![{C'}^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!] @ (x_1) & \text{(by case (5) and (E_LetE))} \\
= & [\![C <: \Sigma \triangleright T \,/\, A]\!]^{\mathsf{e}} @ ([\![M']\!]^{\mathsf{e}}_{\mathcal{D}'''}, \overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) & \\
= & [\![M']\!]^{\mathsf{e}}_{\mathcal{D}'} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) &
\end{array}$$

for some $\mathcal{D}'$ such that

$$\frac{\emptyset \vdash_{\mathcal{D}'''} M' : C \quad C <: \Sigma \triangleright T \,/\, A}{\emptyset \vdash M' : \Sigma \triangleright T \,/\, A}\ (\text{HT\_SubC}).$$

Case $C.\Sigma \neq \Sigma$ or $C.T \neq T$ or ${C'}^{\mathrm{ini}} \neq C^{\mathrm{ini}}$: We have $\Box(C.\Sigma) <: \Box(\Sigma)$ by Lemma 3. Let x_1, y_1, and z_2 be fresh variables, and $\overline{h_1^{\Box}}$ be a sequence of fresh variables such that $|\overline{h_1^{\Box}}| = |\Box(\Sigma)|$. Also, let $(\overline{v^{\mathsf{h}\not\Box}}, \overline{v^{\mathsf{h}\Box}}) = \mathit{split}(\overline{v^{\mathsf{h}}}, \Sigma)$. By the definition of the CPS transformation,

$$\begin{array}{cl}
 & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & [\![C <: \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}]\!]^{\mathsf{e}} @ ([\![M]\!]^{\mathsf{e}}_{\mathcal{D}''}, \overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & \mathsf{let}\, x_1 = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}''} @ ([\![\not\Box(\Sigma) <: \not\Box(C.\Sigma)]\!] @ (\overline{v^{\mathsf{h}\not\Box}}), \overline{v^{\mathsf{h}\Box}}, v'^{\mathsf{k}})\ \mathsf{in}\ \mathsf{return}\ [\![{C'}^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!] @ (x_1)
\end{array}$$

where

$$v'^{\mathsf{k}} = \lambda y_1, \overline{h_1^{\Box}}.\mathsf{let}\, z_2 = v^{\mathsf{k}}\,([\![C.T <: T]\!] @ (y_1))\,\overline{h_1^{\Box}}\ \mathsf{in}\ \mathsf{return}\ [\![C^{\mathrm{ini}} <: {C'}^{\mathrm{ini}}]\!] @ (z_2)\ .$$

Let $\overline{v'^{\mathsf{h}\not\Box}} = [\![\not\Box(\Sigma) <: \not\Box(C.\Sigma)]\!] @ (\overline{v^{\mathsf{h}\not\Box}})$. Because $|\overline{v'^{\mathsf{h}\not\Box}}, \overline{v^{\mathsf{h}\Box}}| = |C.\Sigma|$ by Lemmas 20 and 28, we can apply the IH on $\emptyset \vdash_{\mathcal{D}''} M : C$. We proceed by case analysis on the result.

Case 1: Contradictory with $C.A \neq \square$.

Case 2: We are given some V, $\mathcal{D}'''$, and $\overline{C'}$ such that

- $M = \mathsf{return}\ V$,
- $[\![\mathsf{return}\ V]\!]^{\mathsf{e}}_{\mathcal{D}''} @ (\overline{v'^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}, v'^{\mathsf{k}}) \longrightarrow^{*} E^{\overline{C'}}[v'^{\mathsf{k}}\, [\![V]\!]_{\mathcal{D}'''}\, \overline{v^{\mathsf{h}\square}}]$,
- $\emptyset \vdash_{\mathcal{D}'''} V : C.T$, and
- the first and last computation types of $\overline{C'}$ are C'^{ini} and C'^{fin}, respectively.

Note that $(\overline{v'^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}) = \mathit{split}((\overline{v'^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}), C.\Sigma)$. Then:

$$\begin{array}{cl}
& [\![\mathsf{return}\ V]\!]^{\mathsf{e}}_{\mathcal{D}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & \mathsf{let}\, x_1 = [\![\mathsf{return}\ V]\!]^{\mathsf{e}}_{\mathcal{D}''} @ (\overline{v'^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}, v'^{\mathsf{k}})\ \mathsf{in}\ \mathsf{return}\ [\![C'^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!] @ (x_1) \\
& \quad \text{(by the above reasoning)} \\
\longrightarrow^{*} & \mathsf{let}\, x_1 = E^{\overline{C'}}[v'^{\mathsf{k}}\, [\![V]\!]_{\mathcal{D}'''}\, \overline{v^{\mathsf{h}\square}}]\ \mathsf{in}\ \mathsf{return}\ [\![C'^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!] @ (x_1) \\
& \quad \text{(by case (2) and (E_LetE))} \\
= & E^{\overline{C'}, C^{\mathrm{fin}}}[v'^{\mathsf{k}}\, [\![V]\!]_{\mathcal{D}'''}\, \overline{v^{\mathsf{h}\square}}] \\
\longrightarrow^{+} & E^{\overline{C'}, C^{\mathrm{fin}}}[\mathsf{let}\, z_2 = v^{\mathsf{k}}\, ([\![C.T <: T]\!] @ ([\![V]\!]_{\mathcal{D}'''}))\, \overline{v^{\mathsf{h}\square}}\ \mathsf{in}\ \mathsf{return}\ [\![C^{\mathrm{ini}} <: C'^{\mathrm{ini}}]\!] @ (z_2)] \\
& \quad \text{(by (E_Beta), (E_LetV), and Lemma 29)} \\
= & E^{C^{\mathrm{ini}}, \overline{C'}, C^{\mathrm{fin}}}[v^{\mathsf{k}}\, [\![V]\!]_{\mathcal{D}'}\, \overline{v^{\mathsf{h}\square}}]
\end{array}$$

for some $\mathcal{D}'$ such that

$$\frac{\emptyset \vdash_{\mathcal{D}'''} V : C.T \qquad C.T <: T}{\emptyset \vdash_{\mathcal{D}'} V : T}\ (\text{HT\_SubV}).$$

We have the conclusion (case (2)) by letting $\overline{C} = C^{\mathrm{ini}}, \overline{C'}, C^{\mathrm{fin}}$.

Case 3: We are given some σ, V', x, M', T^{par}, T^{ari}, v', $\mathcal{D}_1$, and $\mathcal{D}'_2$ such that

- $M \longrightarrow^{*} \sigma(V'; x.\, M')$,
- $[\![M]\!]^{\mathsf{e}}_{\mathcal{D}''} @ (\overline{v'^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}, v'^{\mathsf{k}}) \hookrightarrow \mathsf{let}\, x = v'\, [\![V']\!]_{\mathcal{D}_1}\ \mathsf{in}\ [\![M']\!]^{\mathsf{e}}_{\mathcal{D}'_2} @ (\overline{v'^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}, v'^{\mathsf{k}})$,
- $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, \square \in C.\Sigma$,
- v' is a value in the sequence $\overline{v'^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}$ that corresponds to σ in $C.\Sigma$,
- $\emptyset \vdash_{\mathcal{D}_1} V' : T^{\mathrm{par}}$, and
- $\emptyset, x : T^{\mathrm{ari}} \vdash_{\mathcal{D}'_2} M' : C$.

Because $\Sigma <: C.\Sigma$ and $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, \square \in C.\Sigma$ and $\square(C.\Sigma) <: \square(\Sigma)$, we have $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, \square \in \Sigma$ by Lemmas 3 and Lemma 6. Let v be a value in the sequence $\overline{v^{\mathsf{h}}}$ that corresponds to σ in Σ (there exists such a value because $|\overline{v^{\mathsf{h}}}| = |\Sigma|$), and let y_2 and y_1 be fresh variables. Because both v and v' are in $\overline{v^{\mathsf{h}\square}}$ and correspond to σ, we have $v = v'$. Then, the conclusion (case (3)) is proven as follows:

$$\begin{array}{cl}
& [\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & \mathsf{let}\, x_1 = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}''} @ (\overline{v'^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}, v'^{\mathsf{k}})\ \mathsf{in}\ \mathsf{return}\ [\![C'^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!] @ (x_1) \\
& \quad \text{(by the above reasoning)} \\
\hookrightarrow & \mathsf{let}\, x_1 = (\mathsf{let}\, x = v\, [\![V']\!]_{\mathcal{D}_1}\ \mathsf{in}\ [\![M']\!]^{\mathsf{e}}_{\mathcal{D}'_2} @ (\overline{v'^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}, v'^{\mathsf{k}}))\ \mathsf{in}\ \mathsf{return}\ [\![C'^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!] @ (x_1) \\
& \quad \text{(by case (3))} \\
\hookrightarrow & \mathsf{let}\, x = v\, [\![V']\!]_{\mathcal{D}_1}\ \mathsf{in}\ \mathsf{let}\, x_1 = [\![M']\!]^{\mathsf{e}}_{\mathcal{D}'_2} @ (\overline{v'^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}, v'^{\mathsf{k}})\ \mathsf{in}\ \mathsf{return}\ [\![C'^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!] @ (x_1) \\
= & \mathsf{let}\, x = v\, [\![V']\!]_{\mathcal{D}_1}\ \mathsf{in}\ [\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})
\end{array}$$

for some $\mathcal{D}_2$ such that

$$\frac{\emptyset, x : T^{\mathrm{ari}} \vdash_{\mathcal{D}'_2} M' : C \qquad C <: \Sigma \triangleright T \,/\, A}{\emptyset, x : T^{\mathrm{ari}} \vdash_{\mathcal{D}_2} M' : \Sigma \triangleright T \,/\, A}\ (\text{HT\_SubC}).$$

Case 4: We are given some σ, V', x, M', T'^{par}, T'^{ari}, $C'^{\sigma\mathrm{ini}}$, $C'^{\sigma\mathrm{fin}}$, $\overline{C'}$, v', $\overline{h'^{\square}}$, $\mathcal{D}'_1$, and $\mathcal{D}'_2$ such that

- $M \longrightarrow^* \sigma(V'; x. M')$,
- $\overline{h'^{\square}}$ is a sequence of fresh variables such that $|\overline{h'^{\square}}| = |\square(C.\Sigma)|$,
- $[\![M]\!]^{\mathsf{e}}_{\mathcal{D}''} @(\overline{v'^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}, v'^{\mathsf{k}}) \hookrightarrow E^{\overline{C'}}[v' \, [\![V']\!]_{\mathcal{D}'_1} \, (\lambda x, \overline{h'^{\square}}.[\![M']\!]^{\mathsf{e}}_{\mathcal{D}'_2} @(\overline{v'^{\mathsf{h}\not\square}}, \overline{h'^{\square}}, v'^{\mathsf{k}}))]$,
- $\sigma : T'^{\mathrm{par}} \rightsquigarrow T'^{\mathrm{ari}} \,/\, C'^{\sigma\mathrm{ini}} \Rightarrow C'^{\sigma\mathrm{fin}} \in C.\Sigma$,
- v' is a value in the sequence $\overline{v'^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}$ that corresponds to σ in $C.\Sigma$,
- the first and last computation types of $\overline{C'}$ are $C'^{\sigma\mathrm{fin}}$ and C'^{fin}, respectively,
- $\emptyset \vdash_{\mathcal{D}'_1} V' : T'^{\mathrm{par}}$, and
- $\emptyset, x : T'^{\mathrm{ari}} \vdash_{\mathcal{D}'_2} M' : C.\Sigma \triangleright C.T \,/\, C'^{\mathrm{ini}} \Rightarrow C'^{\sigma\mathrm{ini}}$.

Note that $\mathit{split}((\overline{v'^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}), C.\Sigma) = (\overline{v'^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}})$.
Because $\Sigma <: C.\Sigma$ and $\sigma : T'^{\mathrm{par}} \rightsquigarrow T'^{\mathrm{ari}} \,/\, C'^{\sigma\mathrm{ini}} \Rightarrow C'^{\sigma\mathrm{fin}} \in C.\Sigma$ and $\square(C.\Sigma) <: \square(\Sigma)$, there exist some T^{par}, T^{ari}, $C^{\sigma\mathrm{ini}}$, $C^{\sigma\mathrm{fin}}$, $\varsigma_1, \cdots, \varsigma_{|\square(\Sigma)|}$, $U_1^{\mathrm{par}}, \cdots, U_{|\square(\Sigma)|}^{\mathrm{par}}$, and $U_1^{\mathrm{ari}}, \cdots, U_{|\square(\Sigma)|}^{\mathrm{ari}}$ such that
- $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, C^{\sigma\mathrm{ini}} \Rightarrow C^{\sigma\mathrm{fin}} \in \Sigma$,
- $T'^{\mathrm{par}} <: T^{\mathrm{par}}$,
- $T^{\mathrm{ari}} <: T'^{\mathrm{ari}}$,
- $C'^{\sigma\mathrm{ini}} <: C^{\sigma\mathrm{ini}}$,
- $C^{\sigma\mathrm{fin}} <: C'^{\sigma\mathrm{fin}}$, and
- $\square(\Sigma) = \square(C.\Sigma) = \{\varsigma_i : U_i^{\mathrm{par}} \rightsquigarrow U_i^{\mathrm{ari}} \,/\, \square\}^{1 \le i \le |\square(\Sigma)|}$

by Lemmas 3, 20 and 6.
Let v be a value in the sequence $\overline{v^{\mathsf{h}}}$ that corresponds to σ in Σ (there exists such a value because $|\overline{v^{\mathsf{h}}}| = |\Sigma|$). Also, let x_2, k_2, x'_1, y_1, y_2, $h_1, \cdots, h_{|\square(\Sigma)|}$ be fresh variables. By the definition of the CPS transformation, we have

$$v' = \lambda x_2, k_2.\mathsf{let}\, x'_1 = v\, ([\![T'^{\mathrm{par}} <: T^{\mathrm{par}}]\!]@(x_2))\, v'' \,\mathsf{in\ return}\, [\![C^{\sigma\mathrm{fin}} <: C'^{\sigma\mathrm{fin}}]\!]@(x'_1)$$

for some v'' and $\overline{w_i}^{1 \le i \le |\square(\Sigma)|}$ such that
- $v'' = \lambda y_1, \overline{h_i}^{1 \le i \le |\square(\Sigma)|}.\mathsf{let}\, y_2 = k_2\, ([\![T^{\mathrm{ari}} <: T'^{\mathrm{ari}}]\!]@(y_1))\, \overline{w_i}^{1 \le i \le |\square(\Sigma)|} \,\mathsf{in\ return}\, [\![C'^{\sigma\mathrm{ini}} <: C^{\sigma\mathrm{ini}}]\!]@(y_2)$
- $\forall\, i \in [1, |\square(\Sigma)|].\ w_i = \lambda z_2.\mathsf{let}\, z_1 = h_i\, ([\![U_i^{\mathrm{par}} <: U_i^{\mathrm{par}}]\!]@(z_2)) \,\mathsf{in\ return}\, [\![U_i^{\mathrm{ari}} <: U_i^{\mathrm{ari}}]\!]@(z_1)$.

Let $v''' = \lambda x, \overline{h'^{\square}}.[\![M']\!]^{\mathsf{e}}_{\mathcal{D}'_2}@(\overline{v'^{\mathsf{h}\not\square}}, \overline{h'^{\square}}, v'^{\mathsf{k}})$. Furthermore, we write $\overline{h_i}$ and $\overline{w_i}$ for the sequences $\overline{h_i}^{1 \le i \le |\square(\Sigma)|}$ and $\overline{w_i}^{1 \le i \le |\square(\Sigma)|}$, respectively. Then,

$$\begin{array}{rl}
& v''[v'''/k_2] \\
= & \lambda y_1, \overline{h_i}.\mathsf{let}\, y_2 = v'''\, ([\![T^{\mathrm{ari}} <: T'^{\mathrm{ari}}]\!]@(y_1))\, \overline{w_i} \,\mathsf{in\ return}\, [\![C'^{\sigma\mathrm{ini}} <: C^{\sigma\mathrm{ini}}]\!]@(y_2) \\
= & \lambda y_1, \overline{h_i}.\mathsf{let}\, y_2 = v'''\, [\![y_1]\!]_{\mathcal{D}^{\mathsf{y}_1}}\, \overline{w_i} \,\mathsf{in\ return}\, [\![C'^{\sigma\mathrm{ini}} <: C^{\sigma\mathrm{ini}}]\!]@(y_2) \\
\hookrightarrow & \lambda y_1, \overline{h_i}.\mathsf{let}\, y_2 = v'''\, [\![y_1]\!]_{\mathcal{D}^{\mathsf{y}_1}}\, \overline{h_i} \,\mathsf{in\ return}\, [\![C'^{\sigma\mathrm{ini}} <: C^{\sigma\mathrm{ini}}]\!]@(y_2) \\
& \quad \text{(as } \overline{w_i} \hookrightarrow \overline{h_i} \text{ by Lemma 26)} \\
\hookrightarrow & \lambda y_1, \overline{h_i}.\mathsf{let}\, y_2 = [\![M']\!]^{\mathsf{e}}_{\mathcal{D}'_2}@(\overline{v'^{\mathsf{h}\not\square}}, \overline{h_i}, v'^{\mathsf{k}})[[\![y_1]\!]_{\mathcal{D}^{\mathsf{y}_1}}/x] \,\mathsf{in\ return}\, [\![C'^{\sigma\mathrm{ini}} <: C^{\sigma\mathrm{ini}}]\!]@(y_2) \\
= & \lambda x, \overline{h_i}.\mathsf{let}\, y_2 = [\![M']\!]^{\mathsf{e}}_{\mathcal{D}''_2}@(\overline{v'^{\mathsf{h}\not\square}}, \overline{h_i}, v'^{\mathsf{k}}) \,\mathsf{in\ return}\, [\![C'^{\sigma\mathrm{ini}} <: C^{\sigma\mathrm{ini}}]\!]@(y_2) \\
& \quad \text{(by Lemmas 30, 31, and 32)} \\
= & \lambda x, \overline{h_i}.[\![C.\Sigma \triangleright C.T \,/\, C'^{\mathrm{ini}} \Rightarrow C'^{\sigma\mathrm{ini}} <: \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C^{\sigma\mathrm{ini}}]\!]^{\mathsf{e}}@([\![M']\!]^{\mathsf{e}}_{\mathcal{D}''_2}, \overline{v^{\mathsf{h}\not\square}}, \overline{h_i}, v^{\mathsf{k}}) \\
= & \lambda x, \overline{h_i}.[\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2}@(\overline{v^{\mathsf{h}\not\square}}, \overline{h_i}, v^{\mathsf{k}})
\end{array}$$

for some $\mathcal{D}^{\mathsf{y}_1}$, $\mathcal{D}''_2$, and $\mathcal{D}_2$ such that

$$\dfrac{\dfrac{}{\emptyset, y_1 : T^{\mathrm{ari}} \vdash y_1 : T^{\mathrm{ari}}}\,(\mathrm{HT\_Var}) \qquad T^{\mathrm{ari}} <: T'^{\mathrm{ari}}}{\emptyset, y_1 : T^{\mathrm{ari}} \vdash_{\mathcal{D}^{\mathsf{y}_1}} y_1 : T'^{\mathrm{ari}}}\,(\mathrm{HT\_SubV}),$$

$$\emptyset, x : T^{\mathrm{ari}} \vdash_{\mathcal{D}''_2} M' : C.\Sigma \triangleright C.T \,/\, C'^{\mathrm{ini}} \Rightarrow C'^{\sigma\mathrm{ini}} \;,$$

and

$$\frac{\emptyset, x : T^{\mathrm{ari}} \vdash_{\mathcal{D}_2''} M' : C.\Sigma \triangleright C.T \,/\, C'^{\mathrm{ini}} \Rightarrow C'^{\sigma\mathrm{ini}} \quad C.\Sigma \triangleright C.T \,/\, C'^{\mathrm{ini}} \Rightarrow C'^{\sigma\mathrm{ini}} <: \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C^{\sigma\mathrm{ini}}}{\emptyset, x : T^{\mathrm{ari}} \vdash_{\mathcal{D}_2} M' : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C^{\sigma\mathrm{ini}}} \text{ (HT\_SubC)}.$$

Therefore, the conclusion (case (4)) is proven as follows:

$$\begin{array}{ll}
& [\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & \mathsf{let}\, x_1 = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}''} @(\overline{v'^{\mathsf{h}\not\Box}}, \overline{v^{\mathsf{h}\Box}}, v'^{\mathsf{k}})\, \mathsf{in\, return}\, [\![C'^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!] @(x_1) \\
& \quad \text{(by the above reasoning)} \\
\hookrightarrow & \mathsf{let}\, x_1 = E^{\overline{C'}}[v' \,[\![V']\!]_{\mathcal{D}_1'}\, (\lambda x, \overline{h'^{\Box}}.[\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2'} @(\overline{v'^{\mathsf{h}\not\Box}}, \overline{h'^{\Box}}, v'^{\mathsf{k}}))]\, \mathsf{in\, return}\, [\![C'^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!] @(x_1) \\
& \quad \text{(by case (4))} \\
= & E^{\overline{C'}, C^{\mathrm{fin}}}[v' \,[\![V']\!]_{\mathcal{D}_1'}\, v'''] \\
\hookrightarrow & E^{\overline{C'}, C^{\mathrm{fin}}}[\mathsf{let}\, x_1' = v\, ([\![T'^{\mathrm{par}} <: T^{\mathrm{par}}]\!] @([\![V']\!]_{\mathcal{D}_1'}))\, v''[v'''/k_2]\, \mathsf{in\, return}\, [\![C^{\sigma\mathrm{fin}} <: C'^{\sigma\mathrm{fin}}]\!] @(x_1')] \\
= & E^{C^{\sigma\mathrm{fin}}, \overline{C'}, C^{\mathrm{fin}}}[v \,[\![V']\!]_{\mathcal{D}_1}\, v''[v'''/k_2]] \\
\hookrightarrow & E^{C^{\sigma\mathrm{fin}}, \overline{C'}, C^{\mathrm{fin}}}[v \,[\![V']\!]_{\mathcal{D}_1}\, (\lambda x, \overline{h_i}.[\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2} @(\overline{v^{\mathsf{h}\not\Box}}, \overline{h_i}, v^{\mathsf{k}}))] \\
& \quad \text{(by the above reasoning)}
\end{array}$$

for some $\mathcal{D}_1$ such that

$$\frac{\emptyset \vdash_{\mathcal{D}_1'} V : T'^{\mathrm{par}} \qquad T'^{\mathrm{par}} <: T^{\mathrm{par}}}{\emptyset \vdash_{\mathcal{D}_1} V : T^{\mathrm{par}}} \text{ (HT\_SubV)}.$$

Case 5: We are given some M' and $\mathcal{D}'''$ such that

- $M \longrightarrow^+ M'$,
- $[\![M]\!]^{\mathsf{e}}_{\mathcal{D}''} @(\overline{v'^{\mathsf{h}\not\Box}}, \overline{v^{\mathsf{h}\Box}}, v'^{\mathsf{k}}) \hookrightarrow\longrightarrow^+\hookrightarrow [\![M']\!]^{\mathsf{e}}_{\mathcal{D}'''} @(\overline{v'^{\mathsf{h}\not\Box}}, \overline{v^{\mathsf{h}\Box}}, v'^{\mathsf{k}})$, and
- $\emptyset \vdash_{\mathcal{D}'''} M' : C$.

Then, the conclusion (case (5)) is proven as follows:

$$\begin{array}{lll}
& [\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) & \\
= & \mathsf{let}\, x_1 = [\![M]\!]^{\mathsf{e}}_{\mathcal{D}''} @(\overline{v'^{\mathsf{h}\not\Box}}, \overline{v^{\mathsf{h}\Box}}, v'^{\mathsf{k}})\, \mathsf{in\, return}\, [\![C'^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!] @(x_1) & \text{(by the above reasoning)} \\
\hookrightarrow\longrightarrow^+\hookrightarrow & \mathsf{let}\, x_1 = [\![M']\!]^{\mathsf{e}}_{\mathcal{D}'''} @(\overline{v'^{\mathsf{h}\not\Box}}, \overline{v^{\mathsf{h}\Box}}, v'^{\mathsf{k}})\, \mathsf{in\, return}\, [\![C'^{\mathrm{fin}} <: C^{\mathrm{fin}}]\!] @(x_1) & \text{(by case (5) and (E_LetE))} \\
= & [\![C <: \Sigma \triangleright T \,/\, A]\!]^{\mathsf{e}} @([\![M']\!]^{\mathsf{e}}_{\mathcal{D}'''}, \overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) & \\
= & [\![M']\!]^{\mathsf{e}}_{\mathcal{D}'} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) &
\end{array}$$

for some $\mathcal{D}'$ such that

$$\frac{\emptyset \vdash_{\mathcal{D}'''} M' : C \quad C <: \Sigma \triangleright T \,/\, A}{\emptyset \vdash M' : \Sigma \triangleright T \,/\, A} \text{ (HT\_SubC)}.$$

Case (HT_Let): We are given

$$\frac{\emptyset \vdash_{\mathcal{D}_1'} M_1 : \Sigma \triangleright T_1 \,/\, \Box \qquad \emptyset, x : T_1 \vdash_{\mathcal{D}_2'} M_2 : \Sigma \triangleright T \,/\, A}{\emptyset \vdash_{\mathcal{D}} \mathsf{let}\, x = M_1\, \mathsf{in}\, M_2 : \Sigma \triangleright T \,/\, A}$$

for some x, M_1, M_2, T_1, $\mathcal{D}_1'$, and $\mathcal{D}_2'$ such that $M = (\mathsf{let}\, x = M_1\, \mathsf{in}\, M_2)$. Without loss of generality, we can assume that $x \notin fv(\overline{v^{\mathsf{h}}}) \cup fv(v^{\mathsf{k}})$. We proceed by case analysis on the result of applying the IH on $\emptyset \vdash_{\mathcal{D}_1'} M_1 : \Sigma \triangleright T_1 \,/\, \Box$.

Case 1: We are given some V_1 and $\mathcal{D}_1''$ such that

- $M_1 = \mathsf{return}\ V_1$,
- $[\![\mathsf{return}\ V_1]\!]^{\mathsf{e}}_{\mathcal{D}_1'} @(\overline{v^{\mathsf{h}}}, \underline{1}) \longrightarrow^* \mathsf{return}\ [\![V_1]\!]_{\mathcal{D}_1''}$, and

- $\emptyset \vdash_{\mathcal{D}_1''} V : T_1$.

By (HE_LetV),

$$M = (\mathsf{let}\, x = \mathsf{return}\ V_1 \,\mathsf{in}\, M_2) \longrightarrow M_2[V_1/x]\ .$$

Then, the conclusion (case (5)) is proven as follows:

$$\begin{array}{cl} & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\ = & \mathsf{let}\, x = [\![\mathsf{return}\ V_1]\!]^{\mathsf{e}}_{\mathcal{D}_1'}@(\overline{v^{\mathsf{h}}}, \underline{1}) \,\mathsf{in}\, [\![M_2]\!]^{\mathsf{e}}_{\mathcal{D}_2'}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\ \longrightarrow^* & \mathsf{let}\, x = \mathsf{return}\, [\![V_1]\!]_{\mathcal{D}_1''} \,\mathsf{in}\, [\![M_2]\!]^{\mathsf{e}}_{\mathcal{D}_2'}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \quad \text{(by case (1) and (E\_LetE))} \\ \longrightarrow & [\![M_2]\!]^{\mathsf{e}}_{\mathcal{D}_2'}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})[[\![V_1]\!]_{\mathcal{D}_1''}/x] \quad \text{(by (E\_LetV))} \\ = & [\![M_2[V_1/x]]\!]^{\mathsf{e}}_{\mathcal{D}'}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \quad \text{(by Lemmas 31 and 32)} \end{array}$$

for some $\mathcal{D}'$ such that $\emptyset \vdash M_2[V_1/x] : \Sigma \triangleright T \,/\, A$.

Case 2: Contradictory.

Case 3: We are given some σ, V', y, M_1', T^{par}, T^{ari}, v, $\mathcal{D}_1$, and $\mathcal{D}_2''$ such that

- $M_1 \longrightarrow^* \sigma(V'; y.\, M_1')$,
- $[\![M_1]\!]^{\mathsf{e}}_{\mathcal{D}_1'}@(\overline{v^{\mathsf{h}}}, \underline{1}) \hookrightarrow \mathsf{let}\, y = v\, [\![V']\!]_{\mathcal{D}_1} \,\mathsf{in}\, [\![M_1']\!]^{\mathsf{e}}_{\mathcal{D}_2''}@(\overline{v^{\mathsf{h}}}, \underline{1})$,
- $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, \Box \in \Sigma$,
- v is a value in the sequence $\overline{v^{\mathsf{h}}}$ that corresponds to σ in Σ,
- $\emptyset \vdash_{\mathcal{D}_1} V' : T^{\mathrm{par}}$, and
- $\emptyset, y : T^{\mathrm{ari}} \vdash_{\mathcal{D}_2''} M_1' : \Sigma \triangleright T_1 \,/\, \Box$.

Without loss of generality, we can assume that $y \notin fv(M_2)$. By (HE_LetE) and (HE_LetOp),

$$M = (\mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2) \longrightarrow^* (\mathsf{let}\, x = \sigma(V'; y.\, M_1') \,\mathsf{in}\, M_2) \longrightarrow \sigma(V'; y.\, \mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2)\ .$$

Then, the conclusion (case (3)) is proven as follows:

$$\begin{array}{cl} & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\ = & \mathsf{let}\, x = [\![M_1]\!]^{\mathsf{e}}_{\mathcal{D}_1'}@(\overline{v^{\mathsf{h}}}, \underline{1}) \,\mathsf{in}\, [\![M_2]\!]^{\mathsf{e}}_{\mathcal{D}_2'}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\ \hookrightarrow & \mathsf{let}\, x = (\mathsf{let}\, y = v\, [\![V']\!]_{\mathcal{D}_1} \,\mathsf{in}\, [\![M_1']\!]^{\mathsf{e}}_{\mathcal{D}_2''}@(\overline{v^{\mathsf{h}}}, \underline{1})) \,\mathsf{in}\, [\![M_2]\!]^{\mathsf{e}}_{\mathcal{D}_2'}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\ \hookrightarrow & \mathsf{let}\, y = v\, [\![V']\!]_{\mathcal{D}_1} \,\mathsf{in}\, \mathsf{let}\, x = [\![M_1']\!]^{\mathsf{e}}_{\mathcal{D}_2''}@(\overline{v^{\mathsf{h}}}, \underline{1}) \,\mathsf{in}\, [\![M_2]\!]^{\mathsf{e}}_{\mathcal{D}_2'}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\ = & \mathsf{let}\, y = v\, [\![V']\!]_{\mathcal{D}_1} \,\mathsf{in}\, [\![\mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2]\!]^{\mathsf{e}}_{\mathcal{D}_2}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \quad \text{(by Lemma 30)} \end{array}$$

for some $\mathcal{D}_2$ and $\mathcal{D}_2'''$ such that

$$\frac{\emptyset, y : T^{\mathrm{ari}} \vdash_{\mathcal{D}_2''} M_1' : \Sigma \triangleright T_1 \,/\, \Box \qquad \emptyset, y : T^{\mathrm{ari}}, x : T_1 \vdash_{\mathcal{D}_2'''} M_2 : \Sigma \triangleright T \,/\, A}{\emptyset, y : T^{\mathrm{ari}} \vdash_{\mathcal{D}_2} \mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2 : \Sigma \triangleright T \,/\, A}\ \text{(HT\_Let)}.$$

Case 4: Contradictory.

Case 5: We are given some M_1' and $\mathcal{D}_1''$ such that

- $M_1 \longrightarrow^+ M_1'$,
- $[\![M_1]\!]^{\mathsf{e}}_{\mathcal{D}_1'}@(\overline{v^{\mathsf{h}}}, \underline{1}) \hookrightarrow\longrightarrow^+\hookrightarrow [\![M_1']\!]^{\mathsf{e}}_{\mathcal{D}_1''}@(\overline{v^{\mathsf{h}}}, \underline{1})$, and
- $\emptyset \vdash_{\mathcal{D}_1''} M_1' : \Sigma \triangleright T_1 \,/\, \Box$.

By (E_LetE),

$$M = (\mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2) \longrightarrow^+ \mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2\ .$$

Then, the conclusion (case (5)) is proven as follows:

$$\begin{array}{cl} & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\ = & \mathsf{let}\, x = [\![M_1]\!]^{\mathsf{e}}_{\mathcal{D}_1'}@(\overline{v^{\mathsf{h}}}, \underline{1}) \,\mathsf{in}\, [\![M_2]\!]^{\mathsf{e}}_{\mathcal{D}_2'}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\ \hookrightarrow\longrightarrow^+\hookrightarrow & \mathsf{let}\, x = [\![M_1']\!]^{\mathsf{e}}_{\mathcal{D}_1''}@(\overline{v^{\mathsf{h}}}, \underline{1}) \,\mathsf{in}\, [\![M_2]\!]^{\mathsf{e}}_{\mathcal{D}_2'}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \quad \text{(by case (5) and (E\_LetE))} \\ = & [\![\mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2]\!]^{\mathsf{e}}_{\mathcal{D}'}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \end{array}$$

for some $\mathcal{D}'$ such that

$$\frac{\emptyset \vdash_{\mathcal{D}_1''} M_1' : \Sigma \triangleright T_1 \,/\, \square \qquad \emptyset, x : T_1 \vdash_{\mathcal{D}_2'} M_2 : \Sigma \triangleright T \,/\, A}{\emptyset \vdash_{\mathcal{D}'} \mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2 : \Sigma \triangleright T \,/\, A} \text{(HT\_LET)}.$$

Case (HT_LETATM): We are given

$$\frac{\emptyset \vdash_{\mathcal{D}_1'} M_1 : \Sigma \triangleright T_1 \,/\, C \Rightarrow C^{\mathrm{fin}} \qquad \emptyset, x : T_1 \vdash_{\mathcal{D}_2'} M_2 : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C}{\emptyset \vdash_{\mathcal{D}} \mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2 : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}}$$

for some x, M_1, M_2, T_1, C, C^{fin}, C^{ini}, $\mathcal{D}_1'$, and $\mathcal{D}_2'$ such that $M = (\mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2)$ and $A = C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}$. Without loss of generality, we can assume that $x \notin fv(\overline{v^{\mathsf{h}}}) \cup fv(v^{\mathsf{k}})$. Let $(\overline{v^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}) = split(\overline{v^{\mathsf{h}}}, \Sigma)$, and $\overline{h^{\square}}$ be a sequence of fresh variables such that $|\overline{h^{\square}}| = |\square(\Sigma)|$. By the definition of the CPS transformation,

$$[\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) = [\![M_1]\!]^{\mathsf{e}}_{\mathcal{D}_1'} @(\overline{v^{\mathsf{h}}}, \lambda x, \overline{h^{\square}}. [\![M_2]\!]^{\mathsf{e}}_{\mathcal{D}_2'} @(\overline{v^{\mathsf{h}\not\square}}, \overline{h^{\square}}, v^{\mathsf{k}})) \;.$$

Let $v'^{\mathsf{k}} = \lambda x, \overline{h^{\square}}. [\![M_2]\!]^{\mathsf{e}}_{\mathcal{D}_2'} @(\overline{v^{\mathsf{h}\not\square}}, \overline{h^{\square}}, v^{\mathsf{k}})$. We proceed by case analysis on the result of applying the IH on $\emptyset \vdash_{\mathcal{D}_1'} M_1 : \Sigma \triangleright T_1 \,/\, C \Rightarrow C^{\mathrm{fin}}$.

Case 1: Contradictory.

Case 2: We are given some V_1, $\mathcal{D}_1$, and $\overline{C'}$ such that

- $M_1 = \mathsf{return}\ V_1$,
- $[\![\mathsf{return}\ V_1]\!]^{\mathsf{e}}_{\mathcal{D}_1'} @(\overline{v^{\mathsf{h}}}, v'^{\mathsf{k}}) \longrightarrow^* E^{\overline{C'}}[v'^{\mathsf{k}}\, [\![V_1]\!]_{\mathcal{D}_1}\, \overline{v^{\mathsf{h}\square}}]$,
- $\emptyset \vdash_{\mathcal{D}_1} V_1 : T_1$, and
- the first and last computation types of $\overline{C'}$ are C and C^{fin}, respectively.

By (HE_LETV),

$$M = (\mathsf{let}\, x = \mathsf{return}\ V_1 \,\mathsf{in}\, M_2) \longrightarrow M_2[V_1/x] \;.$$

Then, the conclusion (case (5)) is proven as follows:

$$\begin{array}{cl}
 & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & [\![\mathsf{return}\ V_1]\!]^{\mathsf{e}}_{\mathcal{D}_1'} @(\overline{v^{\mathsf{h}}}, v'^{\mathsf{k}}) \\
\longrightarrow^* & E^{\overline{C'}}[v'^{\mathsf{k}}\, [\![V_1]\!]_{\mathcal{D}_1}\, \overline{v^{\mathsf{h}\square}}] \quad \text{(by case (2))} \\
\longrightarrow^+ & E^{\overline{C'}}[[\![M_2]\!]^{\mathsf{e}}_{\mathcal{D}_2'} @(\overline{v^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}, v^{\mathsf{k}})[[\![V_1]\!]_{\mathcal{D}_1}/x]] \quad \text{(by Lemma 29)} \\
= & E^{\overline{C'}}[[\![M_2[V_1/x]]\!]^{\mathsf{e}}_{\mathcal{D}_2''} @(\overline{v^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}, v^{\mathsf{k}})] \quad \text{(by Lemmas 31 and 32)} \\
= & [\![M_2[V_1/x]]\!]^{\mathsf{e}}_{\mathcal{D}'} @(\overline{v^{\mathsf{h}\not\square}}, \overline{v^{\mathsf{h}\square}}, v^{\mathsf{k}}) \quad \text{(by Lemma 27)}
\end{array}$$

for some $\mathcal{D}_2''$ and $\mathcal{D}'$ such that

$$\emptyset \vdash_{\mathcal{D}_2''} M_2[V_1/x] : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C$$

and

$$\emptyset \vdash_{\mathcal{D}'} M_2[V_1/x] : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}} \;.$$

Case 3: We are given some σ, V', y, M_1', T^{par}, T^{ari}, v, $\mathcal{D}_1$, and $\mathcal{D}_2''$ such that

- $M_1 \longrightarrow^* \sigma(V'; y.\, M_1')$,
- $[\![M_1]\!]^{\mathsf{e}}_{\mathcal{D}_1'} @(\overline{v^{\mathsf{h}}}, v'^{\mathsf{k}}) \hookrightarrow \mathsf{let}\, y = v\, [\![V']\!]_{\mathcal{D}_1} \,\mathsf{in}\, [\![M_1']\!]^{\mathsf{e}}_{\mathcal{D}_2''} @(\overline{v^{\mathsf{h}}}, v'^{\mathsf{k}})$,
- $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, \square \in \Sigma$,
- v is a value in the sequence $\overline{v^{\mathsf{h}}}$ that corresponds to σ in Σ,
- $\emptyset \vdash_{\mathcal{D}_1} V' : T^{\mathrm{par}}$, and
- $\emptyset, y : T^{\mathrm{ari}} \vdash_{\mathcal{D}_2''} M_1' : \Sigma \triangleright T_1 \,/\, C \Rightarrow C^{\mathrm{fin}}$.

Without loss of generality, we can assume that $y \notin fv(M_2)$. By (HE_LetE) and (HE_LetOp),

$$M = (\mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2) \longrightarrow^* \mathsf{let}\, x = \sigma(V'; y.\, M_1') \,\mathsf{in}\, M_2 \longrightarrow \sigma(V'; y.\, \mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2) \ .$$

Then, the conclusion (case (3)) is proven as follows:

$$\begin{array}{ll} & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\ = & [\![M_1]\!]^{\mathsf{e}}_{\mathcal{D}_1'} @ (\overline{v^{\mathsf{h}}}, v'^{\mathsf{k}}) \\ \hookrightarrow & \mathsf{let}\, y = v\, [\![V']\!]_{\mathcal{D}_1} \,\mathsf{in}\, [\![M_1']\!]^{\mathsf{e}}_{\mathcal{D}_2''} @ (\overline{v^{\mathsf{h}}}, \lambda x, \overline{h^{\square}}. [\![M_2]\!]^{\mathsf{e}}_{\mathcal{D}_2'} @ (\overline{v^{\mathsf{h}\not\square}}, \overline{h^{\square}}, v^{\mathsf{k}})) \\ = & \mathsf{let}\, y = v\, [\![V']\!]_{\mathcal{D}_1} \,\mathsf{in}\, [\![\mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2]\!]^{\mathsf{e}}_{\mathcal{D}_2} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \quad \text{(by Lemma 30)} \end{array}$$

for some $\mathcal{D}_2$ and $\mathcal{D}_2'''$ such that

$$\frac{\emptyset, y : T^{\mathrm{ari}} \vdash_{\mathcal{D}_2''} M_1' : \Sigma \triangleright T_1 \,/\, C \Rightarrow C^{\mathrm{fin}} \qquad \emptyset, y : T^{\mathrm{ari}}, x : T_1 \vdash_{\mathcal{D}_2'''} M_2 : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C}{\emptyset, y : T^{\mathrm{ari}} \vdash_{\mathcal{D}_2} \mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2 : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}} \text{(HT\_LetATM)}.$$

Case 4: We are given some σ, V', y, M_1', T^{par}, T^{ari}, $C^{\sigma\mathrm{ini}}$, $C^{\sigma\mathrm{fin}}$, $\overline{C'}$, v, $\overline{h'^{\square}}$, $\mathcal{D}_1$, and $\mathcal{D}_2''$ such that

- $M_1 \longrightarrow^* \sigma(V'; y.\, M_1')$,
- $\overline{h'^{\square}}$ is a sequence of fresh variables such that $|\overline{h'^{\square}}| = |\square(\Sigma)|$,
- $[\![M_1]\!]^{\mathsf{e}}_{\mathcal{D}_1'} @ (\overline{v^{\mathsf{h}}}, v'^{\mathsf{k}}) \hookrightarrow E^{\overline{C'}}[v\, [\![V']\!]_{\mathcal{D}_1} (\lambda y, \overline{h'^{\square}}. [\![M_1']\!]^{\mathsf{e}}_{\mathcal{D}_2''} @ (\overline{v^{\mathsf{h}\not\square}}, \overline{h'^{\square}}, v'^{\mathsf{k}}))]$,
- $\sigma : T^{\mathrm{par}} \rightsquigarrow T^{\mathrm{ari}} \,/\, C^{\sigma\mathrm{ini}} \Rightarrow C^{\sigma\mathrm{fin}} \in \Sigma$,
- v is a value in the sequence $\overline{v^{\mathsf{h}}}$ that corresponds to σ in Σ,
- the first and last computation types of $\overline{C'}$ are $C^{\sigma\mathrm{fin}}$ and C^{fin},
- $\emptyset \vdash_{\mathcal{D}_1} V' : T^{\mathrm{par}}$, and
- $\emptyset, y : T^{\mathrm{ari}} \vdash_{\mathcal{D}_2''} M_1' : \Sigma \triangleright T_1 \,/\, C \Rightarrow C^{\sigma\mathrm{ini}}$.

Without loss of generality, we can assume that $y \notin fv(M_2)$. By (HE_LetE) and (HE_LetOp),

$$M = (\mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2) \longrightarrow^* \mathsf{let}\, x = \sigma(V'; y.\, M_1') \,\mathsf{in}\, M_2 \longrightarrow \sigma(V'; y.\, \mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2) \ .$$

Then, the conclusion (case (4)) is proven as follows:

$$\begin{array}{ll} & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\ = & [\![M_1]\!]^{\mathsf{e}}_{\mathcal{D}_1'} @ (\overline{v^{\mathsf{h}}}, v'^{\mathsf{k}}) \\ \hookrightarrow & E^{\overline{C'}}[v\, [\![V']\!]_{\mathcal{D}_1} (\lambda y, \overline{h'^{\square}}. [\![M_1']\!]^{\mathsf{e}}_{\mathcal{D}_2''} @ (\overline{v^{\mathsf{h}\not\square}}, \overline{h'^{\square}}, v'^{\mathsf{k}}))] \\ = & E^{\overline{C'}}[v\, [\![V']\!]_{\mathcal{D}_1} (\lambda y, \overline{h'^{\square}}. [\![M_1']\!]^{\mathsf{e}}_{\mathcal{D}_2''} @ (\overline{v^{\mathsf{h}\not\square}}, \overline{h'^{\square}}, \lambda x, \overline{h^{\square}}. [\![M_2]\!]^{\mathsf{e}}_{\mathcal{D}_2'} @ (\overline{v^{\mathsf{h}\not\square}}, \overline{h^{\square}}, v^{\mathsf{k}})))] \\ = & E^{\overline{C'}}[v\, [\![V']\!]_{\mathcal{D}_1} (\lambda y, \overline{h'^{\square}}. [\![\mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2]\!]^{\mathsf{e}}_{\mathcal{D}_2} @ (\overline{v^{\mathsf{h}\not\square}}, \overline{h'^{\square}}, v^{\mathsf{k}}))] \quad \text{(by Lemma 30)} \end{array}$$

for some $\mathcal{D}_2$ and $\mathcal{D}_2'''$ such that

$$\frac{\emptyset, y : T^{\mathrm{ari}} \vdash_{\mathcal{D}_2''} M_1' : \Sigma \triangleright T_1 \,/\, C \Rightarrow C^{\sigma\mathrm{ini}} \qquad \emptyset, y : T^{\mathrm{ari}}, x : T_1 \vdash_{\mathcal{D}_2'''} M_2 : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C}{\emptyset, y : T^{\mathrm{ari}} \vdash_{\mathcal{D}_2} \mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2 : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C^{\sigma\mathrm{ini}}} \text{(HT\_LetATM)}.$$

Case 5: We are given some M_1' and $\mathcal{D}_1''$ such that

- $M_1 \longrightarrow^+ M_1'$,
- $[\![M_1]\!]^{\mathsf{e}}_{\mathcal{D}_1'} @ (\overline{v^{\mathsf{h}}}, v'^{\mathsf{k}}) \hookrightarrow\longrightarrow^+\hookrightarrow [\![M_1']\!]^{\mathsf{e}}_{\mathcal{D}_1''} @ (\overline{v^{\mathsf{h}}}, v'^{\mathsf{k}})$, and
- $\emptyset \vdash_{\mathcal{D}_1''} M_1' : \Sigma \triangleright T_1 \,/\, C \Rightarrow C^{\mathrm{fin}}$.

By (HE_LetE),

$$M = (\mathsf{let}\, x = M_1 \,\mathsf{in}\, M_2) \longrightarrow^+ \mathsf{let}\, x = M_1' \,\mathsf{in}\, M_2 \ .$$

Then, the conclusion (case (5)) is proven as follows:

$$\begin{array}{cl} & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\ = & [\![M_1]\!]^{\mathsf{e}}_{\mathcal{D}'_1}@(\overline{v^{\mathsf{h}}}, v'^{\mathsf{k}}) \\ \hookrightarrow\longrightarrow^{+}\hookrightarrow & [\![M'_1]\!]^{\mathsf{e}}_{\mathcal{D}''_1}@(\overline{v^{\mathsf{h}}}, v'^{\mathsf{k}}) \quad \text{(by case (5))} \\ = & [\![\mathsf{let}\, x = M'_1 \,\mathsf{in}\, M_2]\!]^{\mathsf{e}}_{\mathcal{D}'}@(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \end{array}$$

for some $\mathcal{D}'$ such that

$$\frac{\emptyset \vdash_{\mathcal{D}''_1} M'_1 : \Sigma \triangleright T_1 \,/\, C \Rightarrow C^{\mathrm{fin}} \qquad \emptyset, x : T_1 \vdash_{\mathcal{D}'_2} M_2 : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C}{\emptyset \vdash \mathsf{let}\, x = M'_1 \,\mathsf{in}\, M_2 : \Sigma \triangleright T \,/\, C^{\mathrm{ini}} \Rightarrow C^{\mathrm{fin}}} \text{(HT\_LetATM)}.$$

Case (HT_App): We are given

$$\frac{\emptyset \vdash_{\mathcal{D}_1} V_1 : T' \to \Sigma \triangleright T \,/\, A \qquad \emptyset \vdash_{\mathcal{D}_2} V_2 : T'}{\emptyset \vdash_{\mathcal{D}} V_1\ V_2 : \Sigma \triangleright T \,/\, A}$$

for some V_1, V_2, T', $\mathcal{D}_1$, and $\mathcal{D}_2$ such that $M = V_1\ V_2$. By case analysis on the result of applying Lemma 7 to $\emptyset \vdash_{\mathcal{D}_1} V_1 : T' \to \Sigma \triangleright T \,/\, A$.

Case $\exists\, x, M_1.\ V_1 = \lambda x.M_1$: The conclusion (case (5)) is proven by (HE_Beta) and Lemma 33.
Case $\exists\, x, V'_1.\ V_1 = \mathsf{fix}\, x.\, V'_1$: The conclusion (case (5)) is proven by (HE_Fix) and Lemma 34.

Case (HT_Case): We are given

$$\frac{\emptyset \vdash_{\mathcal{D}'} V : \mathsf{n} \qquad \forall\, i \in [1, n].\ \emptyset \vdash_{\mathcal{D}_i} M_i : \Sigma \triangleright T \,/\, A}{\emptyset \vdash_{\mathcal{D}} \mathsf{case}(V; M_1, \cdots, M_n) : \Sigma \triangleright T \,/\, A}$$

for some V, n, $M_1, \cdots, M_n$, $\mathcal{D}'$, and $\mathcal{D}_1, \cdots, \mathcal{D}_n$ such that $M = \mathsf{case}(V; M_1, \cdots, M_n)$. The conclusion (case (5)) is proven by Lemma 7, (HE_Case), and Lemma 35.

Case (HT_Op): The conclusion (case (3)) holds obviously.

Case (HT_OpATM): The conclusion (case (4)) holds obviously.

Case (HT_Handle): We are given

$$\frac{\begin{array}{c} H = \{\mathsf{return}\, x \ \mapsto\ L\} \uplus \{\sigma_i(y_i; k_i) \ \mapsto\ M_i\}^{1 \le i \le m} \uplus \{\varsigma_i(z_i) \ \mapsto\ N_i\}^{1 \le i \le n} \\ \Sigma' = \{\sigma_i : T_i^{\mathrm{par}} \rightsquigarrow T_i^{\mathrm{ari}} \,/\, C_i^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{fin}}\}^{1 \le i \le m} \uplus \{\varsigma_i : U_i^{\mathrm{par}} \rightsquigarrow U_i^{\mathrm{ari}} \,/\, \square\}^{1 \le i \le n} \\ \emptyset \vdash_{\mathcal{D}_0} M_0 : \Sigma' \triangleright T' \,/\, C'^{\mathrm{ini}} \Rightarrow C'^{\mathrm{fin}} \qquad \emptyset, x : T' \vdash_{\mathcal{D}^{\mathsf{e}}} L : C'^{\mathrm{ini}} \\ \forall\, i \in [1, m].\ \emptyset, y_i : T_i^{\mathrm{par}}, k_i : T_i^{\mathrm{ari}} \to C_i^{\mathrm{ini}} \vdash_{\mathcal{D}^{\sigma_i}} M_i : C_i^{\mathrm{fin}} \qquad \forall\, i \in [1, n].\ \emptyset, z_i : U_i^{\mathrm{par}} \vdash_{\mathcal{D}^{\varsigma_i}} N_i : \Sigma_i \triangleright U_i^{\mathrm{ari}} \,/\, \square \\ \forall\, C \in \{\overline{C_i^{\mathrm{ini}}}^{1 \le i \le m}, C'^{\mathrm{fin}}\}.\ \forall\, i \in [1, n].\ C.\Sigma <: \Sigma_i \qquad C'^{\mathrm{fin}} = \Sigma \triangleright T \,/\, A \end{array}}{\emptyset \vdash_{\mathcal{D}} \mathsf{with}\, H \,\mathsf{handle}\, M_0 : \Sigma \triangleright T \,/\, A}$$

for some H, M_0, x, L, Σ', $\sigma_1, \cdots, \sigma_m$, $y_1 \cdots, y_m$ $k_1 \cdots, k_m$, $M_1, \cdots, M_m$, $T_1^{\mathrm{par}}, \cdots, T_m^{\mathrm{par}}$, $T_1^{\mathrm{ari}}, \cdots, T_m^{\mathrm{ari}}$, $C_1^{\mathrm{ini}}, \cdots, C_m^{\mathrm{ini}}$, $C_1^{\mathrm{fin}}, \cdots, C_m^{\mathrm{fin}}$, $\varsigma_1, \cdots, \varsigma_n$, $z_1, \cdots, z_n$, $N_1, \cdots, N_n$, $U_1^{\mathrm{par}}, \cdots, U_n^{\mathrm{par}}$, $U_1^{\mathrm{ari}}, \cdots, U_n^{\mathrm{ari}}$, $\Sigma_1, \cdots, \Sigma_n$, T', C'^{ini}, C'^{fin}, $\mathcal{D}_0$, $\mathcal{D}^{\mathsf{e}}$, $\mathcal{D}^{\sigma_1}, \cdots, \mathcal{D}^{\sigma_m}$, and $\mathcal{D}^{\varsigma_1}, \cdots, \mathcal{D}^{\varsigma_n}$ such that $M = \mathsf{with}\, H \,\mathsf{handle}\, M_0$.

For $i \in [1, n]$, let $\mathcal{D}_i^{\square}$ be a typing derivation such that

$$\frac{\emptyset, z_i : U_i^{\mathrm{par}} \vdash_{\mathcal{D}^{\varsigma_i}} N_i : \Sigma_i \triangleright U_i^{\mathrm{ari}} \,/\, \square \qquad \Sigma_i \triangleright U_i^{\mathrm{ari}} \,/\, \square <: \Sigma \triangleright U_i^{\mathrm{ari}} \,/\, \square}{\emptyset, z_i : U_i^{\mathrm{par}} \vdash_{\mathcal{D}_i^{\square}} N_i : \Sigma \triangleright U_i^{\mathrm{ari}} \,/\, \square} \text{(HT\_SubC)},$$

and, for $i \in [1, m], j \in [1, n]$, $\mathcal{D}_{ij}^{\boxslash}$ be a typing derivation such that

$$\frac{\emptyset, z_j : U_j^{\mathrm{par}} \vdash_{\mathcal{D}^{\varsigma_j}} N_j : \Sigma_j \triangleright U_j^{\mathrm{ari}} \,/\, \square \qquad \Sigma_j \triangleright U_j^{\mathrm{ari}} \,/\, \square <: C_i^{\mathrm{ini}}.\Sigma \triangleright U_j^{\mathrm{ari}} \,/\, \square}{\emptyset, z_j : U_j^{\mathrm{par}} \vdash_{\mathcal{D}_{ij}^{\boxslash}} N_j : C_i^{\mathrm{ini}}.\Sigma \triangleright U_j^{\mathrm{ari}} \,/\, \square} \text{(HT\_SubC)}.$$

Furthermore, let

- $\forall\, i \in [1,m].\ w_i^{\boxslash} = \lambda y_i, k'_i.\mathsf{let}\, k_i = \mathsf{return}\,(\lambda y, \overline{h}, k. k'_i\, y\, \overline{v_{ij}}^{1 \le j \le n}\, \overline{h}\, k)\,\mathsf{in}\,\mathsf{return}\, [\![M_i]\!]_{\mathcal{D}^{\sigma_i}}$ (where k'_i, y, k are fresh variables, and $\overline{h}$ is a sequence of fresh variables such that $|\overline{h}| = |C_i^{\mathrm{ini}}.\Sigma|$),
- $\forall\, i \in [1,m].\ \forall\, j \in [1,n].\ v_{ij} = \lambda z_j.[\![N_j]\!]^{\mathsf{e}}_{\mathcal{D}_{ij}^{\boxslash}} @(\overline{h}, \underline{1})$,
- $\forall\, j \in [1,n].\ w_i^{\square} = \lambda z_i.[\![N_i]\!]^{\mathsf{e}}_{\mathcal{D}_i^{\square}} @(\overline{v^{\mathsf{h}}}, \underline{1})$, and
- $\overline{h^{\square}}$ be a sequence of fresh variables such that $|\overline{h^{\square}}| = n$.

Then,

$$[\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) = [\![M_0]\!]^{\mathsf{e}}_{\mathcal{D}_0} @(\overline{w_i^{\boxslash}}^{1 \le i \le m}, \overline{w_i^{\square}}^{1 \le i \le n}, \lambda x, \overline{h^{\square}}.\mathsf{return}\, [\![L]\!]_{\mathcal{D}^{\mathsf{e}}})\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}}\ .$$

Let

- $(\overline{v^{\mathsf{h}\boxslash}}, \overline{v^{\mathsf{h}\square}}) = \mathit{split}(\overline{v^{\mathsf{h}}}, \Sigma)$,
- $\overline{w^{\boxslash}} = \overline{w_i^{\boxslash}}^{1 \le i \le m}$, and
- $\overline{w^{\square}} = \overline{w_i^{\square}}^{1 \le i \le n}$.

We proceed by case analysis on the result of the IH on $\emptyset \vdash_{\mathcal{D}_0} M_0 : \Sigma' \triangleright T' \,/\, C'^{\mathrm{ini}} \Rightarrow C'^{\mathrm{fin}}$.

Case 1: Contradictory.

Case 2: We are given some V_0, $\mathcal{D}'_0$, and $\overline{C'}$ such that

- $M_0 = \mathsf{return}\ V_0$,
- $[\![\mathsf{return}\ V_0]\!]^{\mathsf{e}}_{\mathcal{D}_0} @(\overline{w^{\boxslash}}, \overline{w^{\square}}, \lambda x, \overline{h^{\square}}.\mathsf{return}\, [\![L]\!]_{\mathcal{D}^{\mathsf{e}}}) \longrightarrow^{*} E^{\overline{C'}}[(\lambda x, \overline{h^{\square}}.\mathsf{return}\, [\![L]\!]_{\mathcal{D}^{\mathsf{e}}})\, [\![V_0]\!]_{\mathcal{D}'_0}\, \overline{v^{\mathsf{h}\square}}]$,
- $\emptyset \vdash_{\mathcal{D}'_0} V_0 : T'$, and
- the first and last computation types of $\overline{C'}$ are C'^{ini} and C'^{fin}, respectively.

By (HE_HandleV),

$$M = \mathsf{with}\ H\ \mathsf{handle}\ \mathsf{return}\ V_0 \longrightarrow L[V_0/x]\ .$$

Then, the conclusion (case (5)) is proven as follows:

$$\begin{array}{ll}
 & [\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & [\![M_0]\!]^{\mathsf{e}}_{\mathcal{D}_0} @(\overline{w^{\boxslash}}, \overline{w^{\square}}, \lambda x, \overline{h^{\square}}.\mathsf{return}\, [\![L]\!]_{\mathcal{D}^{\mathsf{e}}})\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \\
\longrightarrow^{*} & E^{\overline{C'}}[(\lambda x, \overline{h^{\square}}.\mathsf{return}\, [\![L]\!]_{\mathcal{D}^{\mathsf{e}}})\, [\![V_0]\!]_{\mathcal{D}'_0}\, \overline{v^{\mathsf{h}\square}}]\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \quad \text{(by case (2) and (E_LetE))} \\
\longrightarrow^{+} & E^{\overline{C'}}[\mathsf{return}\, [\![L]\!]_{\mathcal{D}^{\mathsf{e}}}[[\![V_0]\!]_{\mathcal{D}'_0}/x]]\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \quad \text{(by (E_Beta), (E_LetV), and Lemma 29)} \\
= & E^{\overline{C'}}[\mathsf{return}\, [\![L[V_0/x]]\!]_{\mathcal{D}'^{\mathsf{e}}}]\, \overline{v^{\mathsf{h}}}\, v^{\mathsf{k}} \quad \text{(by Lemma 31)} \\
\hookrightarrow & [\![L[V_0/x]]\!]^{\mathsf{e}}_{\mathcal{D}'} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \quad \text{(by Lemma 36)}
\end{array}$$

for some $\mathcal{D}'^{\mathsf{e}}$ and $\mathcal{D}'$ such that $\emptyset \vdash_{\mathcal{D}'^{\mathsf{e}}} L[V_0/x] : C'^{\mathrm{ini}}$ and $\emptyset \vdash_{\mathcal{D}'} L[V_0/x] : C'^{\mathrm{fin}}$. Note that $C'^{\mathrm{fin}} = \Sigma \triangleright T \,/\, A$.

Case 3: We are given some $i \in [1,n]$, V', y, M'_0, $\mathcal{D}_1$, and $\mathcal{D}_2$ such that

- $M_0 \longrightarrow^{*} \varsigma_i(V'; y.\, M'_0)$,
- $[\![M_0]\!]^{\mathsf{e}}_{\mathcal{D}_0} @(\overline{w^{\boxslash}}, \overline{w^{\square}}, \lambda x, \overline{h^{\square}}.\mathsf{return}\, [\![L]\!]_{\mathcal{D}^{\mathsf{e}}}) \hookrightarrow \mathsf{let}\, y = w_i^{\square}\, [\![V']\!]_{\mathcal{D}_1}\, \mathsf{in}\, [\![M'_0]\!]^{\mathsf{e}}_{\mathcal{D}_2} @(\overline{w^{\boxslash}}, \overline{w^{\square}}, \lambda x, \overline{h^{\square}}.\mathsf{return}\, [\![L]\!]_{\mathcal{D}^{\mathsf{e}}})$,
- $\emptyset \vdash_{\mathcal{D}_1} V' : U_i^{\mathrm{par}}$, and
- $\emptyset, y : U_i^{\mathrm{ari}} \vdash_{\mathcal{D}_2} M'_0 : \Sigma' \triangleright T' \,/\, C'^{\mathrm{ini}} \Rightarrow C'^{\mathrm{fin}}$.

By (HE_HandleE) and (HE_HandleOpTail),

$$M = \mathsf{with}\ H\ \mathsf{handle}\ M_0 \longrightarrow^{*} \mathsf{with}\ H\ \mathsf{handle}\, \varsigma_i(V'; y.\, M'_0) \longrightarrow \mathsf{let}\, y = N_i[V'/z_i]\, \mathsf{in}\, \mathsf{with}\ H\ \mathsf{handle}\ M'_0\ .$$

Then, the conclusion (case (5)) is proven as follows:

$$
\begin{array}{rl}
& \llbracket M \rrbracket^{\mathsf{e}}_{\mathcal{D}} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
= & \llbracket M_0 \rrbracket^{\mathsf{e}}_{\mathcal{D}_0} @ (\overline{w^{\boxslash}}, \overline{w^{\Box}}, \lambda x, \overline{h^{\Box}}.\mathsf{return}\ \llbracket L \rrbracket_{\mathcal{D}^{\mathsf{e}}})\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} \\
\hookrightarrow & (\mathsf{let}\ y = w_i^{\Box}\ \llbracket V' \rrbracket_{\mathcal{D}_1}\ \mathsf{in}\ \llbracket M_0' \rrbracket^{\mathsf{e}}_{\mathcal{D}_2} @ (\overline{w^{\boxslash}}, \overline{w^{\Box}}, \lambda x, \overline{h^{\Box}}.\mathsf{return}\ \llbracket L \rrbracket_{\mathcal{D}^{\mathsf{e}}}))\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} \quad \text{(by case (3))} \\
\longrightarrow & (\mathsf{let}\ y = \llbracket N_i \rrbracket^{\mathsf{e}}_{\mathcal{D}_i^{\Box}} @ (\overline{v^{\mathsf{h}}}, \underline{1})[\llbracket V' \rrbracket_{\mathcal{D}_1}/z_i]\ \mathsf{in}\ \llbracket M_0' \rrbracket^{\mathsf{e}}_{\mathcal{D}_2} @ (\overline{w^{\boxslash}}, \overline{w^{\Box}}, \lambda x, \overline{h^{\Box}}.\mathsf{return}\ \llbracket L \rrbracket_{\mathcal{D}^{\mathsf{e}}}))\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} \\
& \quad \text{(by (E\_LetE) and (E\_Beta))} \\
= & (\mathsf{let}\ y = \llbracket N_i[V'/z_i] \rrbracket^{\mathsf{e}}_{\mathcal{D}_1'} @ (\overline{v^{\mathsf{h}}}, \underline{1})\ \mathsf{in}\ \llbracket M_0' \rrbracket^{\mathsf{e}}_{\mathcal{D}_2} @ (\overline{w^{\boxslash}}, \overline{w^{\Box}}, \lambda x, \overline{h^{\Box}}.\mathsf{return}\ \llbracket L \rrbracket_{\mathcal{D}^{\mathsf{e}}}))\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} \\
& \quad \text{(by Lemmas 31 and 32)} \\
\hookrightarrow & \mathsf{let}\ y = \llbracket N_i[V'/z_i] \rrbracket^{\mathsf{e}}_{\mathcal{D}_1'} @ (\overline{v^{\mathsf{h}}}, \underline{1})\ \mathsf{in}\ \llbracket M_0' \rrbracket^{\mathsf{e}}_{\mathcal{D}_2} @ (\overline{w^{\boxslash}}, \overline{w^{\Box}}, \lambda x, \overline{h^{\Box}}.\mathsf{return}\ \llbracket L \rrbracket_{\mathcal{D}^{\mathsf{e}}})\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} \\
= & \mathsf{let}\ y = \llbracket N_i[V'/z_i] \rrbracket^{\mathsf{e}}_{\mathcal{D}_1'} @ (\overline{v^{\mathsf{h}}}, \underline{1})\ \mathsf{in}\ \llbracket \mathsf{with}\ H\ \mathsf{handle}\ M_0' \rrbracket^{\mathsf{e}}_{\mathcal{D}_2'} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\
& \quad \text{(by Lemma 30)} \\
= & \llbracket \mathsf{let}\ y = N_i[V'/z_i]\ \mathsf{in}\ \mathsf{with}\ H\ \mathsf{handle}\ M_0' \rrbracket^{\mathsf{e}}_{\mathcal{D}'} @ (\overline{v^{\mathsf{h}}}, v^{\mathsf{k}})
\end{array}
$$

for some $\mathcal{D}_1'$, $\mathcal{D}_2'$, and $\mathcal{D}'$ such that

$$\emptyset \vdash_{\mathcal{D}_1'} N_i[V'/z_i] : \Sigma \triangleright U_i^{\mathrm{ari}} \,/\, \Box \ ,$$

$$y : U_i^{\mathrm{ari}} \vdash \mathsf{with}\ H\ \mathsf{handle}\ M_0' : {C'}^{\mathrm{fin}} \ ,$$

and

$$\frac{\emptyset \vdash_{\mathcal{D}_1'} N_i[V'/z_i] : \Sigma \triangleright U_i^{\mathrm{ari}} \,/\, \Box \quad y : U_i^{\mathrm{ari}} \vdash \mathsf{with}\ H\ \mathsf{handle}\ M_0' : {C'}^{\mathrm{fin}} \quad {C'}^{\mathrm{fin}} = \Sigma \triangleright T \,/\, A}{\emptyset \vdash_{\mathcal{D}'} \mathsf{let}\ y = N_i[V'/z_i]\ \mathsf{in}\ \mathsf{with}\ H\ \mathsf{handle}\ M_0' : \Sigma \triangleright T \,/\, A} \ \text{(HT\_Let)}.$$

Case 4: We are given some $i \in [1, m]$, V', y, M_0', $\overline{C}$, $\overline{h'^{\Box}}$, $\mathcal{D}_1$, and $\mathcal{D}_2$ such that

- $M_0 \longrightarrow^* \sigma_i(V'; y.\, M_0')$,
- $\overline{h'^{\Box}}$ is a sequence of fresh variables such that $|\overline{h'^{\Box}}| = n$,
- $\llbracket M_0 \rrbracket^{\mathsf{e}}_{\mathcal{D}_0} @ (\overline{w^{\boxslash}}, \overline{w^{\Box}}, \lambda x, \overline{h^{\Box}}.\mathsf{return}\ \llbracket L \rrbracket_{\mathcal{D}^{\mathsf{e}}}) \hookrightarrow E^{\overline{C}}[w_i^{\boxslash}\ \llbracket V' \rrbracket_{\mathcal{D}_1}\ (\lambda y, \overline{h'^{\Box}}.\llbracket M_0' \rrbracket^{\mathsf{e}}_{\mathcal{D}_2} @ (\overline{w^{\boxslash}}, \overline{h'^{\Box}}, \lambda x, \overline{h^{\Box}}.\mathsf{return}\ \llbracket L \rrbracket_{\mathcal{D}^{\mathsf{e}}}))]$,
- the first and last computation types of $\overline{C}$ are C_i^{fin} and ${C'}^{\mathrm{fin}}$, respectively,
- $\emptyset \vdash_{\mathcal{D}_1} V' : T_i^{\mathrm{par}}$, and
- $\emptyset, y : T_i^{\mathrm{ari}} \vdash_{\mathcal{D}_2} M_0' : \Sigma' \triangleright T' \,/\, {C'}^{\mathrm{ini}} \Rightarrow C_i^{\mathrm{ini}}$.

By (HE_HandleE) and (HE_HandleOp),

$$M = \mathsf{with}\ H\ \mathsf{handle}\ M_0 \longrightarrow^* \mathsf{with}\ H\ \mathsf{handle}\ \sigma_i(V'; y.\, M_0') \longrightarrow M_i[V'/y_i][\lambda y.\mathsf{with}\ H\ \mathsf{handle}\ M_0'/k_i] \ .$$

Then, we have

$$
\begin{array}{rl}
& w_i^{\boxslash}\ \llbracket V' \rrbracket_{\mathcal{D}_1}\ (\lambda y, \overline{h'^{\Box}}.\llbracket M_0' \rrbracket^{\mathsf{e}}_{\mathcal{D}_2} @ (\overline{w^{\boxslash}}, \overline{h'^{\Box}}, \lambda x, \overline{h^{\Box}}.\mathsf{return}\ \llbracket L \rrbracket_{\mathcal{D}^{\mathsf{e}}})) \\
\longrightarrow^+ & \mathsf{let}\ k_i = \mathsf{return}\ (\lambda y, \overline{h}, k.(\lambda y, \overline{h'^{\Box}}.\llbracket M_0' \rrbracket^{\mathsf{e}}_{\mathcal{D}_2} @ (\overline{w^{\boxslash}}, \overline{h'^{\Box}}, \lambda x, \overline{h^{\Box}}.\mathsf{return}\ \llbracket L \rrbracket_{\mathcal{D}^{\mathsf{e}}}))\ y\ \overline{v_{ij}}^{1 \le j \le n}\ \overline{h}\ k)\ \mathsf{in}\ \mathsf{return}\ \llbracket M_i \rrbracket_{\mathcal{D}^{\sigma_i}}[\llbracket V' \rrbracket_{\mathcal{D}_1}/y_i] \\
\hookrightarrow & \mathsf{let}\ k_i = \mathsf{return}\ (\lambda y, \overline{h}, k.\llbracket M_0' \rrbracket^{\mathsf{e}}_{\mathcal{D}_2} @ (\overline{w^{\boxslash}}, \overline{v_{ij}}^{1 \le j \le n}, \lambda x, \overline{h^{\Box}}.\mathsf{return}\ \llbracket L \rrbracket_{\mathcal{D}^{\mathsf{e}}})\ \overline{h}\ k)\ \mathsf{in}\ \mathsf{return}\ \llbracket M_i \rrbracket_{\mathcal{D}^{\sigma_i}}[\llbracket V' \rrbracket_{\mathcal{D}_1}/y_i] \\
= & \mathsf{let}\ k_i = \mathsf{return}\ (\lambda y, \overline{h}, k.\llbracket \mathsf{with}\ H\ \mathsf{handle}\ M_0' \rrbracket^{\mathsf{e}}_{\mathcal{D}''} @ (\overline{h}, k))\ \mathsf{in}\ \mathsf{return}\ \llbracket M_i \rrbracket_{\mathcal{D}^{\sigma_i}}[\llbracket V' \rrbracket_{\mathcal{D}_1}/y_i] \quad \text{(by Lemma 30)} \\
= & \mathsf{let}\ k_i = \mathsf{return}\ \llbracket \lambda y.\mathsf{with}\ H\ \mathsf{handle}\ M_0' \rrbracket_{\mathcal{D}'''}\ \mathsf{in}\ \mathsf{return}\ \llbracket M_i \rrbracket_{\mathcal{D}^{\sigma_i}}[\llbracket V' \rrbracket_{\mathcal{D}_1}/y_i] \\
\hookrightarrow & \mathsf{return}\ \llbracket M_i \rrbracket_{\mathcal{D}^{\sigma_i}}[\llbracket V' \rrbracket_{\mathcal{D}_1}/y_i][\llbracket \lambda y.\mathsf{with}\ H\ \mathsf{handle}\ M_0' \rrbracket_{\mathcal{D}'''}/k_i] \\
= & \mathsf{return}\ \llbracket M_i[V'/y_i][\lambda y.\mathsf{with}\ H\ \mathsf{handle}\ M_0'/k_i] \rrbracket_{\mathcal{D}'^{\sigma_i}} \quad \text{(by Lemmas 31 and 32)}
\end{array}
$$

for some $\mathcal{D}''$, $\mathcal{D}'''$, and $\mathcal{D}'^{\sigma_i}$ such that

$$\emptyset, y : T_i^{\mathrm{ari}} \vdash_{\mathcal{D}''} \mathsf{with}\ H\ \mathsf{handle}\ M_0' : C_i^{\mathrm{ini}} \ ,$$

$$\emptyset \vdash_{\mathcal{D}'''} \lambda y.\mathsf{with}\ H\ \mathsf{handle}\ M_0' : T_i^{\mathrm{ari}} \to C_i^{\mathrm{ini}} \ ,$$

and

$$\emptyset \vdash_{\mathcal{D}'^{\sigma_i}} M_i[V'/y_i][\lambda y.\mathsf{with}\ H\ \mathsf{handle}\ M'_0/k_i] : C_i^{\mathrm{fin}} \ .$$

Therefore, the conclusion (case (5)) is proven as follows:

$$\begin{array}{cl} & \llbracket M \rrbracket^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\ = & \llbracket M_0 \rrbracket^{\mathsf{e}}_{\mathcal{D}_0} @(\overline{w^{\not\square}}, \overline{w^{\square}}, \lambda x, \overline{h^{\square}}.\mathsf{return}\ \llbracket L \rrbracket_{\mathcal{D}^{\mathsf{e}}})\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} \\ \hookrightarrow & E^{\overline{C}}[w_i^{\not\square}\ \llbracket V' \rrbracket_{\mathcal{D}_1}\ (\lambda y, \overline{h'^{\square}}.\llbracket M'_0 \rrbracket^{\mathsf{e}}_{\mathcal{D}_2} @(\overline{w^{\not\square}}, \overline{h'^{\square}}, \lambda x, \overline{h^{\square}}.\mathsf{return}\ \llbracket L \rrbracket_{\mathcal{D}^{\mathsf{e}}}))]\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} \\ & \quad \text{(by case (4))} \\ \longrightarrow^{+}\hookrightarrow & E^{\overline{C}}[\mathsf{return}\ \llbracket M_i[V'/y_i][\lambda y.\mathsf{with}\ H\ \mathsf{handle}\ M'_0/k_i] \rrbracket_{\mathcal{D}'^{\sigma_i}}]\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} \\ & \quad \text{(by the above reasoning and Lemma 29)} \\ \hookrightarrow & \llbracket M_i[V'/y_i][\lambda y.\mathsf{with}\ H\ \mathsf{handle}\ M'_0/k_i] \rrbracket^{\mathsf{e}}_{\mathcal{D}'} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\ & \quad \text{(by Lemma 36)} \end{array}$$

for some $\mathcal{D}'$ such that $\emptyset \vdash_{\mathcal{D}'} M_i[V'/y_i][\lambda y.\mathsf{with}\ H\ \mathsf{handle}\ M'_0/k_i] : C'^{\mathrm{fin}}$.

Case 5: We are given some M'_0 and $\mathcal{D}'_0$ such that

- $M_0 \longrightarrow^{+} M'_0$,
- $\llbracket M_0 \rrbracket^{\mathsf{e}}_{\mathcal{D}_0} @(\overline{w^{\not\square}}, \overline{w^{\square}}, \lambda x, \overline{h^{\square}}.\mathsf{return}\ \llbracket L \rrbracket_{\mathcal{D}^{\mathsf{e}}}) \hookrightarrow\longrightarrow^{+}\hookrightarrow \llbracket M'_0 \rrbracket^{\mathsf{e}}_{\mathcal{D}'_0} @(\overline{w^{\not\square}}, \overline{w^{\square}}, \lambda x, \overline{h^{\square}}.\mathsf{return}\ \llbracket L \rrbracket_{\mathcal{D}^{\mathsf{e}}})$, and
- $\emptyset \vdash_{\mathcal{D}'_0} M'_0 : \Sigma' \triangleright T' \,/\, C'^{\mathrm{ini}} \Rightarrow C'^{\mathrm{fin}}$.

By (HE_HandleE),

$$M = \mathsf{with}\ H\ \mathsf{handle}\ M_0 \longrightarrow^{+} \mathsf{with}\ H\ \mathsf{handle}\ M'_0 \ .$$

The conclusion (case (5)) is proven as follows:

$$\begin{array}{cl} & \llbracket M \rrbracket^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \\ = & \llbracket M_0 \rrbracket^{\mathsf{e}}_{\mathcal{D}_0} @(\overline{w^{\not\square}}, \overline{w^{\square}}, \lambda x, \overline{h^{\square}}.\mathsf{return}\ \llbracket L \rrbracket_{\mathcal{D}^{\mathsf{e}}})\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} \\ \hookrightarrow\longrightarrow^{+}\hookrightarrow & \llbracket M'_0 \rrbracket^{\mathsf{e}}_{\mathcal{D}'_0} @(\overline{w^{\not\square}}, \overline{w^{\square}}, \lambda x, \overline{h^{\square}}.\mathsf{return}\ \llbracket L \rrbracket_{\mathcal{D}^{\mathsf{e}}})\ \overline{v^{\mathsf{h}}}\ v^{\mathsf{k}} \quad \text{(by case (5))} \\ = & \llbracket \mathsf{with}\ H\ \mathsf{handle}\ M'_0 \rrbracket^{\mathsf{e}}_{\mathcal{D}'} @(\overline{v^{\mathsf{h}}}, v^{\mathsf{k}}) \end{array}$$

for some $\mathcal{D}'$ such that $\emptyset \vdash_{\mathcal{D}'} \mathsf{with}\ H\ \mathsf{handle}\ M'_0 : C'^{\mathrm{fin}}$.

□

Theorem 2 (Contextual Equivalence of Evaluation)**.** *If $\Xi \parallel \Delta \vdash e_1 : \tau$ and $\Xi \parallel \Delta \vdash e_2 : \tau$ and $e_1 \longrightarrow^{+} e_2$, then $\Xi \parallel \Delta \vdash e_1 \overset{\mathsf{ctx}}{=} e_2 : \tau$.*

Proof. This can be proved directly following the strategy in [1], where the underlying monad is the free one, namely the one of (possibly infinite) trees whose nodes are labelled with values from finite types. The idea is that if $e_1 \longrightarrow^{+} e_2$, then e_1 and e_2 either both diverge or they reduce to the same value, and are thus trivially bisimilar, applicatively. This then implies that they are contextual equivalent, given the soundness of applicative bisimilarity for contextual equivalence. □

Theorem 3 (Contextual Improvement of Full $\beta\eta$ Monadic Reduction)**.** *If $\Xi \parallel \Delta \vdash e_1 : \tau$ and $\Xi \parallel \Delta \vdash e_2 : \tau$ and $e_1 \hookrightarrow e_2$, then $\Xi \parallel \Delta \vdash e_2 \overset{\mathsf{ctx}}{\leq} e_1 : \tau$.*

Proof. Again, this can be proved following the strategy in [1]. More specifically:

- The fact that if $e_1 \hookrightarrow e_2$, then $\Xi \parallel \Delta \vdash e_1 \overset{\mathsf{ctx}}{=} e_2 : \tau$ can be proved by observing that if $e_1 \hookrightarrow e_2$, then e_1 and e_2 are applicatively bisimilar where the underlying monad is the one trees whose nodes are labelled with values from finite types, exactly as in Theorem 2.

- We also have to prove that if $e_1 \hookrightarrow e_2$ then for every context P, it holds that $P[e_2]$ does not take more steps to be evaluated than $P[e_1]$ can be proved by considering *another* monad, namely the monad $X \mapsto 1+(\mathbb{N}\times(X+1))$, whereas a value in the left component of the coproduct signals divergence, while a pair (n,r) in the right component of the coproduct signals convergence in n steps to r, which can be either a value of an unspecified operation σ. In such a monad, e_1 and e_2 are applicatively *similar*, and since applicative similarity is included in the contextual preorder, we are done.

$\square$

Lemma 38 (Evaluation in $\mathsf{HEPCF}^{\mathsf{ATM}}_{\square}$ is Deterministic)**.** If $M \longrightarrow M_1$ and $M \longrightarrow M_2$, then $M_1 = M_2$.

Proof. Straightforward by induction on the derivation of $M \longrightarrow M_1$. $\square$

Lemma 39 (Well-Definedness of $\mathsf{HEPCF}^{\mathsf{ATM}}_{\square}$ Effect Trees)**.** If $\emptyset \vdash M : \Sigma \triangleright T \,/\, A$ and Σ is top-level, then $\mathbf{ET}(M)$ is well defined and uniquely determined, and it is in $\mathbf{Tree}_{S^{\Sigma}_{T}}$.

Proof. We show that $\mathbf{ET}(M) \in \mathbf{Tree}_{S^{\Sigma}_{T}}$ by coinduction. We proceed by case analysis on the evaluation of M.

Case $M \longrightarrow^{\omega}$: Obvious.

Case $\exists\, V.\ M \longrightarrow^{*} \mathsf{return}\ V$: By the definition, $\mathbf{ET}(M) = \mathsf{return}\ V$. By Lemma 14, $\emptyset \vdash \mathsf{return}\ V : \Sigma \triangleright T \,/\, A$. By Lemmas 10 and 3 and (HT_SubV), we have $\emptyset \vdash V : T$. Thus, $\mathsf{return}\ V \in \mathbf{Tree}_{S^{\Sigma}_{T}}$.

Case $\exists\, \sigma, V, x, M'.\ M \longrightarrow^{*} \sigma(V; x.\, M')$: By Lemma 14, $\emptyset \vdash \sigma(V; x.\, M') : \Sigma \triangleright T \,/\, A$. By Lemma 8 to $\emptyset \vdash \sigma(V; x.\, M') : \Sigma \triangleright T \,/\, A$, the assumption that Σ is top-level, and (HT_SubC) we have

- $\sigma : B \rightsquigarrow \mathsf{n} \,/\, \square \in \Sigma$,
- $\emptyset \vdash V : B$, and
- $x : \mathsf{n} \vdash M' : \Sigma \triangleright T \,/\, A$

for some B and n. By Lemma 7, $V = c$ such that $ty(c) = B$. Then, by the definition,

$$\mathbf{ET}(M) = \sigma(c, \mathbf{ET}(M'[\underline{1}/x]), \cdots, \mathbf{ET}(M'[\underline{\mathsf{n}}/x]))\ .$$

Thus, by the coinduction principle, it suffices to show that, for any $i \in [1, n]$, $\emptyset \vdash M'[\underline{\mathsf{i}}/x] : \Sigma \triangleright T \,/\, A$, which is shown by Lemma 2 with $x : \mathsf{n} \vdash M' : \Sigma \triangleright T \,/\, A$ and $\emptyset \vdash \underline{\mathsf{i}} : \mathsf{n}$.

Otherwise: Contradictory with Lemmas 14 and 9.

The uniqueness of $\mathbf{ET}(M)$ is shown by Lemma 38. $\square$

Lemma 40 (Evaluation in EPCF is Deterministic)**.** If $e \longrightarrow e_1$ and $e \longrightarrow e_2$, then $e_1 = e_2$.

Proof. Straightforward by induction on the derivation of $e \longrightarrow e_1$. $\square$

Lemma 41 (Well-Definedness of EPCF Effect Trees)**.** If $\Xi \parallel \emptyset \vdash e : \tau$, then $\mathbf{ET}(e)$ is well defined and uniquely determined, and it is in $\mathbf{Tree}_{S^{\Xi}_{\tau}}$.

Proof. We show that $\mathbf{ET}(e) \in \mathbf{Tree}_{S^{\Xi}_{\tau}}$ by coinduction. We proceed by case analysis on the evaluation of e.

Case $e \longrightarrow^{\omega}$: Obvious.

Case $\exists\, v.\ e \longrightarrow^{*} \mathsf{return}\, v$: By the definition, $\mathbf{ET}(e) = \mathsf{return}\, v$. By Lemma 19, $\Xi \parallel \emptyset \vdash \mathsf{return}\, v : \tau$. By its inversion, $\Xi \parallel \emptyset \vdash v : \tau$. Thus, $\mathsf{return}\, v \in \mathbf{Tree}_{S^{\Xi}_{\tau}}$.

Case $\exists\, \sigma, v, x, e'.\ e \longrightarrow^{*} \sigma(v; x.\, e')$: By Lemma 19, $\Xi \parallel \emptyset \vdash \sigma(v; x.\, e') : \tau$. By its inversion and Lemma 17,

- $\sigma : B \rightsquigarrow \mathsf{n} \in \Xi$,
- $v = c$, and
- $\Xi \parallel x : \mathsf{n} \vdash e' : \tau$

for some B n, and c. Then, by the definition, $\mathbf{ET}(e) = \sigma(c, \mathbf{ET}(e'[\underline{1}/x]), \cdots, \mathbf{ET}(e'[\underline{\mathsf{n}}/x]))$. Thus, by the coinduction principle, it suffices to show that, for any $i \in [1, n]$, $\Xi \parallel \emptyset \vdash e'[\underline{\mathsf{i}}/x] : \tau$, which is shown by Lemma 16 with $\Xi \parallel x : \mathsf{n} \vdash e' : \tau$ and $\Xi \parallel \emptyset \vdash \underline{\mathsf{i}} : \mathsf{n}$.

Otherwise: Contradictory with Lemmas 19 and 18.

The uniqueness of $\mathbf{ET}(e)$ is shown by Lemma 40. □

Lemma 42 (Evaluation Preserves Effect Trees in EPCF)**.** If $\Xi \parallel \emptyset \vdash e : \tau$ and $e \longrightarrow^* e'$, then $\mathbf{ET}(e) = \mathbf{ET}(e')$.

Proof. By Lemmas 19 and 41, $\mathbf{ET}(e), \mathbf{ET}(e') \in \mathbf{Tree}_{S^{\Xi}_{\tau}}$. We show that $\mathbf{ET}(e) = \mathbf{ET}(e')$ by case analysis on the evaluation of e.

Case $e \longrightarrow^{\omega}$: By Lemma 40, $e' \longrightarrow^{\omega}$. Therefore, $\mathbf{ET}(e) = \mathbf{ET}(e') = \bot$.

Case $\exists\, v.\ e \longrightarrow^* \mathsf{return}\, v$: By Lemma 40, $e' \longrightarrow^* \mathsf{return}\, v$. Therefore, $\mathbf{ET}(e) = \mathbf{ET}(e') = \mathsf{return}\, v$.

Case $\exists\, \sigma, v, x, e_0.\ e \longrightarrow^* \sigma(v; x.\, e_0)$: Because $\mathbf{ET}(e)$ is well defined, we have $\sigma : B \rightsquigarrow \mathsf{n} \in \Xi$ and $v = c$ for some B, n, and c. By Lemma 40, $e' \longrightarrow^* \sigma(c; x.\, e_0)$. Therefore, $\mathbf{ET}(e) = \mathbf{ET}(e') = \sigma(c, \mathbf{ET}(e_0[\underline{1}/x]), \cdots, \mathbf{ET}(e_0[\underline{\mathsf{n}}/x]))$.

Otherwise: Contradictory with Lemmas 19 and 18.

□

Lemma 43 (Strengthening Typing in EPCF)**.**

- If $\Xi \parallel \Delta_1, \Delta_2, \Delta_3 \vdash v : \tau$ and $dom(\Delta_2) \cap fv(v) = \emptyset$, then $\Xi \parallel \Delta_1, \Delta_3 \vdash v : \tau$.
- If $\Xi \parallel \Delta_1, \Delta_2, \Delta_3 \vdash e : \tau$ and $dom(\Delta_2) \cap fv(v) = \emptyset$, then $\Xi \parallel \Delta_1, \Delta_3 \vdash e : \tau$.

Proof. Straightforward by mutual induction on the typing derivations. □

Lemma 44 (Subject Reduction for Full $\beta\eta$ Monadic Reduction)**.**

1. If $\Xi \parallel \Delta \vdash e : \tau$ and $e \hookrightarrow e'$, then $\Xi \parallel \Delta \vdash e' : \tau$.
2. If $\Xi \parallel \Delta \vdash v : \tau$ and $v \hookrightarrow v'$, then $\Xi \parallel \Delta \vdash v' : \tau$.

Proof. By mutual induction on $e \hookrightarrow e'$ and $v \hookrightarrow v'$.

1. By case analysis on how $e \hookrightarrow e'$ is derived.

   Case $e = e'$: Obvious.

   Case $\exists\, e''.\ e \hookrightarrow e'' \wedge e'' \hookrightarrow e'$: By the IH, $\Xi \parallel \Delta \vdash e'' : \tau$, so again by the IH, $\Xi \parallel \Delta \vdash e' : \tau$.

   Case $\exists\, x, e_1, v_2.\ e = (\lambda x.e_1)\, v_2 \wedge e' = e_1[v_2/x]$: Without loss of generality, we can assume that $x \notin dom(\Delta)$. By the inversion of $\Xi \parallel \Delta \vdash (\lambda x.e_1)\, v_2 : \tau$, there exists some $\Xi \parallel \Delta, x : \tau' \vdash e_1 : \tau$ and $\Xi \parallel \Delta \vdash v_2 : \tau'$ for some τ'. By Lemma 16, we have the conclusion $\Xi \parallel \Delta \vdash e_1[v_2/x] : \tau$.

   Case $\exists\, x, v, e_0.\ e = (\mathsf{let}\, x = \mathsf{return}\, v\, \mathsf{in}\, e_0) \wedge e' = e_0[v/x]$: Without loss of generality, we can assume that $x \notin dom(\Delta)$. By the inversion of $\Xi \parallel \Delta \vdash \mathsf{let}\, x = \mathsf{return}\, v\, \mathsf{in}\, e_0 : \tau$, we have $\Xi \parallel \Delta \vdash v : \tau'$ and $\Xi \parallel \Delta, x : \tau' \vdash e_0 : \tau$ for some τ'. By Lemma 16, we have the conclusion $\Xi \parallel \Delta \vdash e_0[v/x] : \tau$.

   Case $\exists\, x.\ e = (\mathsf{let}\, x = e'\, \mathsf{in}\, \mathsf{return}\, x)$: By the inversion of $\Xi \parallel \Delta \vdash \mathsf{let}\, x = e'\, \mathsf{in}\, \mathsf{return}\, x : \tau$, we have the conclusion $\Xi \parallel \Delta \vdash e' : \tau$.

   Case $\exists\, x, y, e_1, e_2, e_3.\ e = (\mathsf{let}\, x = (\mathsf{let}\, y = e_1\, \mathsf{in}\, e_2)\, \mathsf{in}\, e_3) \wedge e' = (\mathsf{let}\, y = e_1\, \mathsf{in}\, \mathsf{let}\, x = e_2\, \mathsf{in}\, e_3) \wedge y \notin fv(e_3)$: By the inversion of $\Xi \parallel \Delta \vdash \mathsf{let}\, x = (\mathsf{let}\, y = e_1\, \mathsf{in}\, e_2)\, \mathsf{in}\, e_3 : \tau$, we have

   - $\Xi \parallel \Delta \vdash e_1 : \tau_1$,
   - $\Xi \parallel \Delta, y : \tau_1 \vdash e_2 : \tau_2$, and
   - $\Xi \parallel \Delta, x : \tau_2 \vdash e_3 : \tau$

for some τ_1 and τ_2. To show the conclusion, by (T_LET) with $\Xi \parallel \Delta \vdash e_1 : \tau_1$, it suffices to show that

$$\Xi \parallel \Delta, y : \tau_1 \vdash \mathsf{let}\, x = e_2 \,\mathsf{in}\, e_3 : \tau \ .$$

By (T_LET) with $\Xi \parallel \Delta, y : \tau_1 \vdash e_2 : \tau_2$, it suffices to show that

$$\Xi \parallel \Delta, y : \tau_1, x : \tau_2 \vdash e_3 : \tau \ ,$$

which is derived by Lemma 15 with $\Xi \parallel \Delta, x : \tau_2 \vdash e_3 : \tau$.

Case compatibility rules: By the IHs.

2. By case analysis on how $v \hookrightarrow v'$ is derived.

   Case $v = v'$: Obvious.

   Case $\exists\, v''.\ v \hookrightarrow v'' \ \wedge\ v'' \hookrightarrow v'$: By the IH, $\Xi \parallel \Delta \vdash v'' : \tau$, so again by the IH, $\Xi \parallel \Delta \vdash v' : \tau$.

   Case $\exists\, x.\ v = \lambda x.v'\, x \ \wedge\ x \notin fv(v')$: Without loss of generality, we can assume that $x \notin dom(\Delta)$. By the inversion of $\Xi \parallel \Delta \vdash \lambda x.v'\, x : \tau$, there exist some τ_1 and τ_2 such that $\tau = \tau_1 \to \tau_2$ and $\Xi \parallel \Delta, x : \tau_1 \vdash v : \tau_1 \to \tau_2$. By Lemma 43, we have the conclusion $\Xi \parallel \Delta \vdash v : \tau_1 \to \tau_2$.

   Case compatibility rules: By the IHs.

□

Lemma 45 (Contextual Equivalence of Evaluation Composition)**.** If $\Xi \parallel \Delta \vdash e : \tau$ and $e \hookrightarrow\longrightarrow^{+}\hookrightarrow e'$, then $\Xi \parallel \Delta \vdash e \stackrel{\mathsf{ctx}}{=} e' : \tau$.

Proof. By the assumption $e \hookrightarrow\longrightarrow^{+}\hookrightarrow e'$, there exist some e_1 and e_2 such that $e \hookrightarrow e_1$ and $e_1 \longrightarrow^{+} e_2$ and $e_2 \hookrightarrow e'$. By Lemma 44 and Theorem 3 with $\Xi \parallel \Delta \vdash e : \tau$ and $e \hookrightarrow e_1$, we have $\Xi \parallel \Delta \vdash e \stackrel{\mathsf{ctx}}{=} e_1 : \tau$. By Lemma 19 and Theorem 2 with $e_1 \longrightarrow^{+} e_2$, we have $\Xi \parallel \Delta \vdash e_1 \stackrel{\mathsf{ctx}}{=} e_2 : \tau$. By Lemma 44 and Theorem 3 with $e_2 \hookrightarrow e'$, we have $\Xi \parallel \Delta \vdash e_2 \stackrel{\mathsf{ctx}}{=} e' : \tau$. Therefore, by the transitivity of contextual equivalence, $\Xi \parallel \Delta \vdash e \stackrel{\mathsf{ctx}}{=} e' : \tau$. □

Lemma 46 (Simulation of Divergence)**.** Let

- T be a ground type,
- $\Sigma = \{\sigma_i : B_i \rightsquigarrow E_i \,/\, \square\}^{1 \le i \le n}$,
- $\Xi = \{\sigma_i : B_i \rightsquigarrow E_i\}^{1 \le i \le n}$, and
- $\overline{v^{\mathsf{h}}} = v^{\mathsf{h}}_1, \cdots, v^{\mathsf{h}}_n$ such that, for any $i \in [1, n]$, $v^{\mathsf{h}}_i = \lambda x.\sigma_i(x; y.\, \mathsf{return}\, y)$.

If $\emptyset \vdash_{\mathcal{D}} M : \Sigma \triangleright T \,/\, \square$ and $M \longrightarrow^{\omega}$ and $\Xi \parallel \emptyset \vdash \overline{v^{\mathsf{h}}} : \Sigma$, then $[\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, \underline{1}) \longrightarrow^{\omega}$.

Proof. The proof proceeds by iteratively, and coinductively applying Lemma 37, at each step observing that in EPCF, if $e_1 \hookrightarrow\longrightarrow^{+}\hookrightarrow e_2$, then the number of reduction steps to an irreducible term from e_2 is strictly less than that of e_1, thanks to Theorem 3. □

Lemma 47 (CPS Transformation of Ground Values)**.** If T is a ground type and $\emptyset \vdash_{\mathcal{D}} V : T$, then $[\![V]\!]_{\mathcal{D}} = V$.

Proof. Straightforward by induction on the typing derivation of $\emptyset \vdash_{\mathcal{D}} V : T$. □

Lemma 48 (Simulation of Termination at Values)**.** Let T be a ground type. If $\emptyset \vdash_{\mathcal{D}} M : \Sigma \triangleright T \,/\, \square$ and $M \longrightarrow^{*} \mathsf{return}\ V$ and $\Xi \parallel \emptyset \vdash \overline{v^{\mathsf{h}}} : \Sigma$, then $[\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, \underline{1}) \longrightarrow^{*} \mathsf{return}\, [\![V]\!]_{\mathcal{D}'}$ for some $\mathcal{D}'$ such that $\emptyset \vdash_{\mathcal{D}'} V : T$.

Proof. By induction on the number of steps of the evaluation $M \longrightarrow^{*} \mathsf{return}\ V$. We proceed by case analysis on the result of applying Lemma 37 to $\emptyset \vdash_{\mathcal{D}} M : \Sigma \triangleright T \,/\, \square$.

Case 1: We are given $M = \mathsf{return}\ V$ and $[\![\mathsf{return}\ V]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, \underline{1}) \longrightarrow^{*} \mathsf{return}\, [\![V]\!]_{\mathcal{D}'}$ for some $\mathcal{D}'$ scuh that $\emptyset \vdash_{\mathcal{D}'} V : T$. Thus, we have the conclusion.

Case 2: Contradictory.

Case 3: Contradictory by Lemma 40.

Case 4: Contradictory.

Case 5: We are given some M' and $\mathcal{D}'$ such that

- $M \longrightarrow^+ M'$,
- $[\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{v^{\mathsf{h}}}, \underline{1}) \hookrightarrow\longrightarrow^+\hookrightarrow [\![M']\!]^{\mathsf{e}}_{\mathcal{D}'}@(\overline{v^{\mathsf{h}}}, \underline{1})$, and
- $\emptyset \vdash_{\mathcal{D}'} M' : \Sigma \triangleright T \,/\, \square$.

By Lemma 38, $M' \longrightarrow^* \mathsf{return}\, V$. By the IH, $[\![M']\!]^{\mathsf{e}}_{\mathcal{D}'}@(\overline{v^{\mathsf{h}}}, \underline{1}) \longrightarrow^* \mathsf{return}\, [\![V]\!]_{\mathcal{D}'}$ for some $\mathcal{D}'$ such that $\emptyset \vdash_{\mathcal{D}'} V : T$. By Lemma 23, $\Xi \parallel \emptyset \vdash [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{v^{\mathsf{h}}}, \underline{1}) : [\![T]\!]$. By Lemma 45, $\Xi \parallel \emptyset \vdash [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{v^{\mathsf{h}}}, \underline{1}) \stackrel{\mathsf{ctx}}{=} [\![M]\!]^{\mathsf{e}}_{\mathcal{D}'}@(\overline{v^{\mathsf{h}}}, \underline{1}) : [\![T]\!]$. Thus, by $[\![M']\!]^{\mathsf{e}}_{\mathcal{D}'}@(\overline{v^{\mathsf{h}}}, \underline{1}) \longrightarrow^* \mathsf{return}\, [\![V]\!]_{\mathcal{D}'}$ and the assumption that T is ground, we have $[\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{v^{\mathsf{h}}}, \underline{1}) \longrightarrow^* \mathsf{return}\, [\![V]\!]_{\mathcal{D}'}$.

∎

Lemma 49 (Simulation of Termination at Operation Calls)**.** Let

- T be a ground type,
- $\Sigma = \{\sigma_i : B_i \rightsquigarrow E_i \,/\, \square\}^{1 \le i \le n}$,
- $\Xi = \{\sigma_i : B_i \rightsquigarrow E_i\}^{1 \le i \le n}$, and
- $\overline{v^{\mathsf{h}}} = v^{\mathsf{h}}_1, \cdots, v^{\mathsf{h}}_n$ such that, for any $i \in [1, n]$, $v^{\mathsf{h}}_i = \lambda x.\sigma_i(x; y.\, \mathsf{return}\, y)$.

If $\emptyset \vdash_{\mathcal{D}} M : \Sigma \triangleright T \,/\, \square$ and $M \longrightarrow^* \sigma(V'; y.\, M')$ and $\Xi \parallel \emptyset \vdash \overline{v^{\mathsf{h}}} : \Sigma$, then there exist some $i \in [1, n]$, $\mathcal{D}_1$, and $\mathcal{D}_2$ such that

- $\sigma = \sigma_i$
- $\Xi \parallel \emptyset \vdash [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{v^{\mathsf{h}}}, \underline{1}) \stackrel{\mathsf{ctx}}{=} \sigma_i([\![V']\!]_{\mathcal{D}_1}; y.\, [\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2}@(\overline{v^{\mathsf{h}}}, \underline{1})) : [\![T]\!]$,
- $\emptyset \vdash_{\mathcal{D}_1} V' : B_i$, and
- $y : E_i \vdash_{\mathcal{D}_2} M' : \Sigma \triangleright T \,/\, \square$.

Proof. By induction on the number of steps of the evaluation $M \longrightarrow^* \sigma(V'; z.\, M')$. We proceed by case analysis on the result of applying Lemma 37 to $\emptyset \vdash_{\mathcal{D}} M : \Sigma \triangleright T \,/\, \square$.

Case 1: Contradictory by Lemma 40.

Case 2: Contradictory.

Case 3: By Lemma 38, we are given some $i \in [1, n]$, $\mathcal{D}_1$, and $\mathcal{D}_2$ such that

- $[\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{v^{\mathsf{h}}}, \underline{1}) \hookrightarrow \mathsf{let}\, y = v^{\mathsf{h}}_i\, [\![V']\!]_{\mathcal{D}_1}\, \mathsf{in}\, [\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2}@(\overline{v^{\mathsf{h}}}, \underline{1})$,
- $\emptyset \vdash_{\mathcal{D}_1} V' : B_i$, and
- $\emptyset, y : E_i \vdash_{\mathcal{D}_2} M' : \Sigma \triangleright T \,/\, \square$.

We have

$$\begin{array}{cll} & \mathsf{let}\, y = v^{\mathsf{h}}_i\, [\![V']\!]_{\mathcal{D}_1}\, \mathsf{in}\, [\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2}@(\overline{v^{\mathsf{h}}}, \underline{1}) & \\ \longrightarrow & \mathsf{let}\, y = \sigma([\![V']\!]_{\mathcal{D}_1}; y.\, \mathsf{return}\, y)\, \mathsf{in}\, [\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2}@(\overline{v^{\mathsf{h}}}, \underline{1}) & \text{(by (E\_LetE)/(E\_Beta))} \\ \longrightarrow & \sigma([\![V']\!]_{\mathcal{D}_1}; y.\, \mathsf{let}\, y = \mathsf{return}\, y\, \mathsf{in}\, [\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2}@(\overline{v^{\mathsf{h}}}, \underline{1})) & \text{(by (E\_LetOp))} \\ \hookrightarrow & \sigma([\![V']\!]_{\mathcal{D}_1}; y.\, [\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2}@(\overline{v^{\mathsf{h}}}, \underline{1}))\ . & \end{array}$$

By Lemmas 23 and 45, we have the conclusion

$$\Xi \parallel \emptyset \vdash [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{v^{\mathsf{h}}}, \underline{1}) \stackrel{\mathsf{ctx}}{=} \sigma([\![V']\!]_{\mathcal{D}_1}; y.\, [\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2}@(\overline{v^{\mathsf{h}}}, \underline{1})) : [\![T]\!]\ .$$

Case 4: Contradictory.

Case 5: We are given some M'' and $\mathcal{D}''$ such that

- $M \longrightarrow^{+} M''$,
- $[\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{v^{\mathsf{h}}},\underline{1}) \hookrightarrow\!\longrightarrow^{+}\!\hookrightarrow [\![M'']\!]^{\mathsf{e}}_{\mathcal{D}''}@(\overline{v^{\mathsf{h}}},\underline{1})$, and
- $\emptyset \vdash_{\mathcal{D}''} M'' : \Sigma \triangleright T \,/\, \square$.

By Lemma 38, $M'' \longrightarrow^{*} \sigma(V'; z.\, M')$. By the IH, there exist some $i \in [1,n]$, $\mathcal{D}_1$, and $\mathcal{D}_2$ such that

- $\sigma = \sigma_i$,
- $\Xi \parallel \emptyset \vdash [\![M'']\!]^{\mathsf{e}}_{\mathcal{D}''}@(\overline{v^{\mathsf{h}}},\underline{1}) \overset{\mathsf{ctx}}{\equiv} \sigma_i([\![V']\!]_{\mathcal{D}_1}; y.\, [\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2}@(\overline{v^{\mathsf{h}}},\underline{1})) : [\![T]\!]$,
- $\emptyset \vdash_{\mathcal{D}_1} V' : B_i$, and
- $y : E_i \vdash_{\mathcal{D}_2} M' : \Sigma \triangleright T \,/\, \square$.

By Lemmas 23 and 45,

$$\Xi \parallel \emptyset \vdash [\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{v^{\mathsf{h}}},\underline{1}) \overset{\mathsf{ctx}}{\equiv} [\![M'']\!]^{\mathsf{e}}_{\mathcal{D}''}@(\overline{v^{\mathsf{h}}},\underline{1}) : [\![T]\!] \;.$$

By the transitivity of the contextual equivalence, we have the conclusion.

□

Theorem 4 (Preservation of Effect Trees)**.** *Let T be a ground type and $\Sigma = \{\sigma_i : B_i \rightsquigarrow E_i \,/\, \square\}^{1 \le i \le n}$ and $\Xi = \{\sigma_i : B_i \rightsquigarrow E_i\}^{1 \le i \le n}$. Assume that $\emptyset \vdash_{\mathcal{D}} M : \Sigma \triangleright T \,/\, \square$ and $\sigma_1, \cdots, \sigma_n$ are ordered. Let $\overline{v^{\mathsf{h}}} = v^{\mathsf{h}}_1, \cdots, v^{\mathsf{h}}_n$ such that, for any $i \in [1,n]$, $v^{\mathsf{h}}_i = \lambda x.\sigma_i(x; y.\, \mathsf{return}\, y)$. Then, $\mathbf{ET}([\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{v^{\mathsf{h}}},\underline{1})) = \mathbf{ET}(M)$.*

Proof. Note that $[\![B]\!] = B$ and $[\![E]\!] = E$ for any B and E, so $[\![T]\!] = T$.

Because $\emptyset \vdash M : \Sigma \triangleright T \,/\, \square$ and Σ is top-level, $\mathbf{ET}(M)$ is well defined and is in $\mathbf{Tree}_{S^{\Sigma}_{T}}$ by Lemma 39.

Next, we show that $\mathbf{ET}([\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{v^{\mathsf{h}}},\underline{1}))$ is well defined and is in $\mathbf{Tree}_{S^{\Xi}_{[\![T]\!]}}$. We have $\Xi \parallel \emptyset \vdash \overline{v^{\mathsf{h}}} : \Sigma$ because we can derive $\Xi \parallel \emptyset \vdash \lambda x.\sigma_i(x; y.\, \mathsf{return}\, y) : B_i \to E_i$ for any $i \in [1,n]$ as follows:

$$\dfrac{\dfrac{\Xi \ni \sigma_i : B_i \rightsquigarrow E_i \qquad \dfrac{(\text{T\_Var})}{\Xi \parallel x : B_i \vdash x : B_i} \qquad \dfrac{\dfrac{(\text{T\_Var})}{\Xi \parallel x : B_i, y : E_i \vdash y : E_i}}{\Xi \parallel x : B_i, y : E_i \vdash \mathsf{return}\, y : E_i}\,(\text{T\_Return})}{\Xi \parallel x : B_i \vdash \sigma_i(x; y.\, \mathsf{return}\, y) : E_i}\,(\text{T\_Op})}{\Xi \parallel \emptyset \vdash \lambda x.\sigma_i(x; y.\, \mathsf{return}\, y) : B_i \to E_i}\,(\text{T\_Abs}).$$

Thus, by Lemma 23, $\Xi \parallel \emptyset \vdash [\![M]\!]^{\mathsf{e}}@(\overline{v^{\mathsf{h}}},\underline{1}) : [\![T]\!]$. Therefore, $\mathbf{ET}([\![M]\!]^{\mathsf{e}}@(\overline{v^{\mathsf{h}}},\underline{1})) \in \mathbf{Tree}_{S^{\Xi}_{[\![T]\!]}}$ by Lemma 41.

Finally, we show that $\mathbf{ET}([\![M]\!]^{\mathsf{e}}@(\overline{v^{\mathsf{h}}},\underline{1})) = \mathbf{ET}(M)$ by coinduction. We proceed by case analysis on the evaluation of M.

Case $M \longrightarrow^{\omega}$: By definition, $\mathbf{ET}(M) = \bot$. By Lemma 46, $\mathbf{ET}([\![M]\!]^{\mathsf{e}}_{\mathcal{D}}@(\overline{v^{\mathsf{h}}},\underline{1})) = \bot$. Therefore, we have the conclusion.

Case $\exists\, V.\ M \longrightarrow^{*} \mathsf{return}\, V$: By Lemma 48, $[\![M]\!]^{\mathsf{e}}@(\overline{v^{\mathsf{h}}},\underline{1}) \longrightarrow^{*} \mathsf{return}\, [\![V]\!]_{\mathcal{D}'}$ for some $\emptyset \vdash_{\mathcal{D}'} V : T$. Thus,

$$\begin{array}{rcl} \mathbf{ET}([\![M]\!]^{\mathsf{e}}@(\overline{v^{\mathsf{h}}},\underline{1})) & = & \mathsf{return}\, [\![V]\!]_{\mathcal{D}'} \\ & = & \mathsf{return}\, V \quad \text{(by Lemma 47)} \\ & = & \mathbf{ET}(M) \;. \end{array}$$

Case $\exists \sigma, V, M'.\ M \longrightarrow^{*} \sigma(V; y.\, M')$: Because $\mathbf{ET}(M) \in \mathbf{Tree}_{S^{\Sigma}_{T}}$, we have $V = c$ and $\sigma = \sigma_i$ for some c and i. Let $E_i = \mathsf{m}$ for some m. By the definition,

$$\mathbf{ET}(M) = \sigma_i(c, \mathbf{ET}(M'[\underline{1}/y]), \cdots, \mathbf{ET}(M'[\underline{\mathsf{m}}/y])) \;.$$

By Lemma 49,

- $\Xi \parallel \emptyset \vdash [\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, \underline{1}) \overset{\mathsf{ctx}}{=} \sigma_i([\![c]\!]_{\mathcal{D}_1}; y.\, [\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2} @(\overline{v^{\mathsf{h}}}, \underline{1})) : [\![T]\!]$,
- $\emptyset \vdash_{\mathcal{D}_1} c : B_i$, and
- $y : E_i \vdash_{\mathcal{D}_2} M' : \Sigma \triangleright T \,/\, \Box$

for some $\mathcal{D}_1$ and $\mathcal{D}_2$. Thus,

$$\begin{array}{rl} & \mathbf{ET}([\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, \underline{1})) \\ = & \mathbf{ET}(\sigma_i([\![c]\!]_{\mathcal{D}_1}; y.\, [\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2} @(\overline{v^{\mathsf{h}}}, \underline{1}))) \\ & \quad \text{(because } \Xi \parallel \emptyset \vdash [\![M]\!]^{\mathsf{e}}_{\mathcal{D}} @(\overline{v^{\mathsf{h}}}, \underline{1}) \overset{\mathsf{ctx}}{=} \sigma_i([\![c]\!]_{\mathcal{D}_1}; y.\, [\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2} @(\overline{v^{\mathsf{h}}}, \underline{1})) : [\![T]\!] \text{ and } [\![T]\!] \text{ is ground)} \\ = & \mathbf{ET}(\sigma_i(c; y.\, [\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2} @(\overline{v^{\mathsf{h}}}, \underline{1}))) \quad \text{(by Lemma 47)} \\ = & \sigma_i(c, \mathbf{ET}([\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2} @(\overline{v^{\mathsf{h}}}, \underline{1})[\underline{1}/y]), \cdots, \mathbf{ET}([\![M']\!]^{\mathsf{e}}_{\mathcal{D}_2} @(\overline{v^{\mathsf{h}}}, \underline{1})[\underline{\mathsf{m}}/y])) \ . \\ = & \sigma_i(c, \mathbf{ET}([\![M'[\underline{1}/y]]\!]^{\mathsf{e}}_{\mathcal{D}_2} @(\overline{v^{\mathsf{h}}}, \underline{1})), \cdots, \mathbf{ET}([\![M'[\underline{\mathsf{m}}/y]]\!]^{\mathsf{e}}_{\mathcal{D}_2} @(\overline{v^{\mathsf{h}}}, \underline{1}))) \quad \text{(by (HT\_EConst) and Lemma 31)} \ . \end{array}$$

Now, it suffices to show that, for any $j \in [1, m]$,

$$\mathbf{ET}([\![M'[\underline{\mathsf{j}}/y]]\!]^{\mathsf{e}} @(\overline{v^{\mathsf{h}}}, \underline{1})) = \mathbf{ET}(M'[\underline{\mathsf{j}}/y]) \ .$$

By the coinduction principle, it suffices to show that

$$\emptyset \vdash M'[\underline{\mathsf{j}}/y] : \Sigma \triangleright T \,/\, \Box \ ,$$

which is shown by Lemma 2 with $y : E_i \vdash_{\mathcal{D}_2} M' : \Sigma \triangleright T \,/\, \Box$ and $\emptyset \vdash \underline{\mathsf{j}} : \mathsf{m}$ (derived by (HT_EConst)).

Otherwise: Contradictory with Lemmas 9 and 14.

$\Box$

3.5 Expressivity

Lemma 50 (GEPCF $\subseteq$ HEPCF$^{\mathsf{ATM}}_{\Box}$)**.** If M is well typed in GEPCF, then M is also well typed in HEPCF$^{\mathsf{ATM}}_{\Box}$.

Proof. The program syntax and semantics of GEPCF is subsumed by HEPCF$^{\mathsf{ATM}}_{\Box}$. The types in GEPCF are defined as follows:

$$R, S \quad \overset{\text{def}}{=} \quad B \mid E \mid R \to_{\Xi} S$$

Types and effects in GEPCF are transformed into ones in HEPCF$^{\mathsf{ATM}}_{\Box}$ as follows:

$$\begin{array}{rcl} [\![B]\!]_{\mathsf{GEPCF}} & \overset{\text{def}}{=} & B \\ [\![E]\!]_{\mathsf{GEPCF}} & \overset{\text{def}}{=} & E \\ [\![R \to_{\Xi} S]\!]_{\mathsf{GEPCF}} & \overset{\text{def}}{=} & [\![R]\!]_{\mathsf{GEPCF}} \to [\![\Xi]\!]_{\mathsf{GEPCF}} \triangleright [\![S]\!]_{\mathsf{GEPCF}} \,/\, \Box \\ [\![\{\sigma_i : B_i \rightsquigarrow E_i\}^{1 \le i \le n}]\!]_{\mathsf{GEPCF}} & \overset{\text{def}}{=} & \{\sigma_i : B_i \rightsquigarrow E_i \,/\, \Box\} \end{array}$$

Then, it is easy to show that, if a term M is well typed in GEPCF, then M is also well typed in HEPCF$^{\mathsf{ATM}}_{\Box}$, by induction on the typing derivation of M in GEPCF using $[\![-]\!]_{\mathsf{GEPCF}}$. $\Box$

Lemma 51 (HEPCF$^{\mathsf{ATM}}$ $\subseteq$ HEPCF$^{\mathsf{ATM}}_{\Box}$)**.** If M is well typed in HEPCF$^{\mathsf{ATM}}$, then M is also well typed in HEPCF$^{\mathsf{ATM}}_{\Box}$.

Proof. The program syntax and semantics of HEPCF$^{\mathsf{ATM}}$ is subsumed by HEPCF$^{\mathsf{ATM}}_{\Box}$. The types in HEPCF$^{\mathsf{ATM}}$ are similar to those in HEPCF$^{\mathsf{ATM}}_{\Box}$ except that an answer type is either a computation type or a value type. In transforming types in HEPCF$^{\mathsf{ATM}}$ to ones in HEPCF$^{\mathsf{ATM}}_{\Box}$, if an answer type is a computation type, then the transformation is applied to the computation type recursively, and if the answer type is a value type, then it is transformed into $\emptyset \triangleright T \,/\, \Box$ where T is the result of transforming the HEPCF$^{\mathsf{ATM}}$ value type. Then, it is easy to show that, if a term M is well typed in HEPCF$^{\mathsf{ATM}}$, then M is also well typed in HEPCF$^{\mathsf{ATM}}_{\Box}$, by induction on the typing derivation of M in HEPCF$^{\mathsf{ATM}}$ using the aforementioned type transformation. $\Box$

Theorem 5 ($\mathsf{GEPCF} \cup \mathsf{HEPCF}^{\mathsf{ATM}} \subset \mathsf{HEPCF}^{\mathsf{ATM}}_{\square}$)**.** *If M is well typed in either* GEPCF *or* $\mathsf{HEPCF}^{\mathsf{ATM}}$*, then it is also well typed in* $\mathsf{HEPCF}^{\mathsf{ATM}}_{\square}$*. Furthermore, there exists a term M that is accepted by* $\mathsf{HEPCF}^{\mathsf{ATM}}_{\square}$ *but neither by* GEPCF *nor* $\mathsf{HEPCF}^{\mathsf{ATM}}$*.*

Proof. By Lemmas 50 and 51, $\mathsf{GEPCF} \cup \mathsf{HEPCF}^{\mathsf{ATM}} \subseteq \mathsf{HEPCF}^{\mathsf{ATM}}_{\square}$. As shown in [3], there exists a term L such that L is accepted by GEPCF but not by $\mathsf{HEPCF}^{\mathsf{ATM}}$, and there exists a term N such that N is accepted by $\mathsf{HEPCF}^{\mathsf{ATM}}$ but not by GEPCF. $\mathsf{HEPCF}^{\mathsf{ATM}}_{\square}$ accept both L and N by Lemmas 50 and 51, and it is easy to construct a term M from L and N such that M is accepted by $\mathsf{HEPCF}^{\mathsf{ATM}}_{\square}$ but neither by GEPCF nor $\mathsf{HEPCF}^{\mathsf{ATM}}$. $\square$

References

[1] Ugo Dal Lago, Francesco Gavazzo, and Paul Blain Levy. Effectful applicative bisimilarity: Monads, relators, and howe's method. In *32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, Reykjavik, Iceland, June 20-23, 2017*, pages 1–12. IEEE Computer Society, 2017.

[2] Naoki Kobayashi. Model checking higher-order programs. *J. ACM*, 60(3):20:1–20:62, 2013.

[3] Taro Sekiyama and Hiroshi Unno. Higher-order model checking of effect-handling programs with answer-type modification. *Proc. ACM Program. Lang.*, 8(OOPSLA2):2662–2691, 2024.